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Alternating Groups as Collineation Groups

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1. INTRODUCTION

Collineation groups of finite projective planes and their geometries have been studied systematically since the beginning of this century. In 1911, H. Mitchell determined ordinary and modular ternary linear groups for an odd characteristic. In 1924, R. Hartley extended this to the even characteristic. More recently, in 1977, Ch. Hering introduced the strongly irreducible collineation groups. In 1987, C. Y. Ho introduced the totally irregular collineation groups. (See, for example, Section 2 below.) Most known results on collineation groups in general require the existence of some perspectivities. In 1982, A. Reifart and G. Stroth determined non-abelian simple strongly irreducible collineation groups containing a perspectivity. (See 2.9 below.) In 1991, A. Gonçalves and C. Y. Ho characterized non-abelian simple totally irregular collineation groups containing involutory perspectivities. (See 2.10 below.)

Not much is known about the case in which all involutions are Baer. (For a definition, see 2.2 below.) The problem of classifying totally irregular (or strongly irreducible) collineation groups with all involutions being Baer is still open. One of the difficulties is that the condition of being Baer is not inductive.

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There are few simple alternating groups as collineation groups of finite projective planes. Via the isomorphisms $A_5 \cong PSL(2,4) \cong PSL(2,5)$ and $A_6 \cong PSL(2,9)$, the alternating groups on five letters and six letters act on infinitely many Desarguesian planes preserving a symmetric bilinear form and leaving invariant a conic. The alternating group on seven letters is a subgroup of $PSU(3,5)$. So A_7 has a unitary action leaving invariant a unital on Desarguesian planes of order of an even power of 5.

In this paper we prove that only finitely many alternating groups can be totally irregular collineation groups of projective planes in the following theorem.

THEOREM 1. *Let R be a subset of the set of positive integers and let $S = \{(G_r, \Pi_r) | r \in R\}$ be a sequence of totally irregular collineation groups $G_r \cong A_r$ on projective planes Π_r. Then R is a finite set. If N is the maximal element in R, then $N \leq 21$. Further, if each G_r contains a perspectivity, then $N \leq 7$, and all involutions are perspectivities.*

We remark that in treating the case in which all involutions are Baer, the totally irregular condition is also not inductive. Our method is not efficient enough to reduce of the value of N to 7. When $r \geq 22$, we are able (in 4.3) to find a subgroup M of G_r isomorphic to A_{r-8}, which induces a totally irregular action on a Baer subplane of a Baer subplane of the plane Π_r. Note that the order of the plane (by 2.6) is bounded. When a collineation group isomorphic to an alternating group on eight or more letters contains a non-trivial perspectivity, the Reifart−Stroth theorem (2.9 below) implies that this group fixes a point or a line. We prove (3.5 below) that this group is a group of perspectivities with a common center and a common axis in 3.5. This is the best result as A_7 acts on a Desarguesian plane of order 25 containing homologies with different centers and axes.

This paper is organized in the following way. General definitions, notations, and some known results are presented in Section 2. Proposition 3.1 determines the structure of a group (not necessarily finite) being a union of three proper subgroups. This might be of some independent interest. We use this to prove in 3.2 that an abelian group of homologies is isomorphic to $Z_2 \times Z_2$ or has a common center and axis. Proposition 3.3 states that if σ is an element of prime order in an alternating group on 4 or more letters, then there is an element $\bar{\sigma}$ in a subgroup conjugated to $\langle \sigma \rangle$ such that $\sigma\bar{\sigma}$ is an involution. A consequence of this is Proposition 3.4: a collineation group isomorphic to an alternating group on 5 or more letters of a projective plane contains a non-trivial perspectivity if and only if all its involutions are perspectivities. Alternating groups on 8 (respectively, 10) or more letters on relatively small planes are dealt with in 3.6 (respectively, 3.7). In the beginning of Section 4, we present three lemmas concerning alternating collineation groups acting totally irregularly with all involutions being Baer. Then we present the proof of Theorem 1 in 5 steps. The finiteness is established in Step 2 by computing the limit of a sequence.

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2. DEFINITIONS, NOTATIONS AND SOME KNOWN RESULTS

The support of a permutation is the set of letters that it moves. A short involution of an alternating group is an involution which has a support of cardinality 4. Two short involutions are disjoint if their supports are disjoint.

For any number r, Let $v(r) = r^2 + r + 1$. The full collineation group of a projective plane $\Pi = (\mathscr{P}, \mathscr{L})$ is denoted by $\mathrm{Aut}(\Pi)$. We sometimes write the order of Π as $o(\Pi)$. For a set of collineation H, let $\mathscr{P}(H)$ (respectively, $\mathscr{L}(H)$) be the set of common fixed points (respectively, lines) of elements of H, and let $\Pi(H)$ be the set of fixed-point-line substructure $(\mathscr{P}(H), \mathscr{L}(H))$ of H. For any $A \in \mathscr{P}$, $[A]$ denotes the set of lines incident with A. For any $a \in \mathscr{L}$, (a) denotes the points incident with a.

We call a collineation g planar if $\Pi(g)$ is a subplane, a generalized perspectivity if $\mathscr{P}(g) \subseteq \{A\} \cup (a)$, for some point A and line a, $\mathscr{L}(g) \subseteq \{a\} \cup [A]$, and $|P(g)| \neq 3$ (A is the center of g, which is denoted by $\mathscr{C}(g)$ and a is the axis of g, which is denoted by $a(g)$), and triangular if $\mathscr{P}(g)$ consists of three non-collinear points.

A generalized perspectivity g is called a generalized homology (respectively, elation) if $\mathscr{C}(g) \notin (a(g))$ (respectively, $\mathscr{C}(g) \in (a(g))$). We remove the term generalized if $(a(g)) \subseteq \mathscr{P}(g)$.

We call a collineation group G totally irregular if each G-orbit of points is irregular (i.e., the stabilizer of any point in G is not 1), and strongly irreducible if G does not leave invariant any point, line, triangle, or proper subplane.

Given a collineation group G, for any point A and line a, we let $G(A, a)$ be the set of all perspectives in G with center A and axis a together with 1. This is a subgroup of G.

Other notation and terminology concerning groups and projective planes can be found in [2, 4, 7, 8]. For the convenience of the reader, we record some known results used in this paper.

LEMMA 2.1 (Baer, [7]). *If $E(A, a)$ and (B, a) are two non-trivial groups of elations of a projective plane Π with $A \neq B$, then the set of elations with common axis a together with 1 is an elementary abelian group.*

LEMMA 2.2 (Baer, [7]). *Let* Π *be a projective plane of order* n. *An involutory collineation* j *is either a perspectivity or a Baer involution, i.e.,* n *is a square and* $\Pi(j)$ *is a subplane of order* $\sqrt{n}$. *An involutory perspectivity is a homology* (*respectively, an elation*) *if* n *is odd* (*respectively, even*).

THEOREM 2.3 (Bruck–Ryser, [7]). *Let* n *be the order of a projective plane. If* $n \equiv 1, 2 \pmod 4$, *then* n *is the sum of two integral squares.*

THEOREM 2.4 (Bloom, [1]). *If* A_r, $r \geq 6$, *is a subgroup of* $PSL(3, q)$, *then* $q \geq 9$, *and* $q = 9$, *only for* $r = 6$. *Also,* A_5 *is not a subgroup of* $PSL(3, 7)$.

LEMMA 2.5 (Hering, [5]). *A product of two perspectivities is not planar unless it is* 1.

In the proof of Theorem A (p. 424) in [6], we have the following result.

LEMMA 2.6 (Ho, [6]). *If* G *is a totally irregular collineation group of a projective plane of order* n, *then* $n < \frac{3}{2}|G| + 1$.

THEOREM 2.7 (Dembowski, [2]). *A projective plane of prime power order* q *admitting* $PSL(2, q)$ *as a collineation group is a Desarguesian plane.*

THEOREM 2.8 (Hering, [5]). *Let* G *be a strongly irreducible collineation group of a projective plane generated by non-trivial perspectivities. Then* G *contains a unique minimal normal subgroup* M, *which is either a non-abelian simple group or* $M \cong Z_3 \times Z_3$.

THEOREM 2.9 (Reifart–Stroth, [9, 10]). *A strongly irreducible non-abelian simple collineation group* G *containing a non-trivial perspectivity is isomorphic to one of the following groups*: $PSL(2, q)$, $PSL(3, q)$, $PSU(3, q)$, A_7, *or* J_2. *Further, if* $G \cong J_2$, *then the order of the plane must be of the form* m^4, m *odd.*

THEOREM 2.10 (Gonçalves–Ho, [3]). *Let* G *be a totally irregular non-abelian simple collineation group of a projective plane of order* n, *containing involutory perspectivities. If* n *is odd, then one of the following holds*:

(1) *G acts strongly irreducible on the subplane generated by the centers and axes of involutory homologies in* G, *and* G *is isomorphic to one of the groups in* 2.9.

(2) $G \cong PSU(3, 2^t)$. *Also* G *fixes a point which is not a center of a perspectivity of* G, *or* G *fixes a line which is not an axis of a perspectivity of* G. *Further, commuting involutory homologies have a common center and a common axis.*

If n *is even, then one of the following holds*:

(1) *G acts strongly irreducibly on the subplane generated by the centers and axes of involutory elations of* G, *and* G *is isomorphic to* $PSL(3, 2^t)$ *or* $PSU(3, 2^t)$.

(2) *G is isomorphic to $PSL(2,2^t)$, $PSU(3,2^t)$, or $Sz(2,2^{2t+1})$, and G fixes a point, which is not a center of a perspectivity of G, or G fixes a line, which is not an axis of a perspectivity of G.*

3. PRELIMINARY LEMMAS

We start with the following elementary result concerning a group which could have infinite order.

PROPOSITION 3.1. *Let H be a group which is the union of three proper subgroups X, Y, Z. Then $X \cap Y = X \cap Y \cap Z$ is a normal subgroup, and $H/(X \cap Y)$ is isomorphic to $Z_2 \times Z_2$.*

Remark. In particular, the above shows that a group which is the disjoint union of three proper subgroups is isomorphic to $Z_2 \times Z_2$. The dihedral group of order 8 shows that $X \cap Y$ could be non-trivial.

Proof. It is clear that we may assume that no two of the three proper subgroups are equal. Also the union of any two of these subgroups is not H. We now show that $X \cap Y = X \cap Z$.

There is $z \in Z \setminus (X \cup Y)$. For $h \in X \cap Y$, we claim that $hz \in Z$. Suppose $hz = k \in X \cup Y$. Then $z = h^{-1}k \in X \cup Y$ as $h \in X \cap Y$, and X, Y are subgroups. This contradicts the choice of z. Therefore hz is not in $X \cup Y$. Since H is the union of X, Y, Z, this proves that $hz \in Z$ as claimed. Thus $(X \cap Y)z \in Z$. So $(X \cap Y) \subseteq Z$ as $z \in Z$. Hence $(X \cap Y) \subseteq Z \cap X$. A similar argument, using an element in Y, shows that $(X \cap Z) \subseteq Y \cap X$. Therefore $X \cap Y = Z \cap X$. By symmetry, this shows that $X \cap Y = X \cap Y \cap Z$. Denote this common intersection by V. For $W \in \{X, Y, Z\}$, let $W_1 = W \setminus V$.

We now show $|Y:V| = 2$. Let $y \in Y_1$. For $x \in X_1$, the element xy is in Z_1. Let $w \in Z_1$. Then $wy^{-1} \in X_1$. This proves $X_1 y = Z_1$. Suppose $u \in Y_1$. Then $X_1 y = Z_1 = X_1 u$. So $xy = hu$, for some $x, h \in X_1$. Thus $h^{-1}x = uy^{-1}$ is an element in both X and Y, i.e., in V. This shows that $Vu = Vy$ and establishes our claim that $|Y:V| = 2$. By symmetry, we see that $|X:V| = 2 = |Z:V|$. Hence V is a normal subgroup of H, and $H/V \cong Z_2 \times Z_2$.

Remark. From now on groups and projective planes will be of finite cardinalities.

COROLLARY 3.2. *An abelian group of homologies of a projective plane either has a common center and a common axis or it is isomorphic to $Z_2 \times Z_2$ such that the three centers (respectively, axes) are the vertices (respectively, sides) of a triangle.*

Remark. The Desarguesian plane of order 7 has a cyclic group of homologies of order 6 and a group of homologies isomorphic to $Z_2 \times Z_2$ with three centers and three axes.

Proof. Let G be an abelian group of homologies. Suppose there are two homologies x, y in G with different centers. Since the product of x, y is not an elation, the axis of x is different from the axis of y, and the center of xy is not in the line $\mathscr{C}(x)\mathscr{C}(y)$. Let $1 \neq g$ be an element in G. As G is abelian, g fixes each of these three centers. Being a homology, this implies that one of these centers is the center of g. This proves that $G = \bigcup_{w \in \{x, y, xy\}} G(\mathscr{C}(w), a(w))$ is the union of three proper subgroups. As the intersection of these subgroups is trivial, $G \cong Z_2 \times Z_2$ by 3.1. Note that the three centers (respectively, axes) are the vertices (respectively, sides) of a triangle.

PROPOSITION 3.3. *Let σ be an element of prime order in $G = A_r$, $r \geq 4$. Then there is $g \in G$ and an element $\bar{\sigma} \in \langle \sigma^g \rangle$ such that $\sigma\bar{\sigma}$ is an involution.*

Proof. Let the order of the element σ be p. The case in which $4 = r$ is trivial. Thus we assume $r \geq 5$. So G is simple. The case $p = 2$ is clear if we take $\bar{\sigma} = 1$. Note that by Glauberman's Z^*-theorem there is a conjugate of σ which commutes with σ. We could take this element for $\bar{\sigma}$ also. Hence we may assume $p \geq 3$.

In S_r, the symmetric group on r letters, two elements with the same cycle structures are conjugated. A conjugate class in S_r may or may not split into two conjugate classes in A_r. Thus the subgroups generated by two elements of the same cycle structures in A_r are conjugated in A_r.

Let $\sigma = ab \cdots f$, where $a = (a_1 \cdots a_p)$, $b = (b_1 \cdots b_p), \dots, f = (f_1 \cdots f_p)$, be the product of m disjoint cycles. If $mp < r$, then $\sigma \in A_{r-1}$. Induction on r implies what we want unless $p = 3$, $m = 1$, and $r = 5$. In this exceptional case the desired result can be seen by inspection in A_5. Therefore we may assume $mp = r$.

Case 1. $p = 3$.

Then $m \geq 2$ as $r = mp \geq 5$. First assume that m is even. Set $\overline{ab} = (a_1b_2b_1)(a_3a_2b_3), \dots, \overline{ef} = (e_1f_2f_1)(e_3e_2f_3)$, and $\bar{\sigma} = \overline{ab} \cdots \overline{ef}$. Then σ and $\bar{\sigma}$ have the same cycle structure. Since $\sigma\bar{\sigma} = (ab)(\overline{ab}) \cdots (ef)(\overline{ef}) = (a_1b_3)(a_2)(a_3b_2)(b_1) \cdots (e_1f_3)(e_2)(e_3f_2)(f_1)$ is an involution, the proof is complete in this situation.

Assume now m is odd. Let $\sigma = ab \cdots def$. We define, as in the case in which m is even, the elements $\overline{ab}, \dots, \overline{de}$. Set $\bar{\sigma} = \overline{ab} \cdots \overline{de}f^{-1}$. Then σ and $\bar{\sigma}$ have the same cycle structure and $\sigma\bar{\sigma} = (ab)(\overline{ab}) \cdots (de)(\overline{de})$ is an involution.

Case 2. $p \geq 5$.

Define $\bar{\sigma} = (a_1 a_2 a_3 a_p \cdots a_5 a_4) \cdots (f_1 f_2 f_3 f_p \cdots f_5 f_4)$. Then σ and $\bar{\sigma}$ have the same cycle structure. Now $\sigma\bar{\sigma} = (a_1 a_3)(a_2 a_p) \cdots (f_1 f_3)(f_2 f_p)$ is an involution as desired. This completes the proof of the proposition.

PROPOSITION 3.4. *Let $G = A_r$, $r \geq 5$ be a collineation group of a projective plane. Then G contains a non-trivial perspectivity if and only if all involutions are perspectivities.*

Proof. If involutions in G are perspectivities, then G contains a non-trivial perspectivity.

Conversely, suppose G contains a non-trivial perspectivity σ. Using 2.2 and 2.5, it suffices to establish the existence of an involutory perspectivity. Replacing σ by a power of it, if necessary, we may assume that σ has prime order. The proposition now follows from 3.3.

THEOREM 3.5. *Let $G = A_r$, $r \geq 8$ be a collineation group of a projective plane. Suppose G contains a non-trivial perspectivity. Then all involutions in G are perspectivities having a common center and a common axis. In particular, G is a group of perspectivities with a common center and axis.*

Proof. Since G contains a non-trivial perspectivity, all involutions in G are perspectivities by 3.4. The order of the projective plane determines whether an involutory perspectivity is a homology or an elation. Thus all involutions of G are elations or all involutions of G are homologies. We will prove that all involutions have a common center and a common axis in each case. A moment of thought shows that it suffices to treat $G \cong A_8$. Let $\Gamma = \{1, 2, 3, 4, 5, 6, 7, 8\}$ be the set of the eight letters that G acts on.

Case 1. All involutions are elations.

We first show that a pair of commuting involutions in G has either a common center or a common axis. Let $i \neq j$ be two commuting involutions in G. Assume that $\mathscr{C}(i) \neq \mathscr{C}(j)$. Since i commutes with j, i fixes the center of j. This proves that $a(i) = \mathscr{C}(i)\mathscr{C}(j)$. Similarly, $a(j) = \mathscr{C}(i)\mathscr{C}(j)$. Hence i and j have a common axis. If i and j have different axes, then the intersection of these two axes is a common center for i and j. This establishes our claim. In particular, two disjoint short involutions in G have either a common center or a common axis.

We now assume that there are two disjoint short involutions with a common axis. We shall prove that all short involutions have a common axis. Since $r \geq 8$ and G is generated by its short involutions, this implies that G is a group of elations with a common axis. The fact that G has a common center follows from 2.1.

As $r \geq 8$, our current hypothesis implies that any two disjoint short involutions in G have a common axis by conjugation in G. Let $x \neq y$ be two distinct short involutions in G. If they are disjoint, then $a(x) = a(y)$.

We now assume that x and y are not disjoint. For typing convenience, we introduce a notation. For any permutation $\sigma \in G$, let $S(\sigma)$ denote the support of σ. We now prove the following.

(3-1) If $S(x) = S(y)$, then $a(x) = a(y)$.

Suppose $S(x) = S(y)$. Then $\Gamma \setminus S(x)$ is a set of four letters. Let i be an involution of G with this set as its support. Then x and i are two disjoint involutions as well as y and i. Hence $a(x) = a(i)$ and $a(y) = a(i)$. Thus $a(x) = a(y)$ as desired. Next we prove the following.

(3-2) If $|S(x) \cap C(y)| = 2$, then $a(x) = a(y)$.

Suppose $|S(x) \cap S(y)| = 2$. We may assume $S(x) = \{1, 2, 3, 4\}$ and $S(y) = \{1, 2, 5, 6\}$. We may assume further that $x = (12)(34)$. We claim that it suffices to treat the case in which x commutes with y.

If y moves the letter 1 to 2, then $y = (12)(56)$ and it commutes with x. Assume now that y moves the letter 1 to a letter different from 2. Without loss of generality, we may assume this letter to be 5. Thus $y = (15)(26)$. Let $z = (12)(56)$. Since $S(z) = S(y)$, $a(z) = a(y)$ by (3-1). Now z commutes with x, and $|S(x) \cap S(z)| = 2$. This establishes our claim.

Thus we may assume that $x = (12)(34)$ and $y = (12)(56)$. By way of contradiction, assume $a(x) \neq a(y)$. Since x commutes with y, this implies that x and y share a common center C. Let $k = xy = (34)(56)$. Then $\mathscr{C}(k) = C$ also. If $a(k) = a(x)$, then y will have this line as a common axis, which is not the case. Hence $a(x)$, $a(y)$, $a(k)$ are three distinct lines through C.

Let $a = a(x)$. Let V be the Klein four group of G with support $\{1, 2, 3, 4\}$. Note that V is a group of elations with common axis a by (3-1). Let $v = (56)(78)$.

Since x and v are disjoint, $a(v) = a(x) = a$. Thus $a(v) \neq a(k)$. As v commutes with k, this implies that v and k share a common center, which is C.

We now show that the three involutions in V have different centers. Deny this. Then these three commuting involutions share a common center, which is C as $x \in V$. Since $\mathscr{C}(k) = C$, this implies that the subgroup $H = \langle k, V \rangle$ has a common center C. Since $a(k) \neq a$, H is abelian by the dual of 2.1. But H is a dihedral group of order 8, which is non-abelian. This contradiction proves that the involutions in V have different centers as asserted.

Let $t = (123)$. Then $t \in G$ and t permutes the involutions in V transitively. Hence the three distinct centers of the involutions in V are in an orbit $\mathscr{O}$ of t. However, t commutes with v. This implies that t fixes the center C of v. But C is also the center of x, which is one of the points in $\mathscr{O}$. This is impossible. This contradiction proves that $a(x) = a(y)$ and establishes (3-2).

In the remaining situation we have $|S(x) \cap S(y)| = 1$, or 3. Suppose $|S(x) \cap S(y)| = 1$. Without loss of generality, we may assume that $S(x) = \{1,2,3,4\}$ and $S(y) = \{1,5,6,7\}$. Let $z = (34)(67)$. Then $|S(x) \cap S(z)| = 2 = |S(y) \cap S(z)|$. Thus $a(x) = a(z) = a(y)$ by (3-2).

Suppose $|S(x) \cap S(y)| = 3$. We may assume that $S(x) = \{1,2,3,4\}$ and $S(y) = \{1,2,3,5\}$. Let $z = (12)(67)$. Then $|S(x) \cap S(z)| = 2 = |S(y) \cap S(z)|$. Again $a(x) = a(z) = a(y)$ by (3-2). This completes the proof that all short involutions in G have a common axis as desired.

A similar argument, using the dual result of 2.1, treats the case in which there are two disjoint short involutions in G with a common center. This completes the proof of Case 1.

Case 2. All involutions are homologies.

The proof of this case is an application of 3.2. We note that $G \cong A_8 \cong GL(4,2)$. First we show that a Sylow 2-subgroup is a group of homologies with a common center and a common axis. Let H, K, L, M be subgroups of G, which we identify with $GL(4,2)$, consisting of matrices of the following forms, respectively:

$$h = \begin{pmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad k = \begin{pmatrix} 1 & * & * & * \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$l = \begin{pmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad m = \begin{pmatrix} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then H is a Sylow 2-subgroup of G, and K, L, M are elementary abelian subgroups of H of order at least 8. By 3.2, each of these subgroups is a group of homologies with a common center and a common axis. Since $K \cap L \neq 1$ and $L \cap M \neq 1$, these three subgroups share a common center and a common axis. As H is generated by these subgroups, this implies that H is a group of homologies with a common center and a common axis.

Let P and Q be subgroups of G consisting of matrices of the following forms, respectively:

$$p = \begin{pmatrix} 1 & * & * & * \\ 0 & 1 & 0 & 0 \\ 0 & * & 1 & 0 \\ 0 & * & 0 & 1 \end{pmatrix}, \qquad q = \begin{pmatrix} 1 & 0 & 0 & 0 \\ * & 1 & 0 & 0 \\ * & * & 1 & 0 \\ * & * & * & 1 \end{pmatrix}.$$

Then P and Q are 2-subgroups of G. Hence they are groups of homologies with a common center and a common axis. Since $H \cap P \neq 1$ and $P \cap Q \neq 1$, these three subgroups share a common center and a common axis. As G is generated by these three subgroups, G is a group of homologies with a common center and a common axis as asserted in Case 2. This completes the proof of 3.5.

LEMMA 3.6. *An alternating group on eight or more letters cannot act as a collineation group on a projective plane of order* 9, 16, *or* 25.

Proof. It suffices to treat the alternating group on eight letters. Let G be an alternating group on eight letters: $\{1, 2, \dots, 7, 8\}$. By way of contradiction, suppose G acts on a projective plane Π of order $n = m^2$, where $m = 3$, 4, or 5. A direct calculation shows that G cannot have a regular orbit of points. This implies that G is totally irregular. By 2.10, we may assume that all involutions in G are Baer.

Let $j = (56)(78)$. Let H be the alternating group on $\{1, 2, 3, 4\}$, and let V be the subgroup of order 4 of H. Then H acts on the Baer subplane $\Pi(j)$ of order m. Since 8 does not divide $n - m$, considering $\langle j \rangle \times V$ we see that V cannot act trivially on $\Pi(j)$. Let h be a 3-cycle of H. Then h permutes the three involutions of V transitively. This implies that V induces a collineation group of order 4 on $\Pi(j)$, which is a Desarguesian plane as $m = 3$, 4, or 5.

Let $k = (57)(68)$. Suppose $\Pi(j) = \Pi(k)$. Let $(l) \cap \mathscr{P}(j)$ be the axis of an involution x of V on $\Pi(j)$. Then the line l is invariant under $\langle j, k, x \rangle$, and a non-trivial element of this subgroup acts fixed-point-freely on the $n - m$ points of l outside $\Pi(j)$. This implies that 8 divides $n - m$, which is impossible. Therefore $\Pi(j) \neq \Pi(k)$.

Assume $m = 3$, or 5. Then $V \cong Z_2 \times Z_2$ induces a group of homologies with three distinct centers forming the vertices of a triangle in $\Pi(j)$. Since $\Pi(k) \neq \Pi(j)$, k induces a homology on $\Pi(j)$. As k commutes with V (respectively, h), the center C of k coincides with one of the three centers of V (respectively, fixed by h). However, h permutes the three centers of V transitively. This contradiction proves that $m \neq 3$, or 5.

Assume $m = 4$. Then V is a group of elations with either a common axis or a common center. Suppose V has a common axis (respectively, center). Then the non-trivial elements of $\langle j, V \rangle$ act fixed-point-freely on the points (respectively, lines) incident with the points of the common axis (respectively, center) outside $\Pi(j)$. This implies 8 divides $16 - 4 = 12$, a contradiction. This contradiction completes the proof of 3.6.

LEMMA 3.7. *An alternating group on* 10 *or more letters cannot be a collineation group of a projective plane of order* s^2, *for* $9 \leq s \leq 31$.

Proof. It suffices to treat an alternating group on 10 letters. Let $G \cong A_{10}$. By way of contradiction, assume G is a collineation group of a projective plane Π of order s^2, where $9 \le s \le 31$. A direct calculation shows that G cannot have a regular orbit of points. This implies that G is totally irregular. By 2.10, we may assume that all involutions in G are Baer.

We now set up some notations. Let W be the subgroup of order 4 of an alternating group on four letters in G (i.e., leaving all other letters fixed), and let j be an involution in W. There is a set of four letters disjoint from the support of j. Let V be the subgroup of order 4 of the alternating group H on these four letters, and let h be a 3-cycle in H. Let L be the alternating group on the six letters not moved by j and fixing the letters in the support of j. Let $F = \Pi(j)$, a Baer subplane of order s.

Since L commutes with j, L acts on $\Pi(j)$. Now $360 = |L|$ does not divide $s^2 - s$ for $9 \le s \le 31$. So L cannot act trivially on F. As L is a non-abelian simple group this implies that L induces a collineation group of F isomorphic to L. We abuse the notation by calling L this collineation group of F. The proof is divided into the following five steps.

Step 1. Let $w \in W$ with $j \neq w$. Then $\Pi(j) \neq \Pi(w)$.

Proof. Deny this. Then $F = \Pi(w) = \Pi(W)$. By conjugation in G, this implies that if α and β are two commuting short involutions with same support, then $\Pi(\alpha) = \Pi(\beta)$. In particular $\Pi(V) = \Pi(i)$ for $1 \neq i \in V$.

Let $1 \neq k \in V$. Since k commutes with j, k acts on F. We claim that k cannot act as a Baer involution on F. By way of contradiction, assume k acts as a Baer involution on F. Then $B = F \cap \Pi(k) = F \cap \Pi(V)$ is a Baer subplane of F. Then s must be a square. This forces $s = 9$, 16, or 25. Let b be a line of B. Then b is invariant under $\langle V, W\rangle = VW$. Thus V acts fixed-point-freely on Θ which consists of the points of $(b) \cap \mathscr{P}(W)$ outside B. Hence 4 divides $s - \sqrt{s}$. This rules out the possibility $s = 9$. Let the support of L be $\Omega = \{1, 2, 3, 4, 5, 6\}$, and let the support of V be $\{1, 2, 3, 4\}$. Since L centralizes W, L acts on F. Let $\tau = (12)(56)$. Then $T = \langle V, \tau\rangle$ is a dihedral group of order 8, and τ induces 1, or a perspectivity or a Baer involution on B (as τ normalizes V). Note that τ induces a Baer involution on F (as τ is conjugated to k in L).

Suppose τ induces 1 on B. Then T acts on Θ without fixed points. Hence 8 divides $s - \sqrt{s}$ which is impossible. So this case cannot occur.

Assume τ induces a perspectivity on B. Let a be the axis of τ on B. Since τ induces a Baer involution on F, this implies that T acts on the points of $(a) \cap F$ outside B without fixed points. Again we obtain 8 divides $s - \sqrt{s}$, a contradiction. Hence this case also cannot occur.

Therefore τ must induce a Baer involution on B. So $\sqrt{s}$ is a square. Hence $\sqrt{s} = 4$. Let $\Pi(\tau) \cap B = D$. Then D is a subplane of order 2. Let U be the group of order 4 of the alternating group with support $\{1, 2, 5, 6\}$. Then $\Pi(U) = \Pi(\tau)$. Hence $\Pi(U) \cap B = D$. Now $R = \langle U, V \rangle$ is transitive on Ω. From $(13)(24) \in V$ and $(15)(26) \in U$, we have $(15)(26)(13)(24) = (153)(642) \in R$. Hence 3 divides $|R|$ as well as 8. If $|R| = 24$, then $O_2(R)$ fixes two letters of Ω and R cannot be transitive on Ω. Thus $|R| > 24$ and $|L : R| < 15$. Since L does not have a subgroup of index 3 or 5, $L = R$. Let d be a line of D. Then L leaves invariant d. Set Γ as the points of $(d) \cap F$ outside D. Then Γ is a set of 14 points invariant under L and each involution fixes exactly 2 points. Since 7 does not divide $|L|$, Γ cannot be an orbit for L. Also L cannot have trivial orbits in Γ only. Let Λ be a non-trivial orbit of L in Γ. Suppose $|\Lambda| \geq 9$. Then L acts trivially on $\Lambda_1 := \Gamma \setminus \Lambda$. This forces $|\Lambda_1| \leq 2$. Hence Λ is an orbit of 12 points. But L has no subgroup of index 12. This contradiction proves that $|\Lambda| \leq 8$. If $|\Lambda| = 8$, then L has a subgroup of order 45, which is impossible. Hence $|\Lambda| = 6$. Then an involution of L fixes two points in Λ. Hence an involution of L acts without fixed points on Λ_1. This implies T acts fixed-point-freely on these 8 points, which implies that Λ_1 is an orbit of L, a contradiction. This contradiction proves that the case in which τ induces a Baer involution on B also cannot occur. This implies that k cannot act as a Baer involution on F as claimed.

Therefore k acts as a perspectivity on F. Let l be a line such that $(l) \cap \mathscr{P}(k)$ is the set of points of the axis of k on F. The group VW leaves l invariant, and a non-trivial element of VW acts fixed-point-freely on (l) outside F. This shows that 16 divides $s^2 - s$, which implies $s = 16$ or 17. We now consider $L \cong A_6$ on F. As $s = 16$ or 17, a direct calculation shows that L is a totally irregular collineation group of F. L contains an involutory perspectivity, k, of F. However, this contradicts the conclusion of 2.10. Hence k cannot be a perspectivity of F. This contradiction establishes Step 1.

Step 2. Let $w \in W$ with $w \neq j$. Then w induces a perspectivity on F.

Proof. Deny this. Then w acts as a Baer involution on F. So s is a square, and $s = 9$, 16, or 25. Since W commutes with L, L acts on the subplane $F \cap \Pi(w)$ of order 3, 4, or 5. As $L \cong A_6$, L acts trivially on $F \cap \Pi(w)$. Since L acts on F non-trivially, this implies 360 divides $s - \sqrt{s}$, which is impossible. This contradiction establishes Step 2.

Step 3. Let $1 \neq k \in V$. Then k induces a perspectivity on F.

Proof. Deny this. Then $B := F \cap \Pi(k)$ is a Baer subplane of F. Let $1 \neq w \in W$ with $w \neq j$. By Step 2, w induces a perspectivity on F. Let b

be a line such that its intersection with F is the axis of w. Then b is invariant under L. Note that b is a line of B also. Set Δ be the set of points of b inside B. Let τ be an involution in L which normalizes V and centralizes k. Thus $T = \langle V, \tau \rangle$ is a dihedral group of order 8. If U is the group of order 4 of the alternating group of L with the same support as τ, then $L = \langle U, V \rangle$ as in Step 1. If $v \in V$ and $v \neq k$, then v induces a perspectivity on $\Pi(k)$ by Step 2. Note that W acts fixed-point-freely on the points of b, which are inside $\Pi(k)$ but outside B. Hence 4 divides $s - \sqrt{s}$. This implies that $s \neq 9$. Hence $s = 16$ or 25. Let Γ be the points of b inside F but outside B.

Suppose $s = 25$. We now consider the action of T on the subplane B of order 5. Either T induces (i) a subgroup of homologies of order 2 or (ii) a subgroup of homologies isomorphic to $Z_2 \times Z_2$. (The centers in case (ii) are the vertices of a triangle.)

We claim case (i) cannot occur. Deny this. Note that T fixes every point of the axis of v on B. The line a of F, whose intersection with B is the axis of v, is invariant under T. Since involutions of L are Baer involutions on F, the element of order 4 in T acts (as k does) fixed-point-freely on the points of a outside B and so T as well. This implies 8 divides $s - \sqrt{s}$, again a contradiction. This contradiction proves that case (i) does not occur as claimed.

Therefore we are in case (ii). So Δ is the set of points of the axis of an involution induced by T on B. Thus T fixes exactly 2 points in Δ. Let A be one of these 2 points and let $\Lambda = A^L$. Also set Γ be the set of points of b in F. Then L acts on the 26 points of Γ. We will show that $|\Lambda| = 1$.

Assume $|\Lambda| > 1$. We know that 8 divides $|L_A|$. If $|L_A| = 8$, then $|\Lambda| = 45$ which is impossible. Hence $|L_A| > 8$ and $|\Lambda| = 15, 9, 5$ or 1. The possibility 9 is ruled out as L has no subgroup of order 40. If $|\Lambda| = 5$, then L fixes all these 5 points. In particular, they belong to Δ. But T only fixes 2 points in Δ. Thus the possibility 5 is out. Assume that $|\Lambda| = 15$. Then $L_A \cong S_4$, and T fixes exactly 1 point in Λ. Hence the other fixed point E of T is in a different orbit of L. The size of this orbit cannot be 15 again as Γ has only 26 points. Hence this orbit has only 1 point. An involution of L has at least 3 fixed points in Λ. Hence an involution has at most 2 fixed points in Γ outside $\Lambda \cup \{E\}$. This rules out any orbit of size 6 or smaller. Since L does not have subgroups of index 7, 8, 9, Γ must have an orbit of size 10 besides E and Λ. The stabilizer of a point of the orbit of size 10 has order $3^2 \cdot 4$, and has an element f of order 4. The square of this element, we may assume without loss of generality, is k. This implies that k fixes E and 3 points in Λ and 2 points in the orbit of size 10. So these 6 points are the points of Δ. Since f acts on the 3 fixed points of k in Λ and no fixed points in the other 12 points (by considering k), it must fix exactly 1 point in Λ (being an even permutation). Therefore f does not fix every point of Δ. In particular the axis of f in B (we are in case (ii), so f induces a homology on B) is not the points of Δ and f only fixes 2 points in Δ: E and 1 point in Λ. But this contradicts the fact that the fixed points of f in the orbit size 10 belong to the fixed points of k, i.e., Δ. This contradiction proves that the orbit of size 10 cannot exist. This implies $|\Lambda|$ cannot be 15. Hence $|\Lambda| = 1$ as desired.

Since A is any one of the 2 fixed points of T in Δ, we see that L fixes both of these 2 points. Hence L acts on the other 24 points Φ in Γ. An involution fixes exactly 4 points in Φ. Since we are in case (ii), a T-orbit in Δ has size 2. Of course a T-orbit in $\Gamma \setminus \Delta$ has size 4 or 8. This implies an L-orbit in Φ has even number of points. In particular L has no trivial orbit in Φ. Suppose L has an orbit in Φ of size 20. Then L will act on the 4 points of Φ not in this orbit. Hence L fixes each of these 4 points, which contradicts the fact that L has no trivial orbit in Φ. Therefore L has no orbit of size 20 in Φ. Since L has no subgroup of index 8, 12, 14, 16, 18, or 24, this implies that an L-orbit in Φ must have size 6 or 10. Since an involution of L fixes exactly 2 points in an orbit of size 6, Φ cannot be the union of four orbits of size 6. Suppose there is an orbit of L of size 6 in Φ. Then Φ has one orbit of size 12 or 18, which is impossible. Therefore each orbit size must be 10. But 24 is not divisible by 10. This contradiction proves that $s \neq 25$.

Suppose $s = 16$. Let Ψ be the 17 points of b in F. If T acts trivially on Δ, then T acts fixed-point-freely on Γ, which implies that 8 divides $s - \sqrt{s} = 12$. This is impossible.

Let $f \in T$ such that $f^2 = k$. Then f acts fixed-point-freely on Γ as k does. Hence an orbit of T in Γ has size 4 or 8. Recall that w induces an elation on F and leaves B invariant. Hence $A := \mathscr{C}(w) \in \Delta$. Thus T fixes A and acts on the other 4 points of Δ. By Step 2, v induces an elation on B. Suppose τ fixes every point of B. Then T fixes every point of the axis of v in B and acts on the 12 points of this line of F outside B fixed-point-freely (as $f^2 = k$). This is impossible.

We claim that τ cannot induce an elation on B. Deny this. Assume $a(v) = a(\tau)$ on B. If these common points are the points of Δ, then we get 8 divides $s - \sqrt{s} = 12$, a contradiction. Thus the common axis passes through A and Δ is an orbit or the union of two orbits of T. In the latter case f induces the identity on B.

Suppose $A^L = A$. Then L acts on the 16 points of b different from A such that each involution fixes exactly 4 points. If L leaves Δ invariant, then L fixes every point in Δ. This again implies T acts fixed-point-freely on Γ, a contradiction. Hence there exists $X \in \Delta$ such that X^L is not in Δ. In particular $X \neq A$. By the action of T we see that $x := |X^L|$ is even and

$x \geq 6$ (as X^L is not in Δ). Since L has no subgroup of index 8, 12, 14, or 16, the possibility for x is 6 or 10. Suppose $x = 6$. Then f has no fixed points on X^L. On the other hand $X \in \Delta$, which implies (as f has no fixed point) $X^{\langle f \rangle} = \Delta \setminus A$. But then X^L is the union of $\langle f \rangle$-orbits each of size 4. This contradicts $x = 6$. In fact, this argument shows that L does not have any orbit of size 6. For otherwise, k will fix exactly 2 points in this orbit and these 2 points are the points of Δ. Now x must be 10. But there are exactly 6 points left and so each of these must be fixed by L. However, this contradicts the fact that each involution of L fixes exactly 4 points in these 16 points. This contradiction proves that $A \neq A^L$.

Since $L_A \geq T$ and there are only 17 points in $\Delta \cup \Gamma, |L_A| > 8$. The action of T shows that $|A^L|$ is odd. Since $A \neq A^L, |A^L| \geq 6$. Since L has no subgroup of index 7, 9, 11, 13, the only possibility for $|A^L|$ is 15. But then L has to fix each of the 2 remaining points. In particular these 2 points are fixed by k. So they belong to Δ. However, T only fixes 1 point in Δ. This contradiction proves that τ does not induce an elation on B as desired.

Therefore τ induces a Baer involution on B. Let Δ_1 be the points of b in B fixed by τ. Now let us look at the action of W on $\Delta(\tau) := (b) \cap \mathscr{P}(k) \cap \mathscr{P}(\tau)$. ($W$ acts on this set as $[W, \langle k, \tau \rangle] = 1$.) W fixes each point of Δ_1 as $\Delta_1 \subseteq \Psi$. Hence W acts on the remaining two points of $\Delta(\tau)$. Therefore there is an involution of W which fixes each of these two points. But this involution fixes at least $|\Psi| + 2 = s + 3$ points on the line b, which contradicts the fact that it is a Baer involution. This contradiction proves that τ does not induce a Baer involution on B, and completes the proof of Step 3.

Step 4. The three involutions of V have different centers and axes on F.

Proof. By Step 3, all three involutions of V induce perspectivities on F. Suppose they share a common axis, which is the intersection of a line l with F. Then l is invariant under the elementary abelian group VW of order 16. From Step 2 and 3.2, l is also an axis of w. Thus a non-trivial element of this subgroup acts fixed-point-freely on the points of l outside F. This implies that 16 divides $s^2 - s$. So $s = 16$ or 17. In the case in which the three involutions of V share a common center, using the lines incident with this common center, we arrive at the same conclusion that $s = 16$ or 17. Therefore we assume $s = 16$ or 17 in the rest of this proof. A direct calculation shows that L cannot have a free orbit of points of F. This contradicts 2.10 as L contains involutory perspectivities of F. Step 4 is now established.

Step 5. Final contradiction.

Proof. Let w be an involution in W with $w \neq j$. Then w induces a perspectivity on F by Step 2. By Step 4, involutions in V have different centers and different axes. As $V \cong Z_2 \times Z_2$, this implies that V is a group of homologies. Now the three centers of V are permuted transitively by the 3-cycle $h \in H$. As w commutes with V, $\mathscr{C}(w)$ coincides with one of these centers of V. On the other hand, w also commutes with h. So $\mathscr{C}(w)$ is fixed by h. This contradiction establishes Step 5 and completes the proof of 3.7.

4. THE PROOF OF THEOREM 1

The next three lemmas concern a totally irregular collineation group M_r of a projective plane Π_r of order n_r such that $M_r \cong A_r$ and the involutions of M_r are Baer involutions. We introduce some notations. Let $r \geq 8$. For a short involution $\sigma \in M_r$, we define $A(\sigma)$ to be the alternating group on the letters not moved by σ and fix each letter in the support of σ. Then $A(\sigma) \cong A_{r-4}$. As $A(\sigma)$ commutes with σ, $A(\sigma)$ acts on the Baer subplane $\Pi(\sigma)$.

LEMMA 4.1. *Let $r \geq 9$, and let σ be a short involution in M_r. Then $A(\sigma)$ acts faithfully on $\Pi(\sigma)$.*

Proof. Deny this. Since $r \geq 9$, $A(\sigma)$ is a non-abelian simple group. So $A(\sigma)$ acts trivially on $\Pi(\sigma)$. Thus for any $\tau \in A(\sigma)$ we have $\Pi(\tau) = \Pi(\sigma)$. We will show that $\Pi(M_r) = \Pi(\sigma)$. (One can use the connectivity of the graph of commuting short involutions to verify this claim as kindly suggested by the referee. We present the following proof.)

The above argument shows that for any alternating subgroup H fixing four letters and moving the other letters, $\Pi(H)$ is a Baer subplane.

Let i be one of the letters in the support of σ. Let Ω be the set of the $r - 4$ letters not in the support of σ. Let $j \in \Omega$, and $\Gamma = \Omega \setminus \{j\}$. Since $r \geq 9, |\Gamma| \geq 4$. Thus there is a short involution α in $A(\sigma)$ with support inside Γ. Let K be the alternating subgroup of M_r fixing j and the three letters in the support of σ different from i. Then $\Pi(K) = \Pi(\alpha)$. Since $\Pi(\alpha) = \Pi(\sigma) = \Pi(A(\sigma))$, we have $\Pi(K) = \Pi(A(\sigma))$. This implies that for any alternating subgroup L fixing three letters and moving the other letters, $\Pi(L)$ is a Baer subplane. Repeat this type of argument three more times and we will reach $\Pi(M_r) = \Pi(\sigma)$ as asserted.

This forces that the point orbits of M_r outside $\Pi(\sigma)$ are regular orbits. However, this contradicts to the fact that M_r is a totally irregular collineation group of Π_r. This contradiction proves 4.1.

LEMMA 4.2. *Let $r \geq 12$. Then every short involution in $A(\sigma)$ induces a Baer involution on $\Pi(\sigma)$.*

Proof. Deny this. Let j be a short involution of $A(\sigma)$ such that it induces a perspectivity on $\Pi(\sigma)$. By 3.4, all involutions of $A(\sigma)$ induce perspectivities on $\Pi(\sigma)$. As $A(\sigma)$ is isomorphic to an alternating group on eight or more letters, it contains an elementary 2-group E of order at least 16. Suppose j induces a homology on $\Pi(\sigma)$. By 3.2, E is a group of homologies having a common center and axis. An easy argument involving conjugates of E shows that $A(\sigma)$ is a group of homologies with a common center and a common axis. Suppose j induces an elation on $\Pi(\sigma)$. Then all involutions of $A(\sigma)$ induce elations on $\Pi(\sigma)$. By 3.5, we see that $A(\sigma)$ is a group of elations with a common center and a common axis. Therefore $A(\sigma)$ is a group of perspectivities with a common center and a common axis in any case.

By 2.6, we see that

$$n_r < \frac{3}{2}|A_r| + 1. \tag{4.1}$$

As $A(\sigma)$ is a group of perspectivities with a common center and axis, $A(\sigma)$ has a regular orbit of points on a line of $\Pi(\sigma)$ incident with the common center. Noting that $o(\Pi(\sigma)) = \sqrt{n_r}$, by 4.1, we obtain the following:

$$\frac{(r-4)!}{2} = |A(\sigma)| \leq \sqrt{n_r}\,. \tag{4.2}$$

However, (4.1) together with (4.2) yields $r \leq 13$. This contradiction proves 4.2.

By 4.2, if $r \geq 12$, then $\Pi(\sigma) \cap \Pi(j)$ is a Baer subplane of $\Pi(\sigma)$ for any short involution $j \in A(\sigma)$. Denote $\Pi(\sigma) \cap \Pi(j)$ by $\Pi(\sigma, j)$. Thus $o(\Pi(\sigma, j)) = \sqrt[4]{n_r}$. Let $A(\sigma, j)$ be the alternating group on the letters not moved by σ or j and fix each letter in support of σ and j. Then $A(\sigma, j) \cong A_{r-8}$ and $A(\sigma, j) \leq A(\sigma) \cap A(j)$.

LEMMA 4.3. *Let $r \geq 22$. Then $A(\sigma, j)$ acts faithfully on $\Pi(\sigma, j)$ as a totally irregular collineation group.*

Proof. Since $A(\sigma, j)$ is a non-abelian simple group, in order to show that the action on $\Pi(\sigma, j)$ is faithful, it suffices to show that the action is not trivial. Assume that the action is trivial. By 4.1, $A(\sigma, j)$ acts non-trivially on $\Pi(\sigma)$. Thus $A(\sigma, j)$ acts fixed-point-freely on the points of a line of $\Pi(\sigma)$ outside $\Pi(\sigma, j)$. This yields $\frac{(r-8)!}{2} = |A_{r-8}| \leq o(\Pi(\sigma)) -$

$o(\Pi(\sigma, j)) < o(\Pi(\sigma)) = \sqrt{n_r}$. Hence the following holds:

$$n_r > \left[\frac{(r-8)!}{2}\right]^2. \tag{4.3}$$

However, (4.1) together with (4.3) yields $r \leq 21$. This contradiction proves that the action of $A(\sigma, j)$ on $\Pi(\sigma, j)$ is faithful as desired.

Next suppose the action of $A(\sigma, j)$ on $\Pi(\sigma, j)$ is not totally irregular. This means $A(\sigma, j)$ has a free orbit of points in $\Pi(\sigma, j)$. Being a free orbit, it must miss the fixed points of any involution. So it misses at least $o(\pi(\sigma, j)) + 1$ points. Hence $|A_{r-8}| = |A(\sigma, j)| \leq o(\Pi(\sigma, j))^2 + o(\Pi(\sigma, j)) + 1 - (o(\Pi(\sigma, j)) + 1)$. Thus $|A_{r-8}| \leq o(\Pi(\sigma, j))^2 = \sqrt{n_r}$. This yields $r \leq 21$ as before. This contradiction proves that $A(\sigma, j)$ is a totally irregular collineation group of $\Pi(\sigma, j)$ and proves 4.3.

Let $S = \{(G_r, \Pi_r) | r \in R \subseteq \{\textit{positive integers}\}\}$ be a sequence of totally irregular collineation groups $G_r \cong A_r$ on finite projective planes Π_r of order n_r. Set N to be the maximal element in R. ($N = \infty$ if $|R| = \infty$.) In proving Theorem 1, we may assume by 2.10, 3.4, and 3.5 that for $r \in R$ and $r \geq 8$, all involutions of G_r are Baer involutions of Π_r. Let $\Gamma = \{14, 15, \dots, 20, 21\}$. For $a \in \Gamma$, set $R_a = \{r \in R | r \equiv a (\bmod 8)\}$. The proof of Theorem 1 is divided into the following five steps.

Step 1. Suppose $a \in \Gamma$ and $r = a + 8t \in R_a$ with $t \geq 0$. Then there is a totally irregular collineation group $M_a \cong A_a$ on a projective plane of order $n_a = \sqrt[4^t]{n_r}$.

Proof. If $t = 0$, then there is nothing to prove. Suppose $t \geq 1$. Then $r \geq 22$. Our conclusion now follows from 4.3.

Step 2. The set R is a finite set, i.e., $N < \infty$.

Proof. Assume $N = \infty$. By the pigeon hole principle, there exists $a \in \Gamma$ such that R_a is an infinite set. Thus $\{t | r = a + 8t,\ t \geq 0,\ r \in R_a\}$ is an infinite set. By Step 1, there is a projective plane of order $n_a = \sqrt[4^t]{n_r}$. From equation (4.1), we have $n_r \leq \frac{3}{2}|A_r| + 1 < r!$. Hence $n_a < \sqrt[4^t]{r!} \leq \sqrt[4^t]{(a+8t)!}$. This implies $n_a \leq \lim_{t \to \infty} \sqrt[4^t]{(a+8t)!} = 1$, a contradiction. This establishes Step 2.

Let $N = a + 8T$ for some $a \in \Gamma$. In proving Theorem 1, we may assume $T \geq 0$. We shall prove $T = 0$, which completes the proof of Theorem 1.

By Step 1, there is a projective plane of order $n_a = \sqrt[4^T]{n_N}$, where $n_N \leq \frac{3}{2}|A_N| + 1 < N! = (a + 8T)!$ by (4.1). Using logarithm in base 2, this yields the following inequality:

$$\log_2 n_a < \frac{\log_2(a + 8T) + \cdots + \log_2 3 + \log_2 2}{4^T} := L_a(T). \quad (4.4)$$

Step 3. If $a \in \Gamma$, then $T \leq 2$.

Proof. As $a \in \Gamma$, $a \geq 14$. By 2.4 and the fact that a projective plane of order at most 8 is Desarguesian, $M_a \cong A_a$ cannot act as a collineation group on a projective plane of order at most 8. This implies that $n_a > 8$. So $\log_2 n_a > 3$.

As $a \in \Gamma$, $a \leq 21$. Let $L_{21}(T) = L(T)$. Since $L_a(T)$ is an increasing function on the variable a, this implies that $\log_2 n_a < L_a(T) \leq L(T)$. The function $L(T)$ is a decreasing function with the variable T. Hence if $T \geq 4$, then $3 < \log_2 n_a < L(4)$.

We now use the following elementary inequalities to compute $L(4)$: $\log_2 2 = 1$, $\log_2 x \leq 2$ for $3 \leq x \leq 4$, $\log_2 x \leq 3$ for $5 \leq x \leq 8$, $\log_2 x \leq 4$ for $9 \leq x \leq 16$, $\log_2 x \leq 5$ for $17 \leq x \leq 32$, $\log_2 x \leq 6$ for $33 \leq x \leq 53$. We obtain $L(4) < (21(6) + 16(5) + 8(4) + 4(3) + 2(2) + 1)/4^4 < 1$. Hence $3 < \log_2 n_a < L(4) < 1$, a contradiction. Therefore $T \leq 3$. A similar computation shows that $L(3) < (13(6) + 16(5) + 8(4) + 4(3) + 2(2) + 1)/4^3 \leq 3.25$. Hence $n_a < 2^{3.25} \leq 9.52$. So $n_a = 9$. However, this contradicts 3.6. This establishes Step 3.

Step 4. If $a \in \Gamma$, then $T \leq 1$.

Proof. We use the same notation as in the proof of Step 3. By Step 3, $T \leq 2$. So $N \leq 21 + 16 = 37$. Hence $\log_2 n_a < L_a(T) \leq (\log_2 37 + \cdots + \log_2 2)/4^2 = L(2)$. Now $L(2) \leq (5(6) + 16(5) + 8(4) + 4(3) + 2(2) + 1)/16 < 10$. Thus $n_a < 2^{10}$.

As $a \in \Gamma$, $a \geq 14$. For a short involution $\sigma \in M_a$, $A(\sigma)$ acts faithfully, by 4.1, on the subplane $\Pi(\sigma)$ of order $\sqrt{n_a} < 2^5$. As $a - 4 \geq 10$ and $A(\sigma) \cong A_{a-4}$, a direct calculation shows that $A(\sigma)$ is a totally irregular collineation group of $\Pi(\sigma)$. As $a \geq 14$, every short involution in $A(\sigma)$ induces a Baer involution on $\Pi(\sigma)$, by 4.2. So $\sqrt{n_a}$ is a square. This forces $\sqrt{n_a} = 9$, 16, or 25. However, this contradicts 3.6. Step 4 is now established.

Step 5. $T = 0$.

Proof. Deny this. We continue to use the notation in the proofs of Steps 1 and 3. By Step 4, we may assume $T = 1$. Thus $N \leq 21 + 8 = 29$ and $\log_2 n_a \leq (\log_2 29 + \cdots + \log_2 3 + \log_2 2)/4 \leq (13(5) + 8(4) + 4(3) + 2(2) + 1)/4 \leq 28.5$. Hence $n_a < 2^{28}\sqrt{2}$. Thus for a short involution $\sigma \in M_a$, $A(\sigma)$ is a collineation group on the subplane $\Pi(\sigma)$ of order

$\sqrt{n_a} < 2^{14}\sqrt[4]{2}$. As $a \geq 14$, by 4.2, all involutions of $A(\sigma)$ induce Baer involutions on $\Pi(\sigma)$. A direct calculation shows that $A(\sigma)$ is a totally irregular collineation group on $\Pi(\sigma)$. Suppose $a \geq 15$. So $a - 4 \geq 11$ and $A(\sigma, j) \cong A_{a-8}$ is a collineation group on $\Pi(\sigma, j)$ for a short involution of M_a disjoint from σ by 4.1. We divide the rest of the proof of $T = 0$ into three cases: Case 1, $18 \leq a \leq 21$; Case 2, $a \in \{16, 17\}$; Case 3, $a \in \{14, 15\}$.

Case 1. From $A(\sigma, j) \cong A_{a-8}$, $a - 8 \geq 10$, and $u := o(\Pi(\sigma, j)) < 2^7\sqrt[8]{2}$, a direct calculation shows that $A(\sigma, j)$ is a totally irregular on $\Pi(\sigma, j)$. By 2.10, the involutions of this group induce Baer involutions on $\Pi(\sigma, j)$. So the order of this subplane is a square. From $u \leq 140$, this forces $u = s^2$ with $9 \leq s \leq 11$ by 2.4 and the fact that A_{10} is not a collineation group of a projective plane of order at most 9. However, this contradicts 3.7. This finishes the proof in Case 1.

Case 2. In this case $N \leq 25$.

So $\log_2 n_a < (\log_2 25 + \cdots + \log_2 3 + \log_2 2)/4 \leq (9(5) + 8(4) + 4(3) + 2(2) + 1)/4 < 24$. Thus $o(\Pi(\sigma)) = \sqrt{n_a} < 2^{12}$. So $A(\sigma, j)$ is a collineation group of $\Pi(\sigma, j)$ of order at most 2^6. By 2.10, as $a - 8 \geq 8$, we may assume involutions in $A(\sigma, j)$ induce Baer involutions on $\Pi(\sigma, j)$. Hence $u := o(\Pi(\sigma, j))$ is a square. This forces $u = 9$, 16, 25, or 49 as A_8 cannot act on a projective plane of order less than 9 by 2.4. By 3.6, we see that $u = 49$. Suppose $a = 16$. Then $\log_2 n_a < L(1) \leq 22$, which yields $n_a < 2^{22}$. Thus $u = \sqrt[4]{n_a} < 2^5\sqrt{2} < 49$. This contradiction proves that $a \neq 16$. Therefore $a = 17$.

Since $a - 8 \geq 9$, there is a short involution k in $A(\sigma, j)$ disjoint from both σ and j. Let K be the alternating group on the letters not in support of σ, j, or k, and fix each letters in support of σ, j, or k. Then $K \cong A_{a-12} = A_5$ acts on the Baer subplace $B := \Pi(\sigma, j) \cap \Pi(k)$ of order $\sqrt{49} = 7$. As K is simple, we see, by 2.4, that K acts trivially on the Desarguesian subplane B. However, this forces $60 = |A_5|$ divides $u - \sqrt{u} = 49 - 7 = 42$. This contradiction proves that $T = 0$ holds in Case 2.

Case 3. In this case, $N \leq 23$.

Thus $\log_2 n_a < (\log_2 23 + \cdots + \log_2 2)/4 \leq 21$. So $n_a < 2^{21}$ and $u := o(\Pi(\sigma, j)) = \sqrt{n_a} < 2^{10}\sqrt{2} < (39)^2$. Suppose $a = 14$. Assume $A(\sigma, j)$ acts trivially on $\Pi(\sigma, j)$. This implies that $A(\sigma, j)$ has a free orbit of points of a line of $\Pi(\sigma)$ outside of $\Pi(\sigma, j)$. As $A(\sigma, j) \cong A_6$ in the current situation, this implies that 360 divides $u^2 - u$, where $2 \leq u \leq 38$. This contradiction proves that $A(\sigma, j)$ is a collineation group of $\Pi(\sigma, j)$ for $a = 14$ or 15. We treat $a = 14$ and $a = 15$ separately.

Case 3.1. $a = 15$.

In this case $a - 4 = 11$. A direct calculation shows that $A(\sigma) \cong A_{11}$ is a totally irregular collineation group on $\Pi(\sigma)$. By 3.7, we see that $u > 31$. Hence $32 \leq u \leq 38$. A direct calculation shows that $A(\sigma, j) \cong A_{a-8} = A_7$ is totally irregular on $\Pi(\sigma, j)$. Assume there is an involution of this action which is a Baer involution. This forces $u = 36$ and the existence of a subplane of order 6. This contradicts 2.3. Therefore all involutions of $A(\sigma, j)$ induce perspectivities on $\Pi(\sigma, j)$. By 2.10, this action is strongly irreducible and the involutions induce homologies. So u is odd, and an element of order 3, which is a product of two involutions, is not planar. This shows $u \equiv 1 \pmod 3$. Thus $u = 37$. Since $37 \equiv 2 \pmod 5$, an element of order 5 of $A(\sigma, j)$ must be planar on $\Pi(\sigma, j)$. However, this contradicts 2.5 as an element of order 5 of A_7 is inverted by an involution. This contradiction completes the proof of Case 3.1.

Case 3.2. $a = 14$.

In this case $\log_2 n_a < (\log_2 22 + \cdots + \log_2 2)/4 < 20$. Hence $A(\sigma)$ is totally irregular, by a direct calculation, on $\Pi(\sigma)$ of order strictly less than 2^{10}. By 2.10, involutions of this group induce Baer involution on this subplane. Now $A(\sigma, j) \cong A_6$ acts on $\Pi(\sigma, j)$. So $u \geq 9$. On the other hand, u is the order of a Baer subplane of $\Pi(\sigma)$ of order strictly less than $2^5 = 32$. Therefore $9 \leq u \leq 31$. This yields that $A(\sigma) \cong A_{10}$ acts as a totally irregular collineation group on $\Pi(\sigma)$ of order u^2, with $9 \leq u \leq 31$. However, this contradicts 3.7. This contradiction completes the proof of Case 3.2 and the proof of Theorem 1.

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