Chalmers–Metcalf operator and uniqueness of minimal projections

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Received 12 October 2005; received in revised form 9 October 2006; accepted 20 October 2006
Communicated by E.W. Cheney
Available online 06 March 2007

Abstract

We know that not all minimal projections in $L_p$ ($1 < p < \infty$) are unique (see [B. Shekhtman, L. Skrzypek, On the non-uniqueness of minimal projections in $L_p$ spaces]). The aim of this paper is examine the connection of the Chalmers–Metcalf operator (introduced in [B.L. Chalmers, F.T. Metcalf, A characterization and equations for minimal projections and extensions, J. Oper. Theory 32 (1994) 31–46]) to the uniqueness of minimal projections. The main theorem of this paper is Theorem 2.2. It relates uniqueness of minimal projections to the invertibility of the Chalmers–Metcalf operator. It is worth mentioning that to a given minimal projection (even unique) we may find many different Chalmers–Metcalf operators, some of them invertible, some not—see Example 2.6. The main application is in Section 3, where we have proven that minimal projections onto symmetric subspaces in smooth Banach spaces are unique (Theorem 3.2). This leads (in Section 4) to the solution of the problem of uniqueness of classical Rademacher projections in $L_p[0, 1]$ for $1 < p < \infty$.

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MSC: 41A65; 41A35; 41A52; 46A22; 42C10; 47A58

Keywords: Approximation theory; Minimal projection; Extension operators; Rademacher functions

\textsuperscript{*} Both authors were supported by Polish State Committee for Scientific Research Grant KBN 1 P03A 01026. The second author was additionally supported by NATO Advanced Research Grant.

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doi:10.1016/j.jat.2006.10.008
0. Introduction

A projection is taken to mean any bounded linear operator $P$ that carries a Banach space $X$ onto a linear subspace $V$ in such a way that it acts as an identity on $V$. We denote the set of all projections from $X$ onto $V$ by $P(X, V)$.

Not every subspace of a given Banach space is the range of a bounded projection. For example, there is no continuous projection from $B[0, 1]$ onto $C[0, 1]$. Subspaces which are the range of bounded projections are called complemented subspaces and are crucial in the study of Banach spaces.

Projections play an important role in numerical analysis, the error of approximation of an element $x$ by $Px$ (i.e., the quantity $\|x - Px\|$) can be estimated by means of the elementary inequality

$$\|x - Px\| \leq \|Id - P\| \cdot \text{dist}(x, V) \leq (1 + \|P\|) \cdot \text{dist}(x, V).$$

Here $\text{dist}(x, V)$ denotes the infimum of $\|x - v\|$ as $v$ ranges over the subspace $V$. The above inequality signifies the consideration of a related problem—making $\|P\|$ small. We are therefore led to make the following definition. A projection $P_0 \in P(X, V)$, is called minimal if

$$\|P_0\| = \lambda(V, X) = \inf\{\|P\| : P \in P(X, V)\}.$$

The constant $\lambda(V, X)$, is called the relative projection constant. If $P$ minimalizes $\|Id - P\|$ then it is called co-minimal.

There is also a significant connection between projections and functional analysis. We can consider a minimal projection as the extension of the Hahn–Banach theorem. Any projection provides us with the way of linearly extending any functional $v^* \in V^*$ to $X^*$ (setting $x^* = v^* \circ P$), or equivalently we can speak of a linear extension of the operator $Id_V : V \to V$ to the operator $P : X \to V$. The smallest the $\|P\|$, the better the extension. And the uniqueness of a minimal projection results in the uniqueness of this extension.

Observe that any projection with norm 1 is automatically minimal, though in general, a given subspace will not be the range of a projection of norm 1. The first problem is to find out whether a considered subspace is complemented; but even if a considered subspace is complemented there could be no minimal projection (as the infimum does not have to be attained); for example see [2]. In many cases the existence of a minimal projection is known a priori (see [16,27]), which is the case when the subspace is finite-dimensional or finite-codimensional. Even in such cases, minimal projections will be difficult to find. As dramatic evidence of the difficult nature of such problems, one may cite the fact that the minimal projections of $C([0, 1])$ onto the subspace of polynomials of degree $>2$ remain unknown. There are rather few situations in which minimal projections are known explicitly or are characterized by some interesting property. Still rarer is the situation in which the minimal projection is known to be unique.

Trivially, in $L_2$, minimal projections are orthogonal projections, have norm 1 and are uniquely minimal. On the other hand, $L_1$ usually lacks uniqueness. For instance, from $\ell^2_1$ onto $V = \{(x, y) : x = 0\}$ we have many minimal projections as $P(x, y) = (0, x + y)$ and $Q(x, y) = (0, x - y)$ both have norm 1. Hence, both are minimal. At the same time we can also have uniqueness in $\ell_1$. There is only one minimal projection from $\ell^2_1$ onto $V = \{(x, y) : x + y = 0\}$ and it is given by the formula $P(x, y) = \left(\frac{-x+y}{2}, -\frac{x+y}{2}\right)$.

As to the uniqueness of minimal projections in $L_p$ ($1 < p < \infty, p \neq 2$), not much is known. The classical result of Cohen–Sullivan [11] says that in $L_p$ ($1 < p < \infty$) the norm 1 projection
is uniquely minimal. As a result, the minimal projections on all one-dimensional subspaces are uniquely minimal in $L_p$ \((1 < p < \infty)\). On the other hand, by the result of Odyniec [28–31], all minimal projections onto codimension 1 subspaces are also uniquely minimal in $L_p$ \((1 < p < \infty)\). Recently, see [36], it has been proven that minimal projections onto two-dimensional subspaces are also uniquely minimal in $L_p$ \((1 < p < \infty)\). To complete the picture, [35] gives the example of minimal projection in $\ell^p_5$ onto three-dimensional subspace, which is not uniquely minimal.

Minimal projections and its properties have been studied by many authors. See, for example, [3,4,10,11,14,15,17–23,25–27,31,32].

One of the main tools to study minimal projections is the so-called Chalmers–Metcalf operator. We can define it as follows.

Below we assume that $X$ is a normed space and $V$ is a finite-dimensional subspace.

**Definition 0.1.** A pair $(x, y) \in S(X^{\ast\ast}) \times S(X^{\ast})$ will be called an extremal pair for $P \in P(X, V)$ iff $y(P^\ast\ast x) = \|P\|$, where $P^\ast\ast : X^{\ast\ast} \to V$ is the second adjoint extension of $P$ to $X^{\ast\ast}$. Let $\mathcal{E}(P)$ be the set of all extremal pairs for $P$.

To each $(x, y) \in \mathcal{E}(P)$ we associate the rank-one operator $x \otimes y$ from $X$ to $X^{\ast\ast}$ given by $(x \otimes y)(z) = y(z) \cdot x$ for $z \in X$.

If $P$ is a projection from $X$ onto a finite-dimensional subspace $V$, then (since $P$ is a compact operator) it has a norming functional, i.e., there is a functional $y \in S(V^{\ast})$ such that $\|y \circ P\| = \|P\|$ (see [31, Theorem III.2.1]). If $X$ is reflexive then any functional attains its norm [33]. Therefore, there is an extremal pair for $P$, i.e. there is $(x, y) \in S(X) \times S(V^{\ast})$, such that $yPx = \|P\|$ (see Definition 0.1). If $X$ is not reflexive then in general, it is not true. For example, the classical Fourier projection does not attain its norm in $C[0, 2\pi]$. But any functional attains its norm in $X^{\ast\ast}$, hence we can always find a extremal pair for $P$ in $S(X^{\ast\ast}) \times S(V^{\ast})$.

**Theorem 0.2 (Chalmers–Metcalf [4, Theorem 1]).** A projection $P \in P(X, V)$ has a minimal norm if and only if the closed convex hull of $\{x \otimes y\}_{(x, y)\in \mathcal{E}(P)}$ contains an operator $E_P$ for which $V$ is an invariant subspace.

The operator $E_P$ is called Chalmers–Metcalf operator and is given by the formula

$$E_P = \int_{\mathcal{E}(P)} x \otimes y \, d\mu(x, y) : X \to X^{\ast\ast},$$

where $\mu$ is a probability Borel measure on $\mathcal{E}(P)$.

The Chalmers–Metcalf theorem has many applications especially in the case of $X = L_1$, e.g., [3–7,13,21–23,35,38,39]. The aim of this paper is to relate some properties of this operator to the uniqueness of minimal projections. We will study several examples of minimal projections and as a result we will prove uniqueness of Rademacher projections in $L_p$.

1. Technical results

**Definition 1.1.** Let $L(X, V)$ denote the set of all continuous linear operators from $X$ to $V$. By $L_V(X, V)$ we will denote the following subspace of $L(X, V)$:

$$L_V(X, V) = \{L \in L(X, V) : L/V \equiv 0\}.$$
Finding the minimal projection, as first observed in [27], can be related to the problem of a best approximation. In fact, we have the following well-known theorem.

**Theorem 1.2.** A projection $P_0 \in P(X, V)$ is minimal if and only if there is a functional $f \in S(L(X, V)^*)$ such that $f/L_{V(X, V)} = 0$ and $f(P_0) = \|P_0\|$.

We will state theorems that connect Chalmers–Metcalf operators to the functionals from the above theorem, thus allowing us to obtain many properties of Chalmers–Metcalf operators that cannot be derived from Theorem 0.2. The proofs and further discussion are contained in [24]. We will also assume that extremal pairs are taken from the set

$$E(P) = \{(x, y) \in S(X^{**}) \times S(V^*), y(P^{**}x) = \|P]\}. \quad (1.2)$$

The difference is that in the original Definition 0.1 the functionals $y$’s are taken from $S(X^*)$ (instead of $S(V^*)$). As a result, the Chalmers–Metcalf operators are linear operators from $V$ to $X^{**}$ (instead of from $X$ to $X^{**}$, as in Theorem 0.2). Observe also that once you have a Chalmers–Metcalf operator from $V$ to $X^{**}$ then, by using Hahn–Banach theorem and extending all functionals $y$’s to $X^*$, we can obtain a Chalmers–Metcalf functional from $X$ to $X^{**}$.

**Theorem 1.3** (Lewicki and Skrzypek [24]). Let $P_{\min}$ be a minimal projection from $X$ to $V$. With additional assumption that $X^*$ is separable and $V$ is finite-dimensional, there is one-to-one correspondence between functionals corresponding to $P_{\min}$ from Theorem 1.2 and Chalmers–Metcalf operators corresponding to $P_{\min}$. In particular each Chalmers–Metcalf operator can be treated as a functional. Namely, each functional can be written as

$$f = \int_{E(P_{\min})} x \otimes y \, d\mu(x, y), \quad (1.3)$$

where $(x \otimes y)(L) = y(L^{**}x)$. And the corresponding Chalmers–Metcalf operator can be written as

$$E_{P_{\min}} = \int_{E(P_{\min})} x \otimes y \, d\mu(x, y), \quad (1.4)$$

where $(x \otimes y)(z) = y(z) \cdot x$.

**Lemma 1.4** (Lewicki and Skrzypek [24]). Let $P$ and $Q$ be two minimal projections from $X$ onto $V$. Let $E_P$ be a Chalmers–Metcalf operator for $P$ and $\mu$ be the measure it represents. Then

$$E(P) \cap \text{supp} \, \mu \subset E(Q) \quad \mu \text{ almost everywhere.} \quad (1.5)$$

**Proof.** Since $\|Q\| = \|P\|$ and by Theorem 1.2,

$$\|Q\| = E_P(Q) = \int_{E(P)} x \otimes y(Q) \, d\mu(x, y)$$

$$= \int_{E(P)} y(Q^{**}x) \, d\mu(x, y) \leq \int_{E(P)} \|Q\| \, d\mu(x, y) = \|Q\|, \quad (1.6)$$

which means that for $(x, y) \in E(P)$ we have $y(Q^{**}x) = \|Q\| \mu$ almost everywhere. Hence, $E(P) \cap \text{supp} \, \mu \subset E(Q) \mu$ almost everywhere. □
Theorem 1.5 (Lewicki and Skrzypek [24]). Let \( P \) and \( Q \) be two minimal projections from \( X \) onto \( V \). Let \( E_P \) be a Chalmers–Metcalf operator for \( P \). Then \( E_P \) is also a Chalmers–Metcalf operator for \( Q \), i.e., any Chalmers–Metcalf operator is “good” for all minimal projections.

Theorem 1.6 (Lewicki and Skrzypek [24]). For a given minimal projection \( P \) the set of Chalmers–Metcalf operators \( E_P \) is convex.

Theorem 1.7 (Lewicki and Skrzypek [24]). Assume that \( X \) is finite-dimensional. Then each Chalmers–Metcalf operator can be written in such way that the measure \( \mu \) is supported on at most \((\dim X \cdot \dim V + 1)\) points.

When \( X \) is finite-dimensional the correspondence between the functional from Theorem 1.2 and the Chalmers–Metcalf operator is given by trace duality (for the full discussion of it see [24]). Trace duality has been used in [17] for estimating the absolute projection constant of finite-dimensional spaces. We will need the following facts (see [40] for details) concerning trace duality.

Let \( X, V \) be finite-dimensional Banach spaces. Then the nuclear norm in \( L(X, V) \) is defined by

\[
v(L) = \inf \left\{ \sum_{i=1}^{n} \| x_i^* \| \cdot \| y_i \| : L = \sum_{i=1}^{n} x_i^* (\cdot) y_i \right\}.
\]

Then we have the following trace duality \((L(V, X), v) = L(X, V)^* \) where “=” means linearly isometric and this isometry is given by

\[
L(V, X) \ni L \mapsto L^{tr} = (K \mapsto \text{tr}(L \circ K)) \in L(X, V)^*.
\]

Additionally, \( L(V, X) \) endowed with the nuclear \( v \) norm forms a Banach operator ideal.

When \( X \) is finite dimensional we have the following:

Theorem 1.8 (Lewicki and Skrzypek [24]). With the assumption that \( X \) is finite dimensional there is a one-to-one correspondence between functionals from Theorem 1.2 and Chalmers–Metcalf operators. This correspondence is given by trace duality. Namely, each functional can be written as

\[
f = \sum_{E(P)} \alpha_i \cdot x_i \otimes y_i,
\]

where \( \alpha_i > 0, \sum \alpha_i = 1 \) and \((x \otimes y)(L) = y(Lx)\). While the corresponding Chalmers–Metcalf operator can be written as

\[
E_P = \sum_{E(P)} \alpha_i \cdot x_i \otimes y_i : V \rightarrow X,
\]

where \( \alpha_i > 0, \sum \alpha_i = 1 \) and \((x \otimes y)(z) = y(z)x\).

It is easy to check that

\[
f = (E_P)^{tr},
\]

i.e., \( f \) and \( E_P \) are related by trace duality (1.8).
The next theorem is actually a part of Theorem 1.8. It shows that each Chalmers–Metcalf operator, by trace duality, generates a functional from Theorem 1.2. This part will be used in Section 2, therefore we will prove it here.

**Theorem 1.9** (Lewicki and Skrzypek [24]). Assume that $X$ is finite dimensional. Let $P$ be a minimal projection from $X$ onto $V$. Then for each Chalmers–Metcalf operator $E : V \to X$ we have

$$E(V) \subset V, \quad \nu(E) = 1$$

and

$$\text{tr}(E \circ Q) = \text{tr}(E) = \|P\| \quad \text{for any projection } Q : X \to V. \quad (1.12)$$

Additionally, each operator $E$ fulfilling the above conditions generates, by trace duality, a linear functional from Theorem 1.3.

**Proof.** Take Chalmers–Metcalf operator $E_P$ ($E_P(V) \subset V$) given by

$$E_P = \sum_{\mathcal{E}(P)} \alpha_i x_i \otimes y_i : V \to X, \quad (1.13)$$

where $\alpha_i > 0$ and $\sum_i \alpha_i = 1$.

Since $(x, y)$ is an extremal pair for $P$ (i.e., $y(Px) = \|P\|$), and the trace of the operator does not depend on its particular representation we have

$$\text{tr}(E_P \circ P) = \text{tr} \left( \sum_{\mathcal{E}(P)} \alpha_i x_i \otimes (y_i \circ P) \right) = \sum_{\mathcal{E}(P)} \alpha_i y_i(Px_i) = \sum_{\mathcal{E}(P)} \alpha_i \|P\| = \|P\|. \quad (1.14)$$

Assume that $e_1, \ldots, e_n$ is a basis for $V$. Since $E_P(V) \subset V$, we can represent $E_P$ as follows:

$$E_P(x) = \sum_{k=1,\ldots,n} e_k^*(x)e_k. \quad (1.15)$$

Since the trace of an operator does not depend on a particular representation then for any projection $R : X \to V$,

$$\text{tr} (E_P \circ R) = \sum_{k=1,\ldots,n} e_k^*(Re_k) = \sum_{k=1,\ldots,n} e_k^*(e_k) = \text{tr} (E_P). \quad (1.16)$$

Therefore, for any projection $R : X \to V$ we get

$$\text{tr}(E_P \circ R) = \text{tr}(E_P) = \|P\|. \quad (1.17)$$

Now it is easy to see that the nuclear norm of a Chalmers–Metcalf operator is 1. Indeed, from (1.13) and the definition of nuclear norm we have $\nu(E_P) \leq 1$. By (1.8) we have

$$\|K \mapsto \text{tr}(E_P \circ K)\| = \nu(E_P) \leq 1. \quad (1.18)$$

But since $\text{tr}(E_P \circ P) = 1$, by the above, the nuclear norm of $E_P$ (as well as its norm as a functional) is equal to 1.
Let \((EP)^{tr}\) denote the functional related to \(EP\) by trace duality. From (1.8) we have \(\|EP^{tr}\| = 1\). Take any \(L \in LV(X, V)\). We can write \(L = (P + L) - P\). Observe that \(P + L \in P(X, V)\), therefore using (1.17) we have
\[
(EP)^{tr}(L) = (EP)^{tr}((P + L) - P) = (EP)^{tr}(P + L) - (EP)^{tr}(P) = \|P\| - \|P\| = 0.
\]
□
(1.19)

2. The Chalmers–Metcalf operator and uniqueness of minimal projections

**Definition 2.1.** Let \(X\) be a Banach space. An element \(x \in X\) is called a smooth point if it has a unique supporting functional \(f_x\). If every \(x\) from the unit sphere is a smooth point, then \(X\) is called a smooth space.

For some basic facts of smoothness as well as some interesting applications see, e.g., [12].

**Theorem 2.2.** Assume \(X\) is a Banach space such that \(X^{**}\) is smooth. Let \(V\) be a finite-dimensional subspace of \(X\). Assume that for a minimal projection \(P\) there is a Chalmers–Metcalf operator \(EP\) such that \(EP/V\) is invertible. Then \(P\) is the unique minimal projection.

**Proof.** Since \(P \in P(X, V)\) is a minimal projection, then by the Chalmers–Metcalf theorem (Theorem 0.2), there is an operator \(EP\) such that \(EP(V) \subset V\), given by the formula
\[
EP = \int_{\mathcal{E}(P)} x \otimes y d\mu(x, y) : X \to X^{**},
\]
where \(\mu\) is a probabilistic Borel measure on \(\mathcal{E}(P)\). Assume now that there is another minimal projection \(Q \in P(X, V)\).

Now we will note the convenience of the smoothness. Take \((x, y) \in \mathcal{E}(P) \cap \text{supp} \mu\). By Lemma 1.4 \((x, y) \in \mathcal{E}(Q) (\mu \text{ almost everywhere})\), we will prove that
\[
y \circ P^{**} = y \circ Q^{**} \text{ as functionals on } X^{**} (\mu \text{ almost everywhere}).
\]
Indeed, \(y(P^{**}x) = \|P\|\) and \(y(Q^{**}x) = \|Q\|\) implies
\[
\frac{y \circ P^{**}}{\|P^{**}\|}(x) = 1 \text{ and } \frac{y \circ Q^{**}}{\|Q^{**}\|}(x) = 1.
\]
Additionally \(\|y \circ Q^{**}\| = \|y \circ P^{**}\| = \|Q^{**}\| = \|P^{**}\|\) and (2.3) gives two supporting functionals for \(x\) in \(X^{**}\). But \(X^{**}\) is smooth. As a result, these two functionals have to be the same, i.e.,
\[
\frac{y \circ P^{**}}{\|P^{**}\|} = \frac{y \circ Q^{**}}{\|Q^{**}\|}.
\]
Applying \(\|Q^{**}\| = \|Q\| = \|P\| = \|P^{**}\|\) we get \(y \circ P^{**} = y \circ Q^{**}\), hence (2.2) is proven.

Now if \(P \neq Q\), then also \(P^{**} \neq Q^{**}\); and there is \(x_0 \in X^{**}\) such that
\[
w_0 := P^{**}(x_0) - Q^{**}(x_0) \neq 0.
\]
From (2.2), for $\mu$ almost every $(x, y) \in \mathcal{E}(P)$ we have

$$y(w_0) = y(P^{**}(x_0) - Q^{**}(x_0)) = y(P^{**}(x_0)) - y(Q^{**}(x_0)) = (y \circ P^{**} - y \circ Q^{**})(x_0) = 0.$$  \hfill (2.6)

Considering now (2.1) and (2.6) we have

$$E_P(w_0) = \int_{\mathcal{E}(P)} y(w_0) \cdot x \, d\mu(x, y) = 0,$$  \hfill (2.7)

a contradiction with the invertibility of $E_P / V$. $\square$

If we know that minimal projection $P$ attains its norm in $X$ (instead of $X^{**}$) then the Chalmers–Metcalf operators act from $X$ to $X$ (instead of $X$ to $X^{**}$). As a result, the proof of Theorem 2.2 holds with the weaker assumption of $X$ being smooth. To summarize we can state the following.

**Corollary 2.3.** Assume $X$ is a smooth Banach space. Let $V$ be a finite-dimensional subspace of $X$. Assume that a minimal projection $P$ attains its norm in $X$ and that there is a Chalmers–Metcalf operator $E_P$ such that $E_P / V$ is invertible. Then $P$ is the unique minimal projection.

For Theorem 2.2 to work we need only one Chalmers–Metcalf operator that is invertible. In general we may have the following.

**Remark 2.4.** It is possible that to one minimal projection there corresponds two or more different Chalmers–Metcalf operators. What is more, it is possible that some of them are invertible and some of them not.

We will construct a projection mentioned in Remark 2.4 in a series of examples below.

**Example 2.5.** Let $f = (1, 1, \ldots, 1) \in \ell^n_\infty$ be a representation of a functional. Then $P : \ell^n_p \to \ker f$ given by

$$P = \text{Id} - \frac{1}{n}(1, 1, \ldots, 1) \otimes (1, 1, \ldots, 1)$$  \hfill (2.8)

is the unique minimal projection for any $1 < p < \infty$. Moreover, there is $E_P$ such that

$$E_P / V = \frac{\|P\|}{n - 1} \text{Id}_V.$$  \hfill (2.9)

**Proof.** Since $P$ is acting on a finite-dimensional space it has a norming pair $(x, y) \in \mathcal{E}(P)$. Now for any $\sigma \in S_n$ a permutation of a set $\{1, 2, \ldots, n\}$ defines the isometries

$$L_\sigma(x_1, x_2, \ldots, x_n) = (x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)}).$$  \hfill (2.10)

Observe that $P$ commutes with $L_\sigma$. Hence, if $(x, y) \in \mathcal{E}(P)$ then $(L_\sigma x, L_\sigma y) \in \mathcal{E}(P)$. Therefore, we can construct $E_P$ as follows:

$$E_P = \frac{1}{n!} \sum_{\sigma \in S_n} L_\sigma x \otimes L_\sigma y.$$  \hfill (2.11)
Now compute $E_P(1, -1, 0, 0, \ldots, 0)$,
\[
n! \cdot E_P(1, -1, 0, 0, \ldots, 0) = \sum_{\sigma \in S_n} L_\sigma y(1, -1, 0, 0, \ldots, 0) \cdot L_\sigma x
\]
\[
= \sum_{\sigma \in S_n} (y_{\sigma(1)} - y_{\sigma(2)}) \cdot L_\sigma x
\]
\[
= \sum_{\sigma \in S_n} y_{\sigma(1)} \cdot L_\sigma x - \sum_{\sigma \in S_n} y_{\sigma(2)} \cdot L_\sigma x
\]
\[
= \sum_{k=1}^n y_k \sum_{\sigma : \sigma(1) = k} L_\sigma x - \sum_{k=1}^n y_k \sum_{\sigma : \sigma(2) = k} L_\sigma x
\]
\[
= \sum_{k=1}^n y_k \left( \sum_{\sigma : \sigma(1) = k} L_\sigma x - \sum_{\sigma : \sigma(2) = k} L_\sigma x \right).
\] (2.12)

Observe that
\[
\sum_{\sigma : \sigma(1) = k} L_\sigma x = (n - 1)! \left( x_k, \sum_{i \neq k} x_i, \sum_{i \neq k} x_i, \ldots, \sum_{i \neq k} x_i \right)
\] (2.13)
and
\[
\sum_{\sigma : \sigma(2) = k} L_\sigma x = (n - 1)! \left( \sum_{i \neq k} x_i, x_k, \sum_{i \neq k} x_i, \ldots, \sum_{i \neq k} x_i \right).
\] (2.14)

Hence,
\[
n \cdot E_P(1, -1, 0, 0, \ldots, 0) = \sum_{k=1}^n y_k \left( \sum_{\sigma : \sigma(1) = k} L_\sigma x - \sum_{\sigma : \sigma(2) = k} L_\sigma x \right)
\]
\[
= \sum_{k=1}^n y_k \cdot \left( x_k - \sum_{i \neq k} x_i, -(x_k - \sum_{i \neq k} x_i), 0, \ldots, 0 \right)
\]
\[
= \left( \sum_{k=1}^n y_k \cdot \left( x_k - \sum_{i \neq k} x_i \right) \right) \cdot (1, -1, 0, \ldots, 0).
\] (2.15)

In the same manner, for any $j = 2, \ldots, n$ we obtain
\[
n \cdot E_P(e_1 - e_j) = \left( \sum_{k=1}^n y_k \cdot \left( x_k - \sum_{i \neq k} x_i \right) \right) \cdot (e_1 - e_j).
\] (2.16)

Therefore, since $\{e_1 - e_k\}$ is a basis for $V = ker(1, 1, \ldots, 1)$, then we proved that
\[
E_P / V = c \cdot Id_V.
\] (2.17)

Applying Theorem 1.9, and since $tr(Id_V) = dim V = n - 1$, we obtain $c = \|P\|_{n-1}$. Hence, $E_P$ defined by (2.11) is a Chalmers–Metcalf operator for $P$ and therefore $P$ is minimal. By Theorem
2.2 \( P \) is the unique minimal projection (actually the uniqueness follows from the fact that in \( \ell^n_p \) \((1 < p < \infty)\) minimal projections onto hyperplanes are unique—see [31, Theorem I.1.3]). □

**Example 2.6.** Take

\[
f_1 = (1, 1, \ldots, 1, 0, 0, \ldots, 0) \in \ell^2_n
\]

and

\[
f_2 = (0, 0, \ldots, 0, 1, 1, \ldots, 1) \in \ell^2_n.
\]

Then \( Q : \ell^n_p \to V = \ker f_1 \cap \ker f_2 \) given by

\[
Q = \text{Id} - \frac{1}{n} f_1 \otimes f_1 - \frac{1}{n} f_2 \otimes f_2
\]

is the unique minimal projection for any \( 1 < p < \infty \). Moreover, there are two Chalmers–Metcalf operators \( E_Q \) and \( \tilde{E}_Q \), such that

\[
E_Q/V \text{ is invertible and } \tilde{E}_Q/V \text{ is not invertible.}
\]

**Proof.** Let \( P \) be a projection as in Example 2.5 (see (2.8)). It can be easily seen that

\[
Q(x_1, \ldots, x_{2n}) = (P(x_1, \ldots, x_n), P(x_{n+1}, \ldots, x_{2n})). \quad (2.19)
\]

Now we will show that \( \| Q \| = \| P \| \), indeed

\[
\| Q(x_1, \ldots, x_{2n}) \|_p^p = \sum_{i=1}^{n} |x_i - \left( \sum_{i=1}^{n} x_i \right) |^p + \sum_{i=n+1}^{2n} |x_i - \left( \sum_{i=n+1}^{2n} x_i \right) |^p
\]

\[
\leq \| P \|_p^p \left( \sum_{i=1}^{n} |x_i|^p \right) + \| P \|_p^p \left( \sum_{i=n+1}^{2n} |x_i|^p \right)
\]

\[
= \| P \|_p^p \cdot \| (x_1, \ldots, x_{2n}) \|_p^p. \quad (2.20)
\]

From the above computation we deduce \( \| Q \| = \| P \| \) and

\[
(x_1, \ldots, x_{2n}) \text{ is a norming point for } Q;
\]

\[
\triangleleft
\]

\[
(x_1, \ldots, x_n) = 0 \text{ or } \frac{(x_1, \ldots, x_n)}{\| (x_1, \ldots, x_n) \|} \text{ is a norming point for } P;
\]

and

\[
(x_{n+1}, \ldots, x_{2n}) = 0 \text{ or } \frac{(x_{n+1}, \ldots, x_{2n})}{\| (x_{n+1}, \ldots, x_{2n}) \|} \text{ is a norming point for } P. \quad (2.21)
\]

Let \( E_P = \frac{1}{k} \sum_{i=1}^{k} x_i \otimes y_i \) be a Chalmers–Metcalf operator for \( P \) given by Example 2.5. By the above statement (2.21) \((x_i, 0), (y_i, 0) \in E(Q)\), and we may define

\[
\tilde{E}_Q = \frac{1}{k} \sum_{i=1}^{k} (x_i, 0) \otimes (y_i, 0). \quad (2.22)
\]
It is easy to check that
\[ \tilde{E}_Q(z_1, \ldots, z_{2n}) = (E_P(z_1, \ldots, z_n), 0). \] (2.23)
Hence, by (2.9) if \((v_1, \ldots, v_n) \in \ker f_1 \cap \ker f_2\) then
\[ \tilde{E}_Q(v_1, \ldots, v_n) = \|Q\| \frac{1}{n} (v_1, \ldots, v_n, 0) \in \ker f_1 \cap \ker f_2. \] (2.24)
So \(\tilde{E}_Q\) is a Chalmers–Metcalf operator for \(Q\) and \(\tilde{E}_Q / V\) is not invertible. In the same manner as \(\tilde{E}_Q\) we may define \(\hat{E}_Q\) by
\[ \hat{E}_Q = \frac{1}{k} \sum_{i=1}^{k} (0, x_i) \otimes (0, x_i). \] (2.25)
As before, if \((v_1, \ldots, v_n) \in \ker f_1 \cap \ker f_2\), then
\[ \hat{E}_Q(v_1, \ldots, v_n) = \|Q\| \frac{1}{n} (0, v_{n+1}, \ldots, v_{2n}) \in \ker f_1 \cap \ker f_2. \] (2.26)
Now consider
\[ E_Q := (\tilde{E}_Q + \hat{E}_Q)/2. \] (2.27)
By Theorem 1.6 the above defined \(E_Q\) is a Chalmers–Metcalf operator for \(Q\). Using (2.24) and (2.26) we have
\[ E_Q / V = \|Q\| \frac{1}{2n} \cdot \text{Id}_V. \]
Hence, \(E_Q\) is an invertible Chalmers–Metcalf operator. As a result, \(Q\) is a unique minimal projection. □

Remark 2.7. It is clear that we can extend the above example to show that the set of Chalmers–Metcalf operators may contain an arbitrary finite number of independent elements.

Example 2.8 (Skrzypek [35]). Consider
\[ V = \{ (x_1, x_2, x_3, x_4, x_5) : x_1 + x_2 = 0 \text{ and } x_3 + x_4 + x_5 = 0 \}. \]
Then the minimal projection from \(\ell_p^5 (p \neq 2)\) onto \(V\) is not uniquely minimal. Moreover, the set of all Chalmers–Metcalf operators contains only non-invertible operators.

Theorem 2.9 (Cohen and Sullivan [11]). Assume that \(X\) is a smooth Banach space and \(V\) is a finite-dimensional subspace. Let \(P\) be a minimal projection of norm 1. Then \(P\) is the unique minimal projection.

Proof. Any projection \(P\) is uniquely determined by its \(\text{Im} P\) and \(\text{Ker} P\). By Auerbach’s Lemma we can choose \(v_1, \ldots, v_k \in S(V)\) and \(v_1^*, \ldots, v_k^* \in S(V^*)\) such that \(v_1, \ldots, v_k\) is a basis for \(V\) and
\[ v_i^*(v_j) = \delta_{ij}. \]
Take $f_i = v_i^* \circ P$. Since $\|P\| = 1$ then $f_i \in S(X^*)$. Define

$$Q = \sum_{i=1}^{k} f_i \otimes v_i.$$ 

Now it is easy to see that $\text{Im} Q = \text{Im} P$ and $\text{Ker} Q = \text{Ker} P$. As a result $P = Q$.

Observe that since $f_i(P v_i) = f_i(v_i) = 1 = \|P\|$ then $(v_i, f_i) \in \mathcal{E}(P)$. We can now define

$$E_P := \frac{1}{n} \sum_{i=1}^{n} f_i \otimes v_i = \frac{1}{n} P.$$ 

Since $E_P / V = \frac{1}{n} \text{Id} / V$, it is a Chalmers–Metcalf operator and it is invertible. Therefore, in view of Corollary 2.3 (note that here $E_P : X \to X$), $P$ is the unique minimal projection. □

**Example 2.10** (Skrzypek [38]). Take the Cheney–Light projection $Q : L_p[m \times n] \to L_p[m] + L_p[n]$ given by the formula

$$Q_{rs}(i, j) = \begin{cases} \frac{n+m-1}{nm}, & i = r, j = s, \\ \frac{m-1}{nm}, & i \neq r, j = s, \\ \frac{n-1}{nm}, & i = r, j \neq s, \\ \frac{-1}{nm}, & i \neq r, j \neq s, \end{cases}$$

where $e_{rs}(i, j) = \delta_{ri} \delta_{sj}$. Then there is a Chalmers–Metcalf operator for this projection equal to $\frac{\|Q\|}{n+m-1} \cdot \text{Id}$. Its invertibility implies that $Q$ is the unique minimal projection for $1 < p < \infty$. It is worth mentioning that for $p = 1$ and $p = \infty$ it has been proved that this projection is not unique [38].

**Theorem 2.11** (Lewicki [21]). Suppose that $X, Y$ are separable Banach spaces and $V, W$ are finite-dimensional subspaces of $X$ and $Y$, respectively. Assume that $P : X \to V$, and $Q : Y \to W$ are minimal projections and let $E_P$ and $E_Q$ be the corresponding Chalmers–Metcalf operators. Consider the tensor product $X \otimes Y$ with some reasonable, uniform crossnorm. Then the projection

$$P \otimes Q : X \otimes Y \to V \otimes W$$

is minimal and its Chalmers–Metcalf operator may be given by

$$E_{P \otimes Q} = E_P \otimes E_Q.$$ 

**Corollary 2.12.** If both $E_P / V : V \to V$ and $E_Q / W : W \to W$ are invertible then $E_{P \otimes Q} / V \otimes W : V \otimes W \to V \otimes W$ is invertible.

**Proof.** We need to show that $\ker \{ E_{P \otimes Q} / V \otimes W \} = 0$. To do this take $0 \neq x \in V \otimes W$, we may write $x$ in the form $x = \sum_{i=1}^{n} v_i \otimes w_i$, where $w_i$ are linearly independent. Now assume for the contrary that $E_{P \otimes Q}(x) = 0$, that is $\sum_{i=1}^{n} E_P(v_i) \otimes E_Q(w_i) = 0$. This implies

$$\sum_{i=1}^{n} f(E_P(v_i)) \otimes E_Q(w_i) = 0 \quad \text{for any } f \in V^*.$$  (2.28)
But since $x \neq 0$, the dimension of $\text{span}\{v_i\}$ is greater than zero and since $E_P$ is invertible the dimension of $\text{span}\{E_P(v_i)\}$ is greater than zero. Hence, by (2.28), $E_Q(w_i)$ are linearly dependent. Since $E_Q$ is invertible, this would imply that $w_i$ are linearly dependent, a contradiction. □

**Remark 2.13.** The above theorem gives us, with the assumptions that the considered tensor norm is smooth, the uniqueness of tensor product of projections for which Chalmers–Metcalf operators are invertible.

### 3. Symmetric case

Recall that a space is called symmetric if there is a basis $e_1, \ldots, e_n$ such that for any permutation $\sigma$ of $\{1, \ldots, n\}$

$$
\| \pm x_{\sigma(1)} e_1 \pm x_{\sigma(2)} e_2 \pm \cdots \pm x_{\sigma(n)} e_n \| = \| x_1 e_1 + \cdots + x_n e_n \|. 
$$

(3.1)

For $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \in \{-1, 1\}^n$ and $\sigma \in S_n$ we will denote by $T_{\sigma, \varepsilon}$ the isometry given by

$$
T_{\sigma, \varepsilon}(x_{\sigma(1)} e_1 + \cdots + x_{\sigma(n)} e_n) = \varepsilon_1 x_1 e_\sigma(1) + \cdots + \varepsilon_n x_n e_\sigma(n).
$$

(3.2)

First we will prove some preliminary results.

**Lemma 3.1.** If $X$ is a symmetric space and $L : X \to X$ is an operator which commutes with the group $\{T_{\sigma, \varepsilon}\}$ (which means that for any $\sigma$ and $\varepsilon$, $L \circ T_{\sigma, \varepsilon} = T_{\sigma, \varepsilon} \circ L$), then $L$ has to be necessarily of the form $L = c \cdot \text{Id}_X$, where $c$ is a constant.

**Proof.** Assume that $L$ is given in basis by $L(e_i) = \sum_{j=1}^{n} a_{ij} e_j$. Take $S_i$, an isometry that leaves all the base elements intact except the $e_i$ and $S_i(e_i) = -e_i$. Then

$$
S_i \circ L(e_i) = S_i \left( \sum_{j=1}^{n} a_{ij} e_j \right) = \sum_{j \neq i} a_{ij} e_j - a_{ii} e_i, 
$$

(3.3)

while

$$
L \circ S_i(e_i) = L(-e_i) = -L(e_i) = \sum_{j=1}^{n} -a_{ij} e_j. 
$$

(3.4)

Comparing (3.3) and (3.4) yields $a_{ij} = 0$ for $i \neq j$. As a result $L$ has to have the form $L(e_i) = a_{ii} e_i$.

Consider $P_{kl}$, an isometry that leaves all the base elements intact except $e_k, e_l$ and $P_{kl}(e_k) = e_l$, $P_{kl}(e_l) = e_k$. Then

$$
P_{kl} \circ L(e_k) = P_{kl}(a_{kk} e_k) = a_{kk} e_l, 
$$

(3.5)

while

$$
L \circ P_{kl}(e_k) = L(e_l) = a_{ll} e_l. 
$$

(3.6)

Hence, $a_{ii} = a_{jj}$, therefore $L = c \cdot \text{Id}_X$. □

**Theorem 3.2.** Assume $X$ is finite-dimensional and $V$ is a symmetric subspace. If $X$ is additionally a smooth space then there is only one minimal projection in $P(X, V)$.
**Proof.** Since $X$ is finite-dimensional then by the classical result of Isbell and Semadeni [16] there is a minimal projection in $P(X, V)$. Take $P \in P(X, V)$ a minimal projection. By Theorems 1.7–1.9 there is an operator $E : V \to X$ such that

$$E(V) \subset V, \quad v(E/V) = 1$$

and

$$\text{tr}(E \circ Q) = \text{tr}(E/V) = \|Q\| \quad \text{for any projection } Q : X \to V. \quad (3.7)$$

Let $G$ be the group of isometries $\{T_{\sigma, \epsilon}\}$ given by (3.2). Observe that $T_{\sigma, \epsilon}^{-1} = T_{\sigma, -\epsilon}$. Put

$$L_0 = \frac{1}{|G|} \sum_{g = (\sigma, \epsilon) \in G} T_g \circ E/V \circ T_g^{-1} : V \to X. \quad (3.8)$$

Since $E/V(V) \subset V$ the above definition is correct.

Because $(L(V, X), v)$ is a Banach operator ideal, $T_g$ are isometries and by (3.7)

$$v(L_0) = v \left( \frac{1}{|G|} \sum_{g \in G} T_g \circ E/V \circ T_g^{-1} \right) \leq \frac{1}{|G|} \sum_{g \in G} v(T_g \circ E/V \circ T_g^{-1})$$

$$\leq \frac{1}{|G|} \sum_{g \in G} \|T_g\| \cdot v(E/V) \cdot \|T_g^{-1}\| = \frac{1}{|G|} \sum_{g \in G} v(E/V) = v(E/V) = 1. \quad (3.9)$$

By the trace duality we will show that $v(L_0) = 1$. Since $L_0(V) \subset V$, therefore $L_0(x) = \sum_{k=1,...,n} f_k^*(x) e_k$ and for any projection $R : X \to V$ we have

$$\text{tr}(L_0) = \sum_{k=1,...,n} f_k^*(e_k) = \sum_{k=1,...,n} f_k^*(Re_k) = \text{tr}(L_0 \circ R). \quad (3.10)$$

Using $\text{tr}(A \circ B) = \text{tr}(B \circ A)$, we have

$$\text{tr}(T_g \circ E/V \circ T_g^{-1}) = \text{tr}(T_g \circ (E/V \circ T_g^{-1}))$$

$$= \text{tr}((E/V \circ T_g^{-1}) \circ T_g) = \text{tr}(E/V). \quad (3.11)$$

Combining (3.10), (3.11) and (3.7) yields

$$\text{tr}(L_0 \circ P) = \text{tr}(L_0) = \text{tr} \left( \frac{1}{|G|} \sum_{g \in G} T_g \circ E/V \circ T_g^{-1} \right)$$

$$= \frac{1}{|G|} \sum_{g \in G} \text{tr}(T_g \circ E/V \circ T_g^{-1}) = \frac{1}{|G|} \sum_{g \in G} \text{tr}(E/V)$$

$$= \frac{1}{|G|} \sum_{g \in G} \text{tr}(E/V \circ P) = \frac{1}{|G|} \sum_{g \in G} \|P\| = \|P\|. \quad (3.12)$$

Therefore, by trace duality

$$v(L_0) = 1, \quad (3.13)$$

and, as a result, $L_0$ can be treated as a functional in $L(X, V)^*$ of norm 1 such that $L_0(R) = \|P\|$ (for any projection $R : X \to V$). Therefore, $L_0$ (by Theorem 1.9) can be viewed as a
Chalmers–Metcalf operator, i.e.,

\[ L_0 = \int_{E(P)} m \otimes n \, d\lambda(m, n) : V \to X. \]

But from the definition of \( L_0 \) (see (3.8)), \( L_0 \) commutes with the group \( \{ T_{\sigma_i} \} \). Hence, by Lemma 3.1 \( \widetilde{L}_0 / V = c \cdot \text{Id}_V \). Applying (3.12)

\[ \| P \| = \text{tr}(L_0) = \text{tr}(c \cdot \text{Id}_V) = c \cdot \text{dim} V, \]

leads to \( c = \frac{\| P \| \cdot \text{dim} V}{\| P \|} \). Applying the Theorem 2.2 yields the result. □

**Remark 3.3.** It is easy to see that Theorem 3.2 also holds true for a subspace \( V \) with enough symmetries. Recall that space \( X \) is a space with enough symmetries if the only operators which commute with all isometries on \( X \) are of the form constant times identity operator.

### 4. Rademacher projections

The well-known Rademacher functions, \( r_0, r_1, \ldots \), defined by \( r_j(t) = (-1)^{\lfloor 2^j t \rfloor} \) for \( 0 \leq t \leq 1 \) plays the central role in many areas of analysis (\( \lfloor \cdot \rfloor \) denotes the integer part of the argument).

For further investigations we will need a notion of **dyadic group**. We shall denote the set of **dyadic rationals** in the unit interval \([0,1)\) by \( Q \). In particular, each element of \( Q \) has a form \( p/2^n \) for some \( p, n \in \mathbb{N}, 0 \leq p < 2^n \).

#### Definition 4.1 (dyadic group on the interval \([0,1])\)

Any \( x \in [0, 1] \) may be written in the form

\[ x = \sum_{k=0}^{\infty} x_k 2^{-(k+1)}, \]

where each \( x_k = 0 \) or \( 1 \). For each \( x \in [0, 1] \setminus Q \) there is only one expression of this form. We shall call it the **dyadic expansion** of \( x \). When \( x \in Q \) there are two expressions of this form, one which terminates in 0’s and one which terminates in 1’s. By the dyadic expansion of \( x \in Q \) we shall mean the one which terminates in 0’s. Notice that \( 1 \notin Q \) so the dyadic expansion of \( x = 1 \) terminates in 1’s.

Now we can define the **dyadic addition** of two numbers \( x, y \) by

\[ x \oplus y = \sum_{k=0}^{\infty} |x_k - y_k| 2^{-(k+1)}. \]

Observe that \( x \oplus x = 0 \); therefore, \( x \oplus y = x \ominus y \).

#### Theorem 4.2 (Shekhtman and Skrzypek [36])

The following holds true:

\[ r_n(x \oplus y) = r_n(x)r_n(y) \quad \text{for } x \oplus y \notin Q. \]

#### Definition 4.3 (Shekhtman and Skrzypek [36])

[dyadic group as operators]. Define the operators \( T_y \) (for \( y \in [0, 1) \)) as

\[ T_y : L_p[0, 1] \ni f \mapsto T_y(f) = \{ x \mapsto f(x \oplus y) \} \in L_p[0, 1]. \]

It is evident that the above operators are isometries.
For many other interesting facts concerning both Walsh and Rademacher functions the reader is referred to [36].

Denote by

$$Rad_n = \text{span}\{r_0, r_1, \ldots, r_{n-1}\}$$

the space spanned by the first $n$ Rademacher functions. The Rademacher projection is defined by

$$R_n = \sum_{i=0}^{n-1} r_i \otimes r_i : L_p[0, 1] \to Rad_n.$$  

(4.6)

We can write the above projection as

$$R_n(f) = \sum_{i=0}^{n-1} \left( \int_0^1 r_i(t) f(t) \, dt \right) r_i,$$  

(4.7)

or using the Dirichlet kernel $D_n^r = r_0 + r_1 + \cdots + r_{n-1}$ and orthogonality of Rademacher functions as

$$(R_n f)(s) = \int_0^1 f(t) D_n^r(t \oplus s) \, dt,$$  

(4.8)

or equivalently

$$R_n(f) = f \ast D_n^r.$$  

(4.9)

Rademacher projections are minimal; we can prove that as follows.

**Theorem 4.4** (Chalmers–Metcalf [3,4]). The Rademacher projection $R_n$ is a minimal projection in the set of all projections from $L_p[0, 1]$ onto $Rad_n$.

**Proof.** Since $Rad_n$ are finite-dimensional they have norming pairs, i.e., there exists $(a, b) \in S(L_p) \times S(L_q)$ such that $b R_n(a) = \|R_n\|$. Observe that dyadic group $T_y$ commutes with $R_n$, indeed

$$T_y(R_n f)(s) = T_y \left( \int_0^1 f(t) D_n^r(t \oplus s) \, dt \right)$$

$$= \int_0^1 f(t) D_n^r(t \oplus s \oplus y) \, dt = \text{(change of variables)}$$

$$= \int_0^1 f(t \oplus y) D_n^r(t \oplus s) \, dt$$

$$= \int_0^1 T_y(f)(t) D_n^r(t \oplus s) \, dt = R_n(T_y f)(s).$$

As a consequence, $(T_y b) R_n((T_y a)) = \|R_n\|$. Therefore, we can define an operator

$$E_{R_n} = \int_0^1 T_y(b) \otimes T_y(a) \, dy.$$
We will prove that the above is a Chalmers–Metcalf operator for $R_n$. We need to show $E_{R_n}(\text{Rad}_n) \subset \text{Rad}_n$. We can see that as follows:

$$E_{R_n}(r_i)(s) = \int_0^1 (T_y b)(r_i) \cdot (T_y a)(s) \, dy$$

$$= \int_0^1 ((T_y b)(t) \cdot r_i(t) \, dt) \cdot (T_y a)(s) \, dy$$

$$= \int_0^1 \left( \int_0^1 b(t \oplus y) \cdot r_i(t) \, dt \right) \cdot (T_y a)(s) \, dy$$

$$= \int_0^1 \left( \int_0^1 b(t) \cdot r_i(t \oplus y) \, dt \right) \cdot (T_y a)(s) \, dy$$

$$= \int_0^1 \left( \int_0^1 b(t) \cdot r_i(t) \cdot r_i(y) \, dt \right) \cdot (T_y a)(s) \, dy$$

$$= \int_0^1 \left( \int_0^1 b(t) \cdot r_i(t) \, dt \right) \cdot r_i(y) \cdot (T_y a)(s) \, dy$$

$$= \left( \int_0^1 b(t) \cdot r_i(t) \, dt \right) \cdot \left( \int_0^1 r_i(y) \cdot (T_y a)(s) \, dy \right)$$

$$= \left( \int_0^1 b(t) \cdot r_i(t) \, dt \right) \cdot \left( \int_0^1 r_i(y) \cdot a(s \oplus y) \, dy \right)$$

$$= \left( \int_0^1 b(t) \cdot r_i(t) \, dt \right) \cdot \left( \int_0^1 r_i(y \oplus s) \cdot a(y) \, dy \right)$$

$$= \left( \int_0^1 b(t) \cdot r_i(t) \, dt \right) \cdot \left( \int_0^1 r_i(y) \cdot r_i(s \cdot a(y)) \, dy \right)$$

$$= \left( \int_0^1 b(t) \cdot r_i(t) \, dt \right) \cdot \left( \int_0^1 r_i(y) \cdot a(y) \, dy \right) \cdot r_i(s).$$

Therefore,

$$E_{R_n}(r_i) = \langle r_i, b \rangle \cdot \langle r_i, a \rangle \cdot r_i,$$

which shows that $E_{R_n}(\text{Rad}_n) \subset \text{Rad}_n$. □

Using Theorem 3.2, since $L_p[0, 1]$ are smooth spaces for $p \in (1, \infty)$, we would like to obtain the unique minimality of Rademacher projections. First, we need to check that the norm of Rademacher projections is strictly greater than 1 (otherwise it would follow easily from the Cohen–Sullivan result [11]).
Remark 4.5. Assume \( n \geq 3 \). Then \( \| R_n \|_p > 1 \), for \( p \neq 2 \).

Proof. Let \( w_3(t) := r_1(t) \cdot r_2(t) \). Consider the following function:
\[
f = \frac{3^{q/p} + 1}{4^{1/q}(3 + 3q)^{1/p}} \cdot (r_0 + r_1 + r_2) + \frac{3^{q/p} - 3}{4^{1/q}(3 + 3q)^{1/p}} \cdot w_3.
\]
One can check that \( \| f \|_p = 1 \). Observe that \( \int_0^1 w_3(t) \cdot r_n(t) \, dt = 0 \), for any \( n \). Therefore,
\[
R_n(f) = \frac{3^{q/p} + 1}{4^{1/q}(3 + 3q)^{1/p}} \cdot (r_0 + r_1 + r_2),
\]
and, as a result,
\[
\| R_n(f) \|_p = \frac{3^{q/p} + 1}{4^{1/q}(3 + 3q)^{1/p}} \cdot (3 + 3p)^{1/q} = \frac{1}{4} (1 + 3p/q)^{1/p} (1 + 3q/p)^{1/q}.
\]
It is easy to check that for \( p \neq 2 \) the latter is strictly greater than 1. \( \square \)

We are now ready to prove the main theorem in this section.

Theorem 4.6. Rademacher projection \( R_n : L_p[0, 1] \to \text{span}\{r_0, \ldots, r_{n-1}\} \), for \( p \in (1, \infty) \), given by
\[
R_n(f) = \sum_{i=0}^{n-1} \left( \int_0^1 r_i(t) f(t) \, dt \right) r_i,
\]
is the unique minimal projection.

Proof. Let \( S_n \) denote the space with a basis of simple functions of intervals \( I_{l,2^k} = [\frac{l}{2^n}, \frac{l+1}{2^n}] \), where \( l \in \{0, 1, \ldots, 2^n - 1\} \). Therefore, for a simple function \( A \in S \), we have
\[
A = \sum_{l=0}^{2^n-1} A_l \chi_{I_{l,2^n}}.
\]
Additionally, we have
\[
\text{Rad}_n \subset S_n.
\]
Consider now the projection
\[
\widetilde{R}_n = R_n/S_n : S_n \to \text{Rad}_n.
\]
Observe that \( \text{span}\{r_0, \ldots, r_{n-1}\} \) is a symmetric space in \( L_p[0, 1] \). Therefore, following Theorem 3.2, there is a Chalmers–Metcalf operator
\[
E_{\widetilde{R}_n} = \int_{E(\widetilde{R}_n)} y \otimes x \, d\mu(x, y) : S_n \to S_n,
\]
such that
\[
E_{\widetilde{R}_n}/\text{Rad}_n = \frac{\| \widetilde{R}_n \|_n}{n} \text{Id}_{\text{Rad}_n}.
\]
Let us consider the projection $L_n : L_p[0, 1] \to S_n$

$$L_n(f) = \sum_{i=0}^{2^n-1} \left( \int_{I_{2^n}} f(x) \, dx \right) x_{I_{2^n}}.$$  \hfill (4.16)

Also observe that $L_n/S_n = \text{Id}$, and since $r_i \in S_n$, we have also

$$r_n(f) = r_n(L_n f).$$

Taking a norming point for $R_n$, we have

$$\|R_n\| = \|R_n(f)\| = \left\| \sum_{i=0}^{n-1} r_i(f)r_i \right\| = \left\| \sum_{i=0}^{n-1} r_i(L_n f)r_i \right\|$$

$$= \|R_n(L_n f)\| = \|\widetilde{R}_n(L_n f)\| \leq \|\widetilde{R}_n\| \cdot \|L_n\| \leq \|\widetilde{R}_n\|.$$  \hfill (4.17)

Hence, since $\widetilde{R}_n = R_n/S_n$ we have

$$\|R_n\| = \|\widetilde{R}_n\|.$$  \hfill (4.18)

Let $\widetilde{y}$ be Hahn–Banach extensions of functionals $y$’s. It is clear now that if $(x, y) \in \mathcal{E}(\widetilde{R}_n)$, then $(x, \widetilde{y}) \in \mathcal{E}(R_n)$. Hence,

$$E_{R_n} = \int_{\mathcal{E}(R_n)} \widetilde{y} \otimes x \, d\mu(x, y) : L_p[0, 1] \to L_p[0, 1]$$

is a Chalmers–Metcalf operator for $R_n$. Additionally,

$$E_{R_n}/\text{Rad}_n = E_{\widetilde{R}_n}/\text{Rad}_n = \frac{\|\widetilde{R}_n\|}{n} \text{Id}_{\text{Rad}_n}.$$  \hfill (4.19)

Therefore, the above Chalmers–Metcalf operator is invertible. Thus by Theorem 2.2 $R_n : L_p[0, 1] \to \text{Rad}_n$ is the unique minimal projection. \hfill $\Box$

The following theorem shows that smoothness cannot be omitted in Theorem 4.6.

**Theorem 4.7** (Chalmers–Metcalf [3]). The Rademacher projection $R_n : L_1[0, 1] \to \text{span}\{r_0, \ldots, r_{n-1}\}$, given by (4.10) is not unique if $n$ is odd.

Using (4.18), (4.19) and in view of Remark 2.13 we immediately obtain the following theorem.

**Theorem 4.8.** Let $1 < p < \infty$. The tensor product of Rademacher projections $R = R_n \otimes R_m : L_p[0, 1]^2 \to \text{Rad}_n \otimes \text{Rad}_m$ is the unique minimal projection.

**References**
