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Orthogonality properties of the Hermite and related polynomials

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Abstract

The authors present a general method of operational nature with a view to investigating the orthogonality properties of several different families of the Hermite and related polynomials. In particular, the classical Hermite polynomials and some of their higher-order and multi-index generalizations are considered here.

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1. Introduction, definitions and preliminaries

The so-called Hermite–Kampé de Fériet (HKdF) polynomials (or, alternatively, the *two-variable* Hermite polynomials) [1, p. 341, Eq. (23)]:

$$H_n(x, y) := n! \sum_{r=0}^{[n/2]} \frac{x^{n-2r} y^r}{(n-2r)!r!} \tag{1}$$

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can be defined by means of the following operational rule (cf. [2]):

$$H_n(x,y) = \exp\left(y\,\frac{\partial^2}{\partial x^2}\right)\{x^n\}, \tag{2}$$

which was applied by Dattoli *et al.* [3] in order to reconsider the orthogonality property of the classical Hermite polynomials from a different point of view. It was indeed shown there [3] that such properties can be derived from fairly straightforward identities, in a direct way, by employing a few elementary properties of the exponential operators.

In terms of the classical Hermite polynomials $H_n(x)$ or $\mathrm{He}_n(x)$, it is easily seen from the definition (1) that

$$H_n(2x,-1) = H_n(x) \quad \text{and} \quad H_n(x,-\tfrac{1}{2}) = \mathrm{He}_n(x). \tag{3}$$

Furthermore, even though there exists the following close relationship [1, p. 341, Eq. (21)]:

$$H_n(x,y) = (-\mathrm{i})^n\; y^{n/2}\; H_n\left(\frac{\mathrm{i}x}{2\sqrt{y}}\right) = \mathrm{i}^n(2y)^{n/2}\;\mathrm{He}_n\left(\frac{x}{\mathrm{i}\sqrt{2y}}\right) \tag{4}$$

with the classical Hermite polynomials, yet the usage of a *second* variable (parameter) y in the Hermite–Kampé de Fériet polynomials $H_n(x,y)$ is found to be convenient from the viewpoint of their applications. Indeed, from an entirely different viewpoint and considerations, Hermite polynomials of *several* variables are introduced and investigated by Erdélyi *et al.* [5, p. 283 *et seq.*].

Limiting ourselves to negative values of the variable y, which is treated in the present context as a parameter, we consider the following polynomial expansion:

$$F(x) = \sum_{n=0}^{\infty} a_n\; H_n(x,-|y|). \tag{5}$$

Then our goal will be that of specifying the coefficients a_n by the use of the operational representation (2) and of the formalism associated with the aforementioned operators. It is easily seen from Eq. (2) that

$$\exp\left(|y|\frac{\partial^2}{\partial x^2}\right)\{F\,(x)\} = \sum_{n=0}^{\infty} a_n\; x^n. \tag{6}$$

Our problem has, therefore, been reduced to that of finding the Taylor expansion of the function

$$\Phi(x) = \exp\left(|y|\frac{\partial^2}{\partial x^2}\right)\{F(x)\}, \tag{7}$$

which, in view of the well-known Gauss–Weierstrass transform, can be written in the following form:

$$\Phi(x) = \frac{1}{2\sqrt{\pi|y|}}\int_{-\infty}^{\infty} \exp\left(-\frac{(x-\sigma)^2}{4|y|}\right) F(\sigma)\,\mathrm{d}\sigma. \tag{8}$$

By recalling that the polynomials $H_n(x,y)$ are specified by means of the generating function:

$$\sum_{n=0}^{\infty} \frac{t^n}{n!}\; H_n(x,y) = \exp(xt + yt^2), \tag{9}$$

we find that

$$\Phi(x) = \frac{1}{2\sqrt{\pi|y|}} \sum_{n=0}^{\infty} \frac{x^n}{n!} \int_{-\infty}^{\infty} H_n\left(\frac{\sigma}{2|y|}, -\frac{1}{4|y|}\right) \exp\left(-\frac{\sigma^2}{4|y|}\right) F(\sigma)\,\mathrm{d}\sigma, \tag{10}$$

which, when compared with Eq. (6), yields

$$a_n = \frac{1}{2 \cdot n!\sqrt{\pi|y|}} \int_{-\infty}^{\infty} H_n\left(\frac{\sigma}{2|y|}, -\frac{1}{4|y|}\right) \exp\left(-\frac{\sigma^2}{4|y|}\right) F(\sigma)\,\mathrm{d}\sigma. \tag{11}$$

This last identity (11) can be suitably exploited to conclude that the functions

$$\varphi_n(x) = \frac{1}{2 \cdot n!\sqrt{\pi|y|}} H_n\left(\frac{x}{2|y|}, -\frac{1}{4|y|}\right) \exp\left(-\frac{x^2}{4|y|}\right) \tag{12}$$

are *biorthogonal* to the polynomials $H_n(x, -|y|)$.

The corresponding expansion in series of the classical Hermite polynomials $H_n(x)$ is easily obtained from the above results by setting $|y| = \frac{1}{2}$.

In these introductory remarks, we have shown how the use of operational tools has allowed the derivation of the orthogonality properties of the Hermite polynomials in a fairly direct way. In the following sections, we will show that the method developed here can be extended appropriately to more involved families of Hermite polynomials as well.

2. Multi-index Hermite polynomials and associated biorthogonal functions

Multi-variable and multi-index Hermite polynomials were introduced by Charles Hermite (1822–1901) himself in his memoirs dated 1864 in which he also investigated the relevant orthogonality properties (cf., e.g., [1, p. 331 *et seq.*]). In this section, we will not follow the original treatment also exploited in [1], but the operational formalism of [2], which will provide a fairly direct understanding of the problem.

According to the operational definition, the two-variable and two-index Hermite polynomials can be defined as follows:

$$H_{m,n}(x, \alpha; y, \gamma|\beta) = \exp\left(\alpha \frac{\partial^2}{\partial x^2} + \beta \frac{\partial^2}{\partial x \partial y} + \gamma \frac{\partial^2}{\partial y^2}\right) \{x^m \; y^n\}, \tag{13}$$

which provides the relevant series expansion:

$$H_{m,n}(x, \alpha; y, \gamma|\beta) = m!\; n! \sum_{s=0}^{\min(m,n)} \frac{\beta^s}{s!(m-s)!(n-s)!} H_{m-s}(x, \alpha) H_{n-s}(y, \gamma) \tag{14}$$

and the generating function

$$\sum_{m,n=0}^{\infty} \frac{u^m}{m!} \frac{v^n}{n!} H_{m,n}(x, \alpha; y, \gamma|\beta) = \exp(xu + \alpha u^2 + yv + \gamma v^2 + \beta uv). \tag{15}$$

Let us now consider a *two-variable* function $F(x, y)$ with the following expansion:

$$F(x, y) = \sum_{m,n=0}^{\infty} a_{m,n} \; H_{m,n}(x, -|\alpha|; y, -|\gamma| \, \Big| -|\beta|). \tag{16}$$

We will evaluate the coefficients $a_{m,n}$ of expansion (16) by following the same procedure as before. Indeed, by using the operational representation (13), we get

$$\begin{aligned} \Phi(x, y) &= \exp\left(|\alpha| \frac{\partial^2}{\partial x^2} + |\beta| \frac{\partial^2}{\partial x \partial y} + |\gamma| \frac{\partial^2}{\partial y^2} \right) \{F(x, y)\} \\ &= \sum_{m,n=0}^{\infty} a_{m,n} \; x^m \; y^n. \end{aligned} \tag{17}$$

The problem now is that of specifying the action of the exponential operator:

$$\widehat{E} = \exp\left(|\alpha| \frac{\partial^2}{\partial x^2} + |\beta| \frac{\partial^2}{\partial x \partial y} + |\gamma| \frac{\partial^2}{\partial y^2} \right). \tag{18}$$

Thus we must look for a generalized version of the Gauss–Weierstrass transform used in the derivation of (8) above. By tacitly assuming, for convenience, that

$$\min(\alpha, \beta, \gamma) > 0,$$

we can omit absolute values and rewrite the exponential operator $\widehat{E}$ in (18) as follows:

$$\widehat{E} = \exp\left(\alpha \left(\frac{\partial}{\partial x} + \frac{1}{2} \frac{\beta}{\alpha} \frac{\partial}{\partial y} \right)^2 + \frac{\Delta}{\alpha} \frac{\partial^2}{\partial y^2} \right) \quad \left(\Delta = \gamma\alpha - \frac{\beta^2}{4} \right). \tag{19}$$

By using the following integral formula:

$$\exp(\lambda \hat{c}^2) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp\left(-\sigma^2 + 2\sigma\sqrt{\lambda} \; \hat{c} \right) \mathrm{d}\sigma, \tag{20}$$

we find from (19) that

$$\widehat{E} := \frac{1}{\pi} \int_{-\infty}^{\infty} \mathrm{d}\xi \int_{-\infty}^{\infty} \mathrm{d}\eta \, \exp\left(-\left[\xi^2 + \eta^2 - 2\sqrt{\alpha} \; \xi \left(\frac{\partial}{\partial x} + \frac{1}{2} \frac{\beta}{\alpha} \frac{\partial}{\partial y} \right) - 2\eta \sqrt{\frac{\Delta}{\alpha}} \frac{\partial}{\partial y} \right] \right). \tag{21}$$

Now the action of the exponential operator $\widehat{E}$ on the function $F(x, y)$ is easily derived by the use of the well-known identity

$$\exp\left(a \frac{\partial}{\partial x} + b \frac{\partial}{\partial y} \right) \{F(x, y)\} = F(x + a, y + b), \tag{22}$$

which, along with the definition (21), yields

$$\widehat{E} F(x, y) = \frac{1}{\pi} \int_{-\infty}^{\infty} \mathrm{d}\xi \int_{-\infty}^{\infty} \mathrm{d}\eta \, \exp\left(-\left(\xi^2 + \eta^2 \right) \right) F\left(x + 2\sqrt{\alpha} \; \xi, \; y + \frac{\beta}{\sqrt{\alpha}} \xi + 2\sqrt{\frac{\Delta}{\alpha}} \eta \right). \tag{23}$$

Finally, upon performing the change of variables given by

$$x + 2\sqrt{\alpha}\,\xi = \sigma \quad \text{and} \quad y + \frac{\beta}{\sqrt{\alpha}}\,\xi + 2\sqrt{\frac{\Delta}{\alpha}}\,\eta = \tau, \tag{24}$$

we obtain the *two-variable* extension of the Gauss–Weierstrass transform as follows:

$$\begin{aligned}\widehat{E}F(x, y) = {} & \frac{1}{4\pi\sqrt{\Delta}} \int_{-\infty}^{\infty} \mathrm{d}\sigma \int_{-\infty}^{\infty} \mathrm{d}\tau \\ & \cdot \exp\left(-\frac{1}{4\Delta}\left[\gamma(x-\sigma)^2 - \beta(x-\sigma)(y-\tau) + \alpha(y-\tau)^2\right]\right) F(\sigma, \tau).\end{aligned} \tag{25}$$

Consequently, the use of the generating function (15) leads us to the following explicit expression for the coefficients $a_{m,n}$ in (16):

$$\begin{aligned}a_{m,n} = {} & \frac{1}{4\pi\sqrt{\Delta}} \frac{1}{m!\, n!} \int_{-\infty}^{\infty} \mathrm{d}\xi \int_{-\infty}^{\infty} \mathrm{d}\eta\ H_{m,n}\left(\frac{\gamma\xi - \frac{\beta}{2}\eta}{2\Delta}, -\frac{\gamma}{4\Delta}; \frac{\alpha\eta - \frac{\beta}{2}\xi}{2\Delta}, -\frac{a}{4\Delta} \middle| -\frac{\beta}{4\Delta}\right) \\ & \cdot \exp\left(-\frac{1}{4\Delta}\left(\gamma\xi^2 - \beta\xi\eta + \alpha\eta^2\right)\right) F(\xi, \eta),\end{aligned} \tag{26}$$

which, while generalizing the previous result, allows us to conclude that the functions

$$\begin{aligned}\varphi_{m,n}(x, y) = {} & \frac{1}{4\pi \cdot m!\, n!\sqrt{\Delta}} H_{m,n}\left(\frac{\gamma x - \frac{\beta}{2}y}{2\Delta}, -\frac{\gamma}{4\Delta}; \frac{\alpha y - \frac{\beta}{2}x}{2\Delta}, -\frac{\alpha}{4\Delta} \middle| \frac{\beta}{4\Delta}\right) \\ & \cdot \exp\left(-\frac{1}{4\Delta}\left(\gamma x^2 - \beta x y + \alpha y^2\right)\right)\end{aligned} \tag{27}$$

are *biorthogonal* to the polynomials

$$H_{m,n}(x, -|\alpha|; y, -|\gamma| \,|\, -|\beta|).$$

In the above case, too, the statement relevant to the orthogonality has been obtained, in a fairly direct way, by using the formalism associated with the exponential operator algebra.

3. Higher-order Hermite polynomials

In this section, we will discuss further generalizations of the orthogonality properties investigated in the preceding sections.

The higher-order Hermite polynomials $H_n^{(m)}(x, y)$ (or, equivalently, the Gould–Hopper polynomials $g_n^m(x, y)$ [9, p. 76, Eq. 1.9 (6)]) are given explicitly in [6, p. 58, Eq. (6.2)]

$$H_n^{(m)}(x, y) := n! \sum_{r=0}^{[n/m]} \frac{x^{n-mr}\, y^r}{(n-mr)!\, r!} =: g_n^m(x, y), \tag{28}$$

where m is the order of the polynomial. The alternative operational definition:

$$H_n^{(m)}(x, y) := \exp\left(y \frac{\partial^m}{\partial x^m}\right)\{x^n\} \tag{29}$$

is particularly useful in (for example) deriving the following generating function:

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} H_n^{(m)}(x, y) = \exp(xt + yt^m). \tag{30}$$

It is easily observed from the definition (28) that

$$H_n^{(1)}(x, y) = (x + y)^n \quad \text{and} \quad H_n^{(2)}(x, y) = H_n(x, y), \tag{31}$$

where $H_n(x, y)$ denotes the Hermite–Kampé de Fériet polynomials defined by (1). Indeed, just as it was pointed out by Srivastava and Manocha [9, pp. 76–77] in connection with the Gould–Hopper polynomials $g_n^m(x, y)$, many obvious variations and special cases of the *higher-order* Hermite polynomials $H_n^{(m)}(x, y)$ have been rediscovered in several different contexts, not only in the mathematical and statistical sciences, but also in the physical and engineering sciences (see also [2,7,8]).

The family of functions biorthogonal to this last family of polynomials $H_n^{(m)}(x, y)$ have already been studied in [7,8], and subsequently (with a different technique) in [4]. Here we will address the problem by following the technique developed in the preceding sections. Therefore, our starting point will be the assumption that the following expansion formula exists (see, for details [7,8]):

$$F(x) = \sum_{n=0}^{\infty} a_n \; H_n^{(2q)}\left(x, (-1)^q |y|\right), \tag{32}$$

which, for $q = 1$, corresponds obviously to the polynomial expansion (5) above.

We will now determine the expansion coefficients a_n in (32) from the identity:

$$\begin{aligned} \Phi(x|q) &= \exp\left((-1)^{q+1}|y| \frac{\partial^{2q}}{\partial x^{2q}}\right)\{F(x)\} \\ &= \sum_{n=0}^{\infty} a_n \; x^n, \end{aligned} \tag{33}$$

where we have made use of the operational definition (29) with $m = 2q$ and y replaced by $(-1)^q|y|$. The existence of the analogue of the Gauss–Weierstrass transform, which is needed in this case, is ensured by the following integral formula (cf. [4,7,8]):

$$\exp\left((-1)^{q+1} \; \lambda \hat{c}^{2q}\right) = \int_{-\infty}^{\infty} \exp(-\hat{c}\xi) S_q(\xi, \lambda) \, \mathrm{d}\xi, \tag{34}$$

where, for convenience,

$$S_q(\xi, \lambda) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp\left(-\lambda u^{2q} + \mathrm{i}u\xi\right) \mathrm{d}u = S_q(-\xi, \lambda).$$

Thus, in view of (34), Eq. (33) readily yields

$$\Phi(x|q) = \int_{-\infty}^{\infty} S_q(\xi, |y|) F(x-\xi)\, \mathrm{d}\xi. \tag{35}$$

By performing the change of variables given by

$$x - \xi = \sigma,$$

we find from (35) that

$$\begin{aligned}\Phi(x|q) &= -\int_{-\infty}^{\infty} S_q(\sigma - x, |y|) F(\sigma)\, \mathrm{d}\sigma \\ &= -\sum_{n=0}^{\infty} \frac{(-x)^n}{n!} \int_{-\infty}^{\infty} S_q^{(n)}(\sigma, |y|) F(\sigma)\, \mathrm{d}\sigma,\end{aligned} \tag{36}$$

where the superscript n denotes the nth-order derivative of $S_q(\xi, |y|)$ with respect to ξ.

By comparing (32) with (36), it is clear that the functions

$$\varphi_n(x|q) = \frac{(-1)^{n+1}}{n!} S_q^{(n)}(x, |y|) \tag{37}$$

are *biorthogonal* to the polynomials

$$H_n^{(2q)}\left(x, (-1)^q |y|\right).$$

This result is in agreement with the analogous conclusions of [4,7,8], which were obtained within a different framework.

4. Concluding remarks and observations

In the preceding sections, we have exploited a general procedure to deal with the orthogonality properties of a large body of Hermite polynomial families. The obtained results and the fairly straightforward underlying formalism are elements proving the usefulness and the generality of the method which can easily be extended to other families of polynomials. The two-variable *simple* Laguerre polynomials can indeed be defined by means of the following operational identity (cf. [2]):

$$L_n(x, y) := \exp\left(-y \frac{\partial}{\partial x} x \frac{\partial}{\partial x}\right) \left\{\frac{(-x)^n}{n!}\right\} = n! \sum_{r=0}^{n} \frac{(-x)^r \; y^{n-r}}{(n-r)!(r!)^2} = y^n L_n\left(\frac{x}{y}\right), \tag{38}$$

which can, in turn, be used to state the relevant orthogonality properties by applying a procedure analogous to that outlined here for the Hermite case. Here, as usual,

$$L_n(x) := L_n^{(0)}(x) = y^{-n} \; L_n(xy, y) = L_n(x, 1)$$

denotes the classical Laguerre polynomial of order 0 and degree n in x.

The extension of the above-detailed method to more involved forms of Hermite polynomials with n variables and n indices, such as those defined below:

$$H_{\{n\}}(\{x\},\{\alpha\}\Big|\{\beta\}) = \exp\left(\sum_{j=1}^{n}\alpha_j\frac{\partial^2}{\partial x_j^2}+\sum_{l<j}^{n}\beta_{l,j}\frac{\partial^2}{\partial x_j\,\partial x_l}+\sum_{j=1}^{n}\alpha_j\frac{\partial^2}{\partial x_j^2}\right)\left\{\prod_{k=1}^{n}x_k^{n_k}\right\}$$
$$(\{x\}=x_1,\ldots,x_n;\ \{\alpha\}=\alpha_1,\ldots,\alpha_n;\ \{n\}=n_1,\ldots,n_n) \tag{39}$$

can be accomplished with some minor algebraic complication, but *without* any further conceptual implication.

In a forthcoming paper, we will consider the problem of a unified approach to the theory of orthogonal polynomials by following the technique discussed in this paper.

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