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Gap vertex-distinguishing edge colorings of graphs

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ABSTRACT

In this paper, we study a new coloring parameter of graphs called the *gap vertex-distinguishing edge coloring*. It consists in an edge-coloring of a graph G which induces a vertex distinguishing labeling of G such that the label of each vertex is given by the difference between the highest and the lowest colors of its adjacent edges. The minimum number of colors required for a gap vertex-distinguishing edge coloring of G is called the *gap chromatic number* of G and is denoted by $\mathrm{gap}(G)$.

We here study the gap chromatic number for a large set of graphs G of order n and prove that $\mathrm{gap}(G) \in \{n-1, n, n+1\}$.

1. Introduction and definitions

All graphs considered in this paper are finite and undirected. For a graph G, we use $V(G)$, $E(G)$, $\Delta(G)$ and $\delta(G)$ to denote its vertex set, edge set, maximum degree and minimum degree, respectively. For any undefined terms, we refer the reader to [4].

A vertex labeling of a graph G is said to be *vertex-distinguishing labeling* if distinct vertices are assigned distinct labels. Let k be a non-negative integer. A *k-edge-coloring* of G is a mapping f from $E(G)$ to $\{1, 2, \dots, k\}$. We say that an edge coloring is *proper* if no two adjacent edges have the same color. Many researchers investigated the question of edge coloring inducing a vertex distinguishing labeling. This is often referred to as *vertex-distinguishing edge colorings*. In the literature, four main different functions have been proposed to label each vertex v of G according to the colors of its incident edges. A vertex labeling l induced by an edge-coloring f is said to be:

(1) vertex-labeling by sum if $l(v) = \sum_{v \ni e} f(e), \forall v \in V$ (see [11,2]).
(2) vertex-labeling by sets if $l(v) = \bigcup_{v \ni e} f(e), \forall v \in V$ (see [8,9,14]).
(3) vertex-labeling by multiset if $l(v) = \biguplus_{v \ni e} f(e), \forall v \in V$ (see [1,6,7,10]).
(4) vertex-labeling by product if $l(v) = \prod_{v \ni e} f(e), \forall v \in V$ (see [17]).

The problem of vertex-distinguishing edge colorings offers many variants and received a great interest during these last years. We refer the interested reader to Chapter 13 of Chartrand and Zhang's book [12]. In this paper, we define a new variant called *gap vertex-distinguishing edge coloring*, which is defined as follows.

Definition 1. Let G be a graph, k be a positive integer and f be a mapping from $E(G)$ to the set $\{1, 2, \dots, k\}$. For each vertex v of G, the label of v is defined as

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$$l(v) = \begin{cases} f(e)_{e\ni v} & \text{if } d(v) = 1 \\ \max\limits_{e\ni v} f(e) - \min\limits_{e\ni v} f(e) & \text{otherwise.} \end{cases}$$

The mapping f is called gap vertex-distinguishing labeling if distinct vertices have distinct labels. Such a coloring is called a *gap-k-coloring*.

The minimum positive integer k for which G admits a gap-k-coloring is called the *gap chromatic number* of G and is denoted by $\text{gap}(G)$. Necessary and sufficient conditions for the existence of such a coloring are given by the following proposition.

Proposition 1. *A graph G admits a gap vertex-distinguishing edge coloring if and only if it has no connected component isomorphic to K_1 or K_2.*

Proof. Since no isolated vertex of a graph G is assigned a label in an edge coloring of G, we may assume that G has no isolated vertices. Furthermore, if G contains a connected component K_2, then the two vertices of K_2 are assigned the same label in any edge coloring of G. Hence, when considering gap vertex-distinguishing edge coloring of a graph G, we may assume that the order of every connected component of G is at least 3. Let G be such a graph and let $E(G) = \{e_1, e_2, \ldots, e_m\}$. The following edge coloring function: $f(e_i) = 2^{i-1}$ for $1 \leqslant i \leqslant m$ induces a gap vertex-distinguishing edge coloring of G. □

The following lemma gives a lower bound on the gap chromatic number.

Lemma 2. *A graph G of order n and without connected component isomorphic to K_1 or K_2 satisfies* $\text{gap}(G) \geq n - 1$. *Moreover, if $\delta(G) \geq 2$ or if any vertex of degree greater than 1 has at least two adjacent vertices of degree 1, then* $\text{gap}(G) \geq n$.

To illustrate these concepts, consider the graph G shown in Fig. 1(a). A 6-edge coloring f_1 of G is given in Fig. 1(b) and a 5-edge coloring f_2 of G is given in Fig. 1(c). For example, in Fig. 1(b), the vertex w is incident to two edges colored 2 and one edge colored 3, then $l_1(w) = 1$, while the vertex z is incident with one edge colored 6, then $l_1(z) = 6$. The resulting vertex labels are distinct for both figures. By Lemma 2, we have $\text{gap}(G) \geq 5$; hence we can immediately conclude that $\text{gap}(G) = 5$.

After a strong analysis of this problem, we raised the conjecture asserting that there is no graph G of order n with $\text{gap}(G) > n + 1$.

Conjecture 3. *A graph G of order n (not necessarily connected), without isolated edges and isolated vertices has* $\text{gap}(G) \in \{n - 1, n, n + 1\}$.

In the following sections, we prove this conjecture for a large set of graphs and we even decide the exact value of $\text{gap}(G)$. The rest of the paper is organized as follows: first, we point out some previous work related to the topic of this paper and give some motivations to investigate this new parameter. The results of Section 3 will confirm our conjecture for a large part of graphs with minimum degree at least 2. In Section 4, we prove our conjecture for some classes of graphs with minimum degree 1, such as paths, complete binary trees and all trees with at least two leaves at distance 2. This classification of our results according to $\delta(G)$ is due to the definition of our parameter, especially to the definition of labels of vertices of degree one. Concluding remarks and some open issues are discussed in the last section.

2. Motivation and related work

In this section, we describe the motivation to study the gap coloring problem. We first introduce the following notation: given a set S of positive integers, we denote by $\text{diam}(S)$ the diameter of S, where $\text{diam}(S) = \max\{x - y : x, y \in S\}$. The following proposition is thus obvious.

Proposition 4. *Let S_1 and S_2 be two sets of positive integers, if* $\text{diam}(S_1) \neq \text{diam}(S_2)$, *then* $S_1 \neq S_2$.

From the gap vertex labeling function (Definition 1), we observe that the label of every vertex v with degree at least 2 is the diameter of the set of colors incident to v. Note that this is not the case for the vertices of degree 1. Then, the gap labeling of a graph G can be seen as a strong version of set and multiset labelings (defined on page 2, in (2) and (3)). Indeed, according to Proposition 4, a gap distinguishing labeling of a graph G is also a multiset distinguishing labeling of G and a set distinguishing labeling (if $\delta(G) > 1$). We here present the main results about these related coloring problems.

Let $\chi'_s(G)$ denote the minimum number of colors required to have a proper edge coloring of G that induces a vertex-distinguishing labeling by sets. This coloring number was introduced and studied by Burris and Schelp in [5,8], and independently called *observability* of a graph by Cerny et al. [9]. The following result has been conjectured by Burris and Schelp [8] and proved in [3].

Theorem 5 *([3]). A graph G with n vertices, without isolated edges and with at most one isolated vertex, has* $\chi'_s(G) \leq n + 1$.

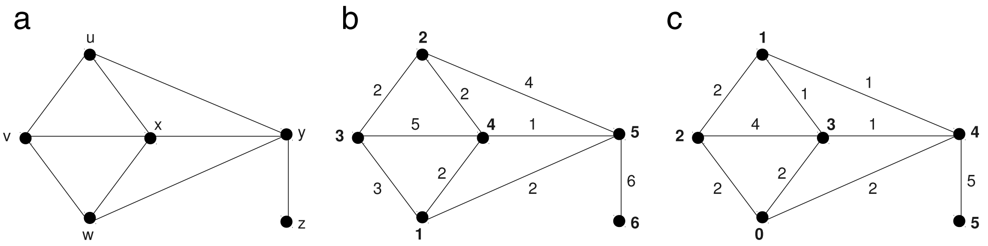

Fig. 1. A gap vertex-distinguishing edge coloring of a graph.

Let $\chi'_0(G)$ denote the minimum number of colors required to have an edge coloring (not necessarily proper) of a graph G that induces a vertex distinguishing labeling by sets. Harary and Plantholt [14] referred to this type of coloring as a *point-distinguishing edge coloring*. They proved, among other things, the exact value of $\chi'_0(P_n)$, $\chi'_0(C_n)$, $\chi'_0(Q_n)$ and $\chi'_0(K_n)$ for $n \geq 3$. Even for bipartite graphs it seems that the problem of determining $\chi'_0(K_{m,n})$ is not easy (see [15,16,18]). Clearly we have $\chi'_0(G) \leq \chi'_s(G)$, and the following result follows from Theorem 5.

Theorem 6. *A graph G with n vertices, without isolated edges and with at most one isolated vertex, has* $\chi'_0(G) \leq n + 1$.

Finally, let $c(G)$ denote the minimum number of colors required to have an edge coloring (not necessarily proper) of G that induces a vertex-distinguishing labeling by multisets. This concept was studied in [6,7,13,10] and the following result stated in [12] will be useful to bound our parameter.

Theorem 7 (*[12]*)**.** *If G is a connected graph of order* $n \geq 4$*, then* $c(G) \leq n - 1$.

We now characterize the relationship between our coloring parameter and the two coloring parameters $\chi'_0(G)$ and $c(G)$ defined previously. The following results follow from Proposition 4 and the definitions of $\chi'_0(G)$ and $c(G)$.

Lemma 8. *For every graph G without components isomorphic to either* K_1 *or* K_2 *and with minimum degree at least* 2*, we have*

$$\chi'_0(G) \leq \mathrm{gap}(G).$$

Lemma 9. *For every graph G, without components isomorphic to either* K_1 *or* K_2*, we have*

$$c(G) \leq \mathrm{gap}(G).$$

We will see in Corollary 20 how the results of the current paper can be connected to the study of $\chi'_0(G)$.

3. Graphs with $\delta(G) \geq 2$

Recall that an *m-edge-connected graph* is a graph in which removing any $m - 1$ edges does not disconnect it. The main result of this section is the following.

Theorem 10. *For every m-edge-connected graph G of order n with* $m \geq 2$,

$$\mathrm{gap}(G) = \begin{cases} n & \text{if } G \text{ is not a cycle of length } \equiv 2, 3 (\mathrm{mod}\ 4) \\ n + 1 & \text{otherwise.} \end{cases}$$

The proof of Theorem 10 is the combination of several results detailed below.

Theorem 11. *Let* C_n *be a cycle of order n, then*

$$\mathrm{gap}(C_n) = \begin{cases} n & \text{if } n \equiv 0, 1 (\mathrm{mod}\ 4) \\ n + 1 & \text{otherwise.} \end{cases}$$

Proof. Let $C_n = (v_1, v_2, \ldots, v_n, v_{n+1} = v_1)$. For each integer i with $1 \leq i \leq n$, let $e_i = v_i v_{i+1}$. We consider two cases as follows:

Case 1: $n \equiv 0, 1 (\mathrm{mod}\ 4)$. By Lemma 2, we have $\mathrm{gap}(C_n) \geq n$, it then suffices to prove that C_n admits a gap-n-coloring. Two subcases are considered:

Subcase 1.1: $n \equiv 0 \pmod 4$. A mapping f from $E(C_n)$ to $\{1, 2, \ldots, n\}$ is defined as follows (see Fig. 2(a)).

$$\text{For } 1 \le i \le n, \qquad f(e_i) = \begin{cases} n+1-i & \text{if } i \text{ is odd} \\ 1 & \text{if } i \equiv 2 \pmod 4 \\ 2 & \text{if } i \equiv 0 \pmod 4. \end{cases}$$

This mapping induces the following gap vertex labeling function.

$$\text{For } 1 \le i \le n, \qquad l(v_i) = \begin{cases} n-i+1 & \text{if } i \equiv 2 \pmod 4 \\ n-i & \text{if } i \equiv 0, 3 \pmod 4 \\ n-i-1 & \text{if } i \equiv 1 \pmod 4. \end{cases}$$

Then, it is easy to check that l is a bijection from $V(C_n)$ to $\{0, 1, \ldots, n-1\}$. Hence $\mathrm{gap}(C_n) = n$.

Subcase 1.2: $n \equiv 1 \pmod 4$. A mapping f from $E(C_n)$ to $\{1, 2, \ldots, n\}$ is defined as follows (see Fig. 2(b)).

$$\text{For } 1 \le i \le n, \qquad f(e_i) = \begin{cases} i & \text{if } i \text{ is odd} \\ n-1 & \text{if } i \equiv 2 \pmod 4 \\ n & \text{if } i \equiv 0 \pmod 4. \end{cases}$$

This mapping induces the following gap vertex labeling function.

$$\text{For } 1 \le i \le n, \qquad l(v_i) = \begin{cases} n-i & \text{if } i \equiv 1, 2 \pmod 4 \\ n-i+1 & \text{if } i \equiv 0 \pmod 4 \\ n-i-1 & \text{if } i \equiv 3 \pmod 4. \end{cases}$$

Then, it is easy to check that l is a bijection from $V(C_n)$ to $\{0, 1, \ldots, n-1\}$. Hence $\mathrm{gap}(C_n) = n$.

Case 2: $n \equiv 2, 3 \pmod 4$. We first prove that $\mathrm{gap}(C_n) > n$. Let $f : V(C_n) \longrightarrow \{1, 2, \ldots, n\}$ be any edge-coloring of C_n which induces a gap vertex-distinguishing function l. Now note that:

$$\sum_{i=1}^{n} l(v_i) = |f(e_1) - f(e_n)| + \sum_{i=2}^{n} |f(e_i) - f(e_{i-1})| = \frac{n(n-1)}{2}.$$

In this formula, each term $f(e_i)$ appears twice with opposite (or same) signs; hence $\frac{n(n-1)}{2}$ is even. But this latter value is odd if $n \equiv 2, 3 \pmod 4$, which is a contradiction. Thus, $\mathrm{gap}(C_n) \ge n+1$. It then remains to show that $\mathrm{gap}(C_n) \le n+1$. Two subcases are considered according to whether $n \bmod 4 = 2$ or 3.

Subcase 2.1: $n \equiv 3 \pmod 4$. We know that C_{n+1} admits a gap-$(n+1)$-coloring. Necessarily, C_{n+1} must contain two successive edges of same color j where $1 \le j \le n+1$. By merging these two edges into a single edge colored by j, we obtain a gap-$(n+1)$-coloring of C_n (see Fig. 2(c)).

Subcase 2.2: $n \equiv 2 \pmod 4$. In this subcase, we define an edge coloring f from $E(C_n)$ to $\{1, 2, \ldots, n, n+1\}$ by (see Fig. 2(d)) : $f(e_n) = f(e_{n-1}) = 2, f(e_{n-2}) = 3$ and

$$\text{for } 1 \le i \le n-3, \qquad f(e_i) = \begin{cases} n+2-i & \text{if } i \text{ is odd} \\ 1 & \text{if } i \equiv 2 \pmod 4 \\ 2 & \text{if } i \equiv 0 \pmod 4. \end{cases}$$

This mapping induces the following gap vertex distinguishing labeling:

$l(v_{n-2}) = 2, l(v_{n-1}) = 1, l(v_n) = 0$ and

$$\text{for } 1 \le i \le n-3, \qquad l(v_i) = \begin{cases} n-i & \text{if } i \equiv 1 \pmod 4 \\ n+2-i & \text{if } i \equiv 2 \pmod 4 \\ n+1-i & \text{if } i \equiv 0, 3 \pmod 4. \end{cases}$$

Then, it is easy to check that l is a bijection from the vertex set of C_n to the set $\{0, 1, \ldots, n\} \setminus \{3\}$. Hence $\mathrm{gap}(C_n) = n+1$. □

We now introduce a definition which plays a pervasive role in this section.

Definition 2. Let G be a graph of order n and let f be an edge coloring of G. For every vertex v of G, we specify an interval $I(v) = [\min f(e)_{e \ni v}, \max f(e)_{e \ni v}]$. We say that f is *balanced* if $I(v_1) \cap I(v_2) \cap \cdots \cap I(v_n) \ne \emptyset$.

The following proposition summarizes an important property of our coloring parameter.

Proposition 12. *Let G be a graph with $\delta(G) \ge 2$. If there exists a spanning subgraph H of G with $\delta(H) \ge 2$ and there exists a gap vertex-distinguishing balanced edge coloring f of H with k colors, then* $\mathrm{gap}(G) \le k$.

Proof. Under the stated hypothesis, the gap vertex-distinguishing labeling of H is induced by a balanced edge coloring f with k colors. Therefore, there exists at least an integer j where $1 \le j \le k$ such that $\forall v \in V$, we have $j \in I(v)$. By coloring the edges of $G \setminus H$ with the color j, we obtain a gap-k-coloring of G. Hence $\mathrm{gap}(G) \le k$. □

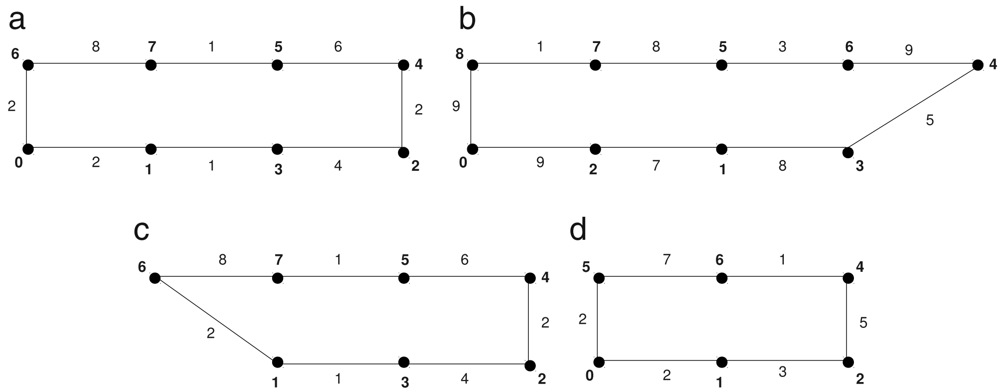

Fig. 2. A gap vertex-distinguishing edge colorings of C_n: (a) $n = 8$, (b) $n = 9$, (c) $n = 7$, (d) $n = 6$.

We illustrate the interest of Proposition 12 by considering the following example: let G be a Hamiltonian graph of order $n \equiv 0 (\text{mod } 4)$. In the proof of Theorem 11 (Subcase 1.1), it is easy to check that the proposed edge coloring of C_n is balanced. Indeed, for each vertex v in G, we have $2 \in I(v)$. Hence, we can extend the cycle C_n to G by weighting the added edges with color 2 without affecting the gap chromatic value of C_n. Thus, for every Hamiltonian graph G of order $n \equiv 0 (\text{mod } 4)$, we have $\text{gap}(G) = n$.

The following proposition is useful for proving Theorem 10. Furthermore, it provides a useful tool for proving other results.

Proposition 13. *If $G = (V, E)$ is an m-edge-connected graph of order n (with $m \geq 2$), different from a cycle of length $\equiv 1$, 2 or 3(mod 4), then for all integer $a \geq 0$, there exists an $(a + n)$-edge-coloring f which induces a gap vertex-distinguishing labeling $l : V \to \{a, a + 1, \ldots, a + n - 1\}$.*

Proof. The proof of this proposition is done by giving a polynomial-time algorithm. Let us begin with some definitions and notations. For every subset S of V, let N_S denote the set of neighboring vertices of S, not included in S.

$$N_S = \{u \in V \setminus S : \exists v \in S \text{ for which } (v, u) \in E\}.$$

For every two adjacent vertices u and v of G such that $v \in S$ and $u \in N_S$, let $P(v, u)$ be a function which returns a path (or cycle) from v to a vertex $w \in S$ that passes through u, such that the set of vertices between v and w does not belong to S.

Let f be an edge coloring of G. For every subgraph R of G, let $g(R)$ be a function defined on the set $E(R)$ as follows:

$$g(R) = \min\{f(e_i) : \forall e_i \in E(R), f(e_i) \neq 1, 2\}.$$

We denote by Q the set of all graphs that are isomorphic to a cycle of order multiple of 4 or to two cycles having at least one vertex in common.

Observation Every m-edge-connected graph G (with $m \geq 2$), different from a cycle of length $\equiv 1$, 2 or 3(mod 4) contains at least one subgraph $H \in Q$.

It is clear that if G is a 2-edge-connected graph, different from a cycle, then $\Delta(G) \geq 3$. Hence, the subgraph H can always be obtained from G. The basic idea of our algorithm is to find a balanced edge-coloring f of a 2-edge-connected spanning subgraph $G' = (V', E')$ of G. Initially, both sets V' and E' are empty set. During the algorithm, the updating of V' and E' is done gradually through a specific edge coloring procedure (which is explained in more detail below). When an edge of G is colored by this procedure it is inserted into E'. A vertex $v \in V$ is inserted into V' if and only if it is incident with at least two colored-edges ($e, s \in E$). Note that when a vertex v is inserted in V', we set the label $l(v)$ as $l(v) = |f(e) - f(s)|$ and the interval $I(v)$ at $[\min(f(e), f(s)), \max(f(e), f(s))]$. Such an edge coloring ensures that for every interval $I(v)$, we have $2 \in I(v)$.

In more details, the proposed algorithm starts by coloring the edges of a subgraph $H \in Q$ of G of order k which induces a gap vertex-distinguishing labeling of H, where the vertices of H are labeled by distinct numbers ranging from $n + a - k$ to $n + a - 1$. We can easily establish this labeling structure for every subgraph H of G which is isomorphic to a member of Q. Then, we have proposed four edge-coloring functions to color the set of edges which constructs a cycle that has an unique vertex in V' or a path between two vertices of V'. This last step is iterated until all vertices are labeled (*i.e.*, $|V'| = |V|$).

In order to color the subgraph H, we need to define several edge-coloring functions. For a proper understanding of our algorithm, we are going to present the algorithm for a graph G which contains at least one cycle of length multiple of 4. Otherwise, all other edge-coloring functions of H are described in detail in Appendix. The different steps of the algorithm are illustrated in the example of Fig. 3, where $a = 12$.

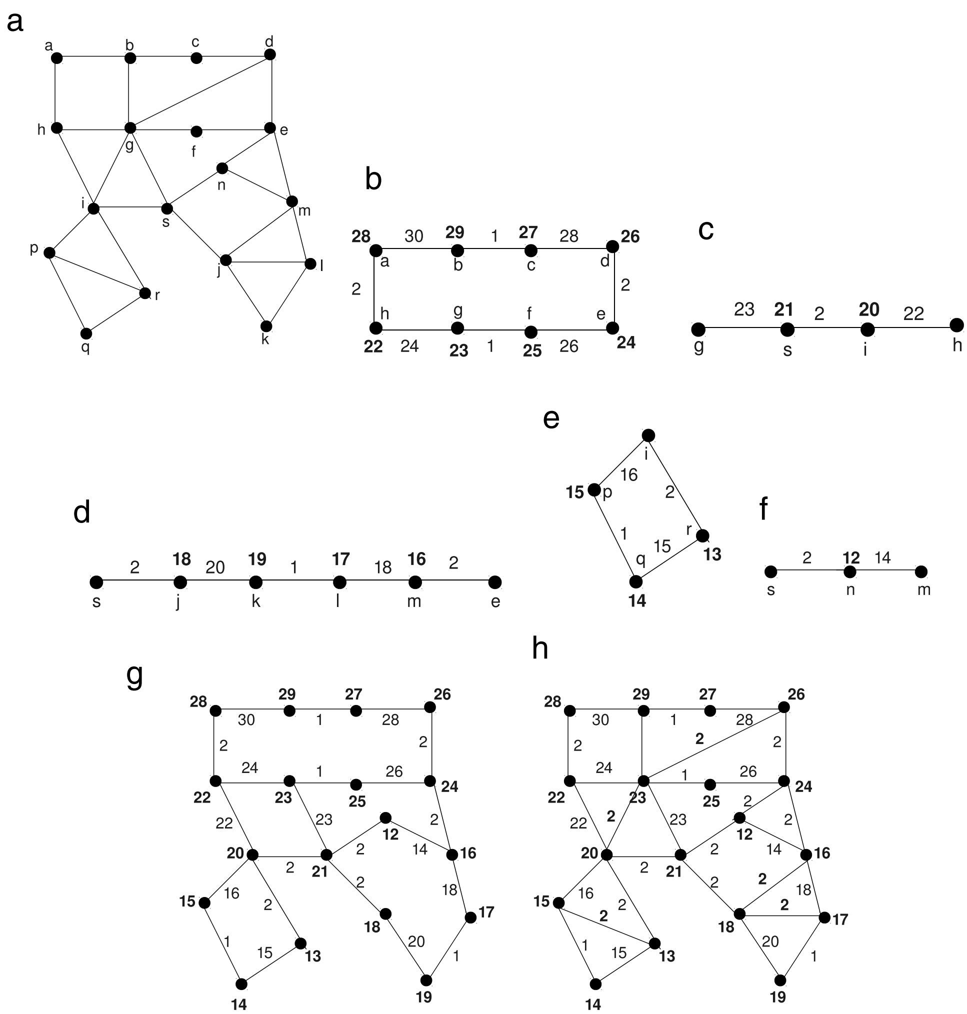

Fig. 3. Illustration of Algorithm 1 ($a = 12$): (a) A 2-edge-connected graph G. (b) Coloring of R_1. (c),(d),(e),(f) illustrate the coloring of R_2, R_3, R_4, R_5, respectively. (g) A balanced gap-30-coloring of a spanning subgraph G' of G. (h) A gap-30-coloring of G which induces a gap vertex-distinguishing function $l : V \to \{12, 13, \ldots, 29\}$.

Algorithm 1

Input: An integer $a \geq 0$ and an m-edge-connected graph $G = (V, E)$ of order n, such that $m \geq 2$ and G is not isomorphic to a cycle of length $\equiv 1, 2$ *or* $3(mod\ 4)$.

Output: A balanced $(a+n)$-edge-coloring f of G which induces a gap vertex-distinguishing function $l : V \to \{a, a+1, \ldots, a+n-1\}$.

Begin of Algorithm

Step 1: $V' \leftarrow \emptyset, E' \leftarrow \emptyset$. Let an index $t = 2$.

Step 2: Take any subgraph $H = R_1 \in Q$ of G.
2.1 If (R_1 is a cycle of length $k \equiv 0(mod\ 4)$) **Then**
Let $H = (v_1, v_2, \ldots, v_k, v_{k+1} = v_1)$. For each integer i with $1 \leq i \leq k$, let $e_i = v_i v_{i+1}$. A mapping f from $E(R_1)$ to $\{1, 2, \ldots, a+n\}$ is defined as follows:

$$\text{For } 1 \leq i \leq k, \quad f(e_i) = \begin{cases} n+a-i+1 & \text{if } i \text{ is odd} \\ 1 & \text{if } i \equiv 2(mod\ 4) \\ 2 & \text{if } i \equiv 0(mod\ 4) \end{cases}$$

This mapping induces the following vertex labeling of R_1:

$$\text{For } 1 \leq i \leq k, \quad l(v_i) = \begin{cases} n+a-i+1 & \text{if } i \equiv 2(mod\ 4) \\ n+a-i & \text{if } i \equiv 0, 3(mod\ 4) \\ n+a-i-1 & \text{if } i \equiv 1(mod\ 4) \end{cases}$$

Then, it is easy to check that l is a bijection from the vertex set of R_1 to the set $\{n+a-1, n+a-2, \ldots, n+a-k\}$.
Otherwise all other edge-coloring functions of R_1 are described in detail in the Appendix.
2.2 $V' \leftarrow V(R_1)$, $E' \leftarrow E(R_1)$ and set $z = g(R_1)$.

Step 3: While $(V' \neq V)$ **do**
Begin while
3.1 Take any two adjacent vertices u and v such that $v \in V'$ and $u \in N_{V'}$.
3.2 Let $R_t = P(v, u)$, we represent the obtained subgraph R_t by the walk $(v_1 = v, v_2 = u, \ldots, v_{k-1}, v_k)$. For each integer i with $1 \leq i \leq k-1$, let $e_i = v_i v_{i+1}$. We now define an edge coloring f of R_t. Four cases are considered according to the value of $k\ mod\ 4$.
Case 1: $k \equiv 0(mod\ 4)$. A mapping f from $E(R_t)$ to $\{1, 2, \ldots, a+n\}$ is defined as follows: $f(e_{k-1}) = z-k+2$ and

$$\text{For } 1 \leq i \leq k-2, \quad f(e_i) = \begin{cases} z-i & \text{if } i \text{ is odd} \\ 1 & \text{if } i \equiv 0(mod\ 4) \\ 2 & \text{if } i \equiv 2(mod\ 4) \end{cases}$$

This mapping induces the following gap vertex labeling of R_t: $l(v_{k-1}) = z-k$ and

$$\text{For } 2 \leq i \leq k-2, \quad l(v_i) = \begin{cases} z-i-1 & \text{if } i \equiv 1, 2(mod\ 4) \\ z-i-2 & \text{if } i \equiv 3(mod\ 4) \\ z-i & \text{if } i \equiv 0(mod\ 4) \end{cases}$$

Case 2: $k \equiv 2(mod\ 4)$. A mapping f from $E(R_t)$ to $\{1, 2, \ldots, a+n\}$ is defined as follows:

$$\text{For } 1 \leq i \leq k-1, \quad f(e_i) = \begin{cases} z-i & \text{if } i \text{ is even} \\ 1 & \text{if } i \equiv 3(mod\ 4) \\ 2 & \text{if } i \equiv 1(mod\ 4) \end{cases}$$

This mapping induces the following gap vertex labeling of R_t.

$$\text{For } 2 \leq i \leq k-1, \quad l(v_i) = \begin{cases} z-i-1 & \text{if } i \equiv 0, 1(mod\ 4) \\ z-i-2 & \text{if } i \equiv 2(mod\ 4) \\ z-i & \text{if } i \equiv 3(mod\ 4) \end{cases}$$

Case 3: $k \equiv 1(mod\ 4)$. A mapping f from $E(R_t)$ to $\{1, 2, \ldots, a+n\}$ is defined as follows: $f(e_1) = z-2$ and

$$\text{For } 2 \leq i \leq k-1, \quad f(e_i) = \begin{cases} z-i & \text{if } i \text{ is odd} \\ 1 & \text{if } i \equiv 2(mod\ 4) \\ 2 & \text{if } i \equiv 0(mod\ 4) \end{cases}$$

This mapping induces the following gap vertex labeling of R_t: $l(v_2) = z-3$ and

$$\text{For } 3 \leq i \leq k-1, \quad l(v_i) = \begin{cases} z-i-1 & \text{if } i \equiv 0, 3(mod\ 4) \\ z-i-2 & \text{if } i \equiv 1(mod\ 4) \\ z-i & \text{if } i \equiv 2(mod\ 4) \end{cases}$$

Case 4: $k \equiv 3(mod\ 4)$. A mapping f from $E(R_t)$ to $\{1, 2, \ldots, a+n\}$ is defined as follows: $f(e_{k-1}) = z-k+2$ and

$$\text{For } 1 \leq i \leq k-2, \quad f(e_i) = \begin{cases} z-i & \text{if } i \text{ is even} \\ 1 & \text{if } i \equiv 3(mod\ 4) \\ 2 & \text{if } i \equiv 1(mod\ 4) \end{cases}$$

This mapping induces the following gap vertex labeling of R_t: $l(v_{k-1}) = z-k$, and

$$\text{For } 2 \leq i \leq k-2, \quad l(v_i) = \begin{cases} z-i-1 & \text{if } i \equiv 0, 1(mod\ 4) \\ z-i-2 & \text{if } i \equiv 2(mod\ 4) \\ z-i & \text{if } i \equiv 3(mod\ 4) \end{cases}$$

Observation: In the previous four cases, it is easy to check that l is a bijection from the vertex set $V(R_t) - \{v_1, v_k\}$ to $\{z-3, z-4, \ldots, z-k\}$.

2.3 $V' \leftarrow V' \cup V(R_t)$, $E' \leftarrow E' \cup E(R_t)$. Set $z = g(R_t)$ and $t = t+1$.
End while

Step 4: For all edges $e \in E \setminus E'$, set $f(e) = 2$.

End of algorithm.

We now present the proof of correctness of the above algorithm. We first show that this algorithm achieves its goal without blocking, *i.e.*, both actions in Step 3 (3.1 and 3.2) satisfy the following assertions:

$$\text{If } |V'| < |V| \quad \text{then } N_{V'} \neq \emptyset. \tag{1}$$

$$\text{For every vertex } u \in N_{V'} \text{there exists a path from } u \text{ to a vertex } v \in V' \text{ of order at least 2.} \tag{2}$$

The assertion (1) follows from the connectivity hypothesis on G. For a vertex $u \in N_{V'}$ there exists, at last, an edge $(u, v) \in E$ such that $v \in V'$. The 2-edge-connectivity hypothesis of G implies that every edge of G belongs to a cycle, then the two vertices u and v belong to the same cycle. Therefore, the assertion (2) also holds.

We now prove that our coloring algorithm gives a gap vertex-distinguishing function $l : V' \to \{a, a+1, \ldots, a+n-1\}$ of G' induced by a balanced edge coloring f with $a+n$ colors. At the end of the loop of Step 3, we obtain a bijection l from the set V' to the set $\{a, a+1, \ldots, a+n-1\}$, *i.e.*, for any two vertices u, v of V', we have $l(u) \neq l(v)$. It then remains to show that f is a balanced edge-coloring and for every vertex v of V', we have $l(v)$ equal to $\max_{e \ni v} f(e) - \min_{e \ni v} f(e)$ in G'. By considering the degree in G' of each vertex v, we have two cases.

Case 1. $d(v) = 2$: from the algorithm, it is clear that the label of each vertex v of degree 2 which is incident with two edges e and s of E' is equal to $|f(e) - f(s)|$.

Case 2. $d(v) > 2$: let $R(v) = \{R_{d_1}, R_{d_2}, \ldots, R_{d_p}\}$ denote the set of all subgraphs having a common vertex v, where $d_1 \leq d_2 \leq \cdots \leq d_p$. From the algorithm, we can observe that (see Fig. 3(g)):

- for any two subgraphs R_i and R_j of $R(v)$, we have $E(R_i) \cap E(R_j) = \emptyset$.
- v is incident with exactly two edges e_{d_1} and s_{d_1} of $E(R_{d_1})$. Let $f(e_{d_1}) \geq f(s_{d_1})$; then the label of v is fixed as $l(v) = f(e_{d_1}) - f(s_{d_1})$.
- for every subgraph R_i of $R(v)$, where $i \geq d_2$, we have v is incident with one or two edges of $E(R_i)$.

Furthermore, according to the edge coloring f, we can easily see that:

- for every vertex v of G', we have $2 \in I(v)$;
- $1 \leq f(s_{d_1}) \leq 2$ and $f(e_{d_1}) \geq g(R_{d_1}) \geq 2$;
- for every subgraph R_i of $R(v)$, where $i \geq d_2$ then $\forall e \in E(R_i)$ with $v \in e$, we have $2 \leq f(e) \leq g(R_{d_1})$.

From these observations we can conclude the following:

- the edge-coloring f is balanced;
- for every vertex v of V', $\max_{e \ni v} f(e) = f(e_{d_1})$ and $\min_{e \ni v} f(e) = f(s_{d_1})$.

At Step 4 of the algorithm, we know that the obtained edge coloring f of G' is balanced. Hence, we can extend G' to G by coloring the added edges with color 2 without affecting the vertex labeling function $l : V \to \{a, a+1, \ldots, a+n-1\}$. □

Now, we can state the proof of Theorem 10. To proceed, we introduce the following result.

Theorem 14. *For every m-edge-connected graph G of order n (with $m \geq 2$), different from a cycle of length $n \equiv 2$ or $3 \pmod 4$, we have*

$$\mathrm{gap}(G) = n.$$

Proof. By Lemma 2, we have $\mathrm{gap}(G) \geq n$. It then suffices to prove that G admits a gap-n-coloring. We know by Theorem 11 that if G is a cycle of length $n \equiv 0, 1 \pmod 4$, then $\mathrm{gap}(G) = n$. Otherwise, it is clear by Proposition 13 that if we set the integer parameter a at 0, we obtain a gap-n-coloring of G induced by a balanced edge coloring. Hence $\mathrm{gap}(G) = n$. □

We can now conclude that the result of Theorem 10 is a direct consequence of Theorems 11 and 14.
Here we generalize the previous results to a special case of disconnected graphs as follows.

Theorem 15. *If G is a graph of order n with connected components $G_1, \ldots, G_t$ such that each component of G is an m-edge connected graph (with $m \geq 2$), different from a cycle of length $\equiv 1, 2, 3 \pmod 4$, then*

$$\mathrm{gap}(G) = n.$$

Proof. Let n_i be the order of G_i $(1 \le i \le t)$. The proof is essentially due to Proposition 13. The idea is to provide a gap vertex distinguishing edge coloring for each component of G_i according to the parameter a of Proposition 13 as follows: by applying this proposition in sequence to $G_1, G_2, \ldots, G_t$, we can obtain the labeling function $l : V(G_i) \to \{a, a+1, \ldots, a+n_i-1\}$ induced by an edge coloring f with $a+n_i$ colors such that $a = n - \sum_{j=1}^{i} n_j$. From this, it is easy to check that l is a bijection from the vertex set of G to the set $\{0, 1, 2, \ldots, n-1\}$. Thus $\mathrm{gap}(G) = n$. □

We believe that the result of Theorem 10 can be extended to all graphs of minimum degree at least 2. But we have not been able to prove it. We suggest the following conjecture.

Conjecture 16. *For every connected graph G of order n with minimum degree $\delta(G) \ge 2$, we have*

$$\mathrm{gap}(G) = \begin{cases} n+1 & \text{if } G \text{ is a cycle of length} \equiv 2, 3 \pmod 4 \\ n & \text{otherwise.} \end{cases}$$

4. Graphs with $\delta(G) = 1$

In this section, we give the value of $\mathrm{gap}(G)$ for some classes of graphs having $\delta(G) = 1$.

Theorem 17. *Let P_n be the path of order n. Then*

$$\mathrm{gap}(P_n) = \begin{cases} n-1 & \text{if } n \equiv 0, 1 \pmod 4 \\ n & \text{otherwise.} \end{cases}$$

Proof. The proof of this theorem is similar to the one of Theorem 11. Let $P_n = (v_1, v_2, \ldots, v_n)$. For each integer i with $1 \le i \le n-1$, let $e_i = v_i v_{i+1}$. We consider two cases as follows:
Case 1: $n \equiv 0, 1 \pmod 4$. By Lemma 2, we have $\mathrm{gap}(P_n) \ge n-1$, it then suffices to prove that P_n admits a gap-$(n-1)$-coloring. Two subcases are considered:
Subcase 1.1: $n \equiv 0 \pmod 4$. A mapping f from $E(P_n)$ to $\{1, 2, \ldots, n-1\}$ is defined as follows (see Fig. 4(a)).

$$\text{For } 1 \le i \le n-1, \qquad f(e_i) = \begin{cases} \dfrac{i}{2} & \text{if } i \text{ even} \\ \dfrac{n-2}{2} & \text{if } i \equiv 3 \pmod 4 \\ n-1 & \text{if } i \equiv 1 \pmod 4. \end{cases}$$

This mapping induces the following vertex labeling function: $l(v_n) = \frac{n-2}{2}$

$$\text{and for } 1 \le i \le n-1, \qquad l(v_i) = \begin{cases} \dfrac{n-i-2}{2} & \text{if } i \equiv 0 \pmod 4 \\ n-1-\dfrac{i-1}{2} & \text{if } i \equiv 1 \pmod 4 \\ n-1-\dfrac{i}{2} & \text{if } i \equiv 2 \pmod 4 \\ \dfrac{n-i-1}{2} & \text{if } i \equiv 3 \pmod 4. \end{cases}$$

Then, it is easy to check that l is a bijection from $V(P_n)$ to $\{0, 1, \ldots, n-1\}$. Hence $\mathrm{gap}(P_n) = n-1$.
Subcase 1.2: $n \equiv 1 \pmod 4$. A mapping f from $E(P_n)$ to $\{1, 2, \ldots, n-1\}$ is defined as follows (see Fig. 4(b)).

$$\text{For } 1 \le i \le n-1, \qquad f(e_i) = \begin{cases} \dfrac{i}{2} & \text{if } i \text{ even} \\ \dfrac{n-1}{2} & \text{if } i \equiv 3 \pmod 4 \\ n-1 & \text{if } i \equiv 1 \pmod 4. \end{cases}$$

This mapping induces the following vertex labeling function:

$$\text{and for } 1 \le i \le n-1, \qquad l(v_i) = \begin{cases} \dfrac{n-1-i}{2} & \text{if } i \equiv 0 \pmod 4 \\ n-1-\dfrac{i-1}{2} & \text{if } i \equiv 1 \pmod 4 \\ n-1-\dfrac{i}{2} & \text{if } i \equiv 2 \pmod 4 \\ \dfrac{n-i}{2} & \text{if } i \equiv 3 \pmod 4. \end{cases}$$

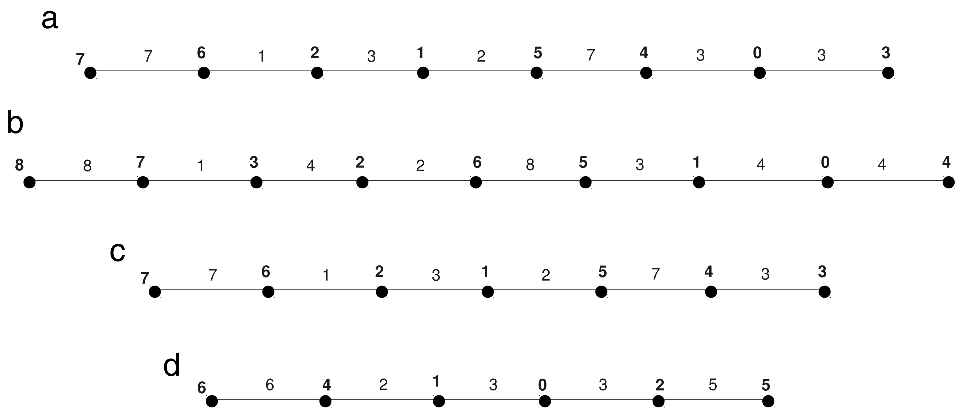

Fig. 4. A gap-coloring of P_n: (a) $n = 8$, (b) $n = 9$, (c) $n = 7$, (d) $n = 6$.

Then, it is easy to check that l is a bijection from $V(P_n)$ to $\{0, 1, \ldots, n-1\}$. Hence $\text{gap}(P_n) = n - 1$.

Case 2: $n \equiv 2, 3 (\text{mod } 4)$. We first prove that $\text{gap}(P_n) > n-1$. Let $f : V(P_n) \longrightarrow \{1, 2, \ldots, n-1\}$ be any edge-coloring of P_n which induces a gap vertex-distinguishing labeling l. We note that:

$$\sum_{i=1}^{n} l(v_i) = f(e_1) + f(e_{n-1}) + \sum_{i=2}^{n-1} |f(e_i) - f(e_{i-1})| = \frac{n(n-1)}{2}.$$

In this formula, each term $f(e_i)$ appears twice with opposite (or same) signs; hence $\frac{n(n-1)}{2}$ is even. But this latter value is odd if $n \equiv 2, 3 (\text{mod } 4)$, which is a contradiction. Thus, $\text{gap}(P_n) \geq n$. It then remains to show that $\text{gap}(P_n) \leq n$. Two subcases are considered according to whether $n \bmod 4 = 2$ or 3.

Subcase 2.1: $n \equiv 3 (\text{mod } 4)$. We know that P_{n+1} admits a gap-n-coloring. Necessarily P_{n+1} must contain two successive edges of same color j where $1 \leq j \leq n$. By merging these two edges into a single edge colored by j, we obtain a gap-n-coloring of P_n (see Fig. 4(c)).

Subcase 2.2: $n \equiv 2 (\text{mod } 4)$. In this subcase, we define an edge coloring f from $E(P_n)$ to $\{1, 2, \ldots, n\}$ (see Fig. 4(d)) by $f(e_{n-1}) = n - 1$ and

$$\text{for } 1 \leq i \leq n-2, \qquad f(e_i) = \begin{cases} \frac{i}{2} + 1 & \text{if } i \text{ even} \\ \frac{n}{2} & \text{if } i \equiv 3 (\text{mod } 4) \\ n & \text{if } i \equiv 1 (\text{mod } 4). \end{cases}$$

This mapping induces the following gap vertex distinguishing labeling: $l(v_1) = n$, $l(v_{n-1}) = \frac{n}{2} - 1$, $l(v_n) = n - 1$ and

$$\text{for } 2 \leq i \leq n-2, \qquad l(v_i) = \begin{cases} \frac{n-i}{2} - 1 & \text{if } i \equiv 0 (\text{mod } 4) \\ n - \frac{i+1}{2} & \text{if } i \equiv 1 (\text{mod } 4) \\ n - \frac{i}{2} - 1 & \text{if } i \equiv 2 (\text{mod } 4) \\ \frac{n-i-1}{2} & \text{if } i \equiv 3 (\text{mod } 4). \end{cases}$$

Then, it is easy to check that l is a bijection from $V(P_n)$ to $\{0, 1, \ldots, n\} \setminus \{\frac{n}{2}\}$. Hence $\text{gap}(P_n) = n$. $\square$

The complete binary tree of height $h > 0$ will be denoted by BT_h, note that BT_h has exactly $2^{h+1} - 1$ vertices. The following theorem gives the gap chromatic number of BT_h.

Theorem 18. *For any complete binary tree BT_h of order n and height $h \geq 2$, we have*

$$\text{gap}(BT_h) = n - 1.$$

Proof. By Theorem 17, we have $\text{gap}(BT_1) = \text{gap}(P_3) = 3$. Then, we may restrict our attention to $h \geq 2$. By Lemma 2, we have $\text{gap}(BT_h) \geq n - 1$, it then suffices to prove that BT_h admits a gap-$(n-1)$-coloring. We define the level $l(u)$ of vertex

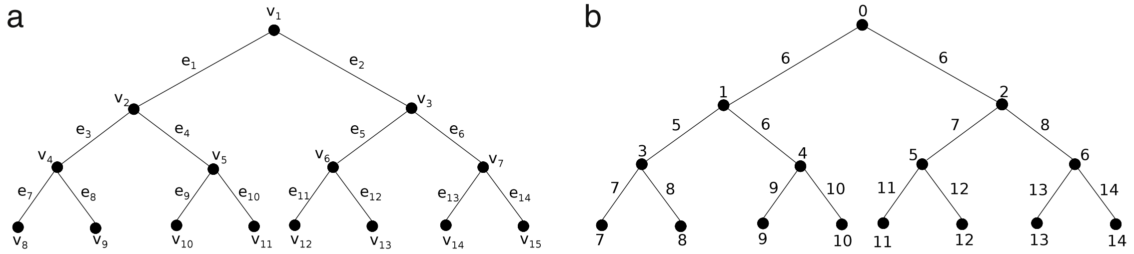

Fig. 5. (a) Notation of BT_3, (b) A gap-14-coloring of BT_3.

u of BT_h as the number of edges along the unique path between it and the root. Similarly, the level of an edge $e = (u, v)$ of BT_h is $l(e) = \max\{l(u), l(v)\}$. We represent the vertices and the edges of BT_h, level by level, left to right by the sequence $v_1, v_2, \ldots, v_n$ and $e_1, e_2, \ldots, e_{n-1}$, respectively (see Fig. 5(a)). We now define a mapping f from $E(BT_h)$ to $\{1, 2, \ldots, n-1\}$ as follows.

$$\text{For } 1 \le i \le n-1, \qquad f(e_i) = \begin{cases} 2h & \text{if } i \le 2 \\ i + 2(h - l(e_i)) & \text{if } i \ge 3. \end{cases}$$

This mapping induces the following gap vertex labeling: $l(v_i) = i - 1$ for $1 \le i \le n$. Then it is easy to check that l is a bijection from $V(BT_h)$ to $\{0, 1, \ldots, n-1\}$. Thus $\text{gap}(BT_h) = n - 1$. □

Theorem 19. *Let $T = (V, E)$ be a tree of order n which has at least two leaves u and v at distance two, then*

$$\text{gap}(T) \le n.$$

Proof. The proof of this theorem is done by giving a polynomial-time algorithm. We first start with some definitions used in the following. Let $R_1 = (u, w, v)$ be a path of T and let R be the subtree of T rooted in w and induced by the set $V \setminus \{u, v\}$ (see Fig. 6(a)). Let h be the depth of R. For every level i of R, let L_i denote the set of leaves at level i. Let S be a subset of $V(R)$ and for every vertex x of $V(R) \setminus S$, let $P(x, S)$ be the function which returns a path from x to a vertex $y \in S$, such that the set of vertices between x and y does not belong to S.

Let l be a vertex labeling on $V(T)$. For every path P of T, let $g(P)$ be a function defined as follows:

$$g(P) = \min\{l(v) : \forall v \in V(P)\}.$$

The different steps of Algorithm 2 are illustrated in the example of Fig. 6.

Algorithm 2
Input: A tree $T = (V, E)$ of order n with two leaves u and v at distance 2.
Output: A gap-n-coloring of T.
Begin of Algorithm
Set a mapping $f : E(R_1) \to \{1, n\}$ as follows: $f(vw) = n, f(uw) = 1$.
This mapping induces the following gap vertex labeling of R_1: $l(v) = n$, $l(w) = n - 1$ and $l(u) = 1$.
Let a set $S = \{w\}$, an integer $z = n - 1$ and an index $t = 2$.
For $i = 1$ to h **do**
Begin For
For every vertex x of L_i in the subtree R **do**
Begin For
Let $R_t = P(x, S)$. We denote the path R_t by the sequence of vertices $v_1 = x, v_2, \ldots, v_{k-1}, v_k$. For each integer i with $1 \le i \le k - 1$, let $e_i = v_i v_{i+1}$. Set an edge coloring f of R_t as follows:

$$\text{For } 1 \le i \le k-1, \quad f(e_i) = \begin{cases} z - \frac{i+1}{2} & \text{if } i \text{ odd} \\ \frac{i}{2} & \text{otherwise} \end{cases}$$

This mapping induces the following gap vertex labeling of R_t.

$$\text{For } 1 \le i \le k-1, l(v_i) = z - i$$

$S \leftarrow S \cup V(R_t), z \leftarrow g(R_t), t \leftarrow t + 1.$
End for
End for
End of Algorithm

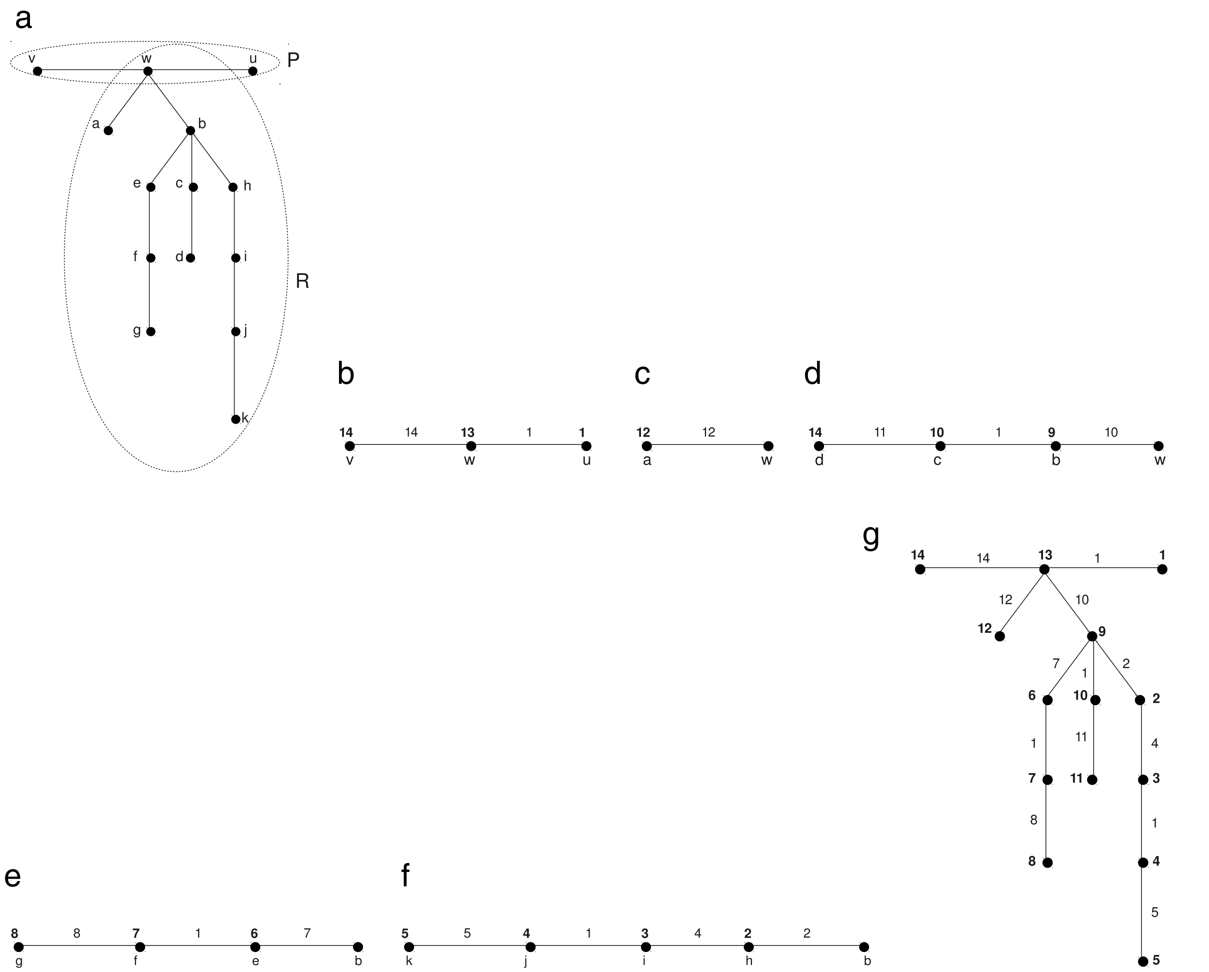

Fig. 6. Illustration of Algorithm 2: (a) A tree T. (b) Coloring of R_1. (c),(d),(e),(f) illustrate the coloring of R_2, R_3, R_4, R_5, respectively. (g) A gap-14-coloring of T.

Now, we present the proof of correctness for the above algorithm. At the end of this algorithm, we obtain a bijection l from V to the set $\{1, 2, \ldots, n\}$. It then remains to show the property of our coloring parameter. By considering the degree of each vertex v of T, we have three cases:

Case 1. $d(v) = 1$: from the algorithm, it is clear that $l(v) = f(e)_{e \ni v}$.

Case 2. $d(v) = 2$: from the algorithm, it is clear that the label of vertex v of degree 2 which is incident with two edges e and s of E equal to $|f(e) - f(s)|$.

Case 3. $d(v) > 2$: let $R(v) = \{R_{d_1}, R_{d_2}, \ldots, R_{d_p}\}$ denote the set of all paths having a common vertex v, where $d_1 \leq d_2 \leq \cdots \leq d_p$. We represent the distance between two vertices $x, y \in V$ by $\mathrm{dist}(x, y)$. From the algorithm, we can observe the following.

- Every path R_i of $R(v)$ contains a leaf l_i of T which is an endpoint of R_i. We can see that $\mathrm{dist}(v, l_{d_1}) \leq \mathrm{dist}(v, l_{d_2}) \leq \cdots \leq \mathrm{dist}(v, l_{d_p})$.
- For any two paths R_i and R_j of $R(v)$, $E(R_i) \cap E(R_j) = \emptyset$.
- The vertex v is incident with exactly two edges e_{d_1} and s_{d_1} of $E(R_{d_1})$. Let $f(e_{d_1}) \geq f(s_{d_1})$; then the label of v is fixed as $l(v) = f(e_{d_1}) - f(s_{d_1})$.
- For every path R_i of $R(v)$, where $i \geq d_2$, the vertex v is incident to exactly one edge e_i of $E(R_i)$.

Furthermore, according to the edge-coloring f, we can see the following.

- $f(s_{d_1}) = \lceil \frac{\mathrm{dist}(v, l_{d_1})}{2} \rceil$.
- For every path R_i of $R(v)$, where $i \geq d_2$, we consider two cases for the value of $f(e_i)$ (with $v \in e_i$) according to the distance between v and l_i:

- $\mathrm{dist}(v, l_i)$ *is even*. We have $f(e_i) = \frac{\mathrm{dist}(v,l_i)}{2}$. Hence $f(s_{d_1}) \le f(e_i) \le g(R_{d_1}) \le f(e_{d_1})$.
- $\mathrm{dist}(v, l_i)$ *is odd*. We have $f(e_i) = g(R_i) + \frac{\mathrm{dist}(v,l_i)-1}{2}$. Hence $f(s_{d_1}) \le f(e_i) \le g(R_{d_1}) \le f(e_{d_1})$.

From these observations, we can conclude that for every vertex v of V, $f(e_{d_1}) = \max_{e \ni v} f(e)$ and $f(s_{d_1}) = \min_{e \ni v} f(e)$. Hence, T admits a gap-n-coloring. □

5. Concluding remarks and open problems

In this paper, we studied a new variant of graph edge colorings that induces a vertex distinguishing labeling. Exact results are given for paths, cycles, some trees and all m-edge-connected graphs with $m \ge 2$. The study of the relationships between our parameter and the point distinguishing problem gives the following result which is a direct consequence of Lemma 8 and Theorem 15.

Corollary 20. *If G is a graph of order n with connected components $G_1, \ldots, G_t$ such that each component of G is an m-edge connected graph (with $m \ge 2$) different from a cycle of length $\equiv 1, 2, 3 \pmod 4$, then $\chi'_0(G) \le n$.*

We would like to end this paper by mentioning three further issues.

(1) We leave as an open question to show that the gap chromatic number of a graph of order n is always in $\{n-1, n, n+1\}$.
(2) The computational complexity of the gap chromatic number is still an open problem (this is the case of the most variants of vertex distinguishing problems derived from an improper edge coloring).
(3) As for the other distinguishing parameters, it would be interesting to consider the variant of the gap coloring problem that distinguishes the adjacent vertices only.

Appendix. Step 2 of Algorithm 1

In Step 2 of Algorithm 1, it remains to handle the case where R_1 is a subgraph of G which is isomorphic to two cycles having at least one vertex in common. Let us recall that the goal is to define an edge coloring of R_1 (of order k) which induces the following gap vertex-distinguishing function $l : V(R_1) \to \{n+a-1, n+a-2, \ldots, n+a-k\}$ such that $\forall \in v$, we have $2 \in I(v)$.

It is clear that the edge set of R_1 can be partitioned into two sets generating a cycle C and a path (cycle) P such that the endpoints of P belong to C. Let $C = (v_1, v_2, \ldots, v_q, v_{q+1} = v_1)$. For each integer i with $1 \le i \le q$, let $e_i = v_i v_{i+1}$. Let $P = (u_1, u_2, \ldots, u_t)$. For each integer i with $1 \le i \le t-1$, let $s_i = u_i u_{i+1}$, we assume that $v_q = u_1$. In the following, we illustrate the coloring of R_1, several cases are considered according to the value of q and t.

Case 1: $q \equiv 1 \pmod 4$ and $t \equiv 1, 2 \pmod 4$. A mapping f of $E(R_1)$ is defined as follows: $f(e_q) = 2$ and

$$\text{For } 1 \le i \le q-1, \qquad f(e_i) = \begin{cases} a+n-i+1 & \text{if } i \text{ is odd} \\ 1 & \text{if } i \equiv 2 \pmod 4 \\ 2 & \text{if } i \equiv 0 \pmod 4. \end{cases}$$

Then, the following cases define the coloring of the remaining edges of R_1.

Subcase 1.1: $t \equiv 1 \pmod 4$.

$$\text{For } 1 \le i \le t-1, \qquad f(s_i) = \begin{cases} g(C)-i-1 & \text{if } i \text{ is odd} \\ 1 & \text{if } i \equiv 2 \pmod 4 \\ 2 & \text{if } i \equiv 0 \pmod 4. \end{cases}$$

Subcase 1.2: $t \equiv 2 \pmod 4$. We use the same coloring scheme as in Subcase 1.1 except that $f(s_{t-1}) = g(C) - t + 1$.

Case 2: $q \equiv 1 \pmod 4$ and $t \equiv 0, 3 \pmod 4$. A mapping f of $E(R_1)$ is defined as follows: $f(e_q) = 1$ and

$$\text{For } 1 \le i \le q-1, \qquad f(e_i) = \begin{cases} a+n-i+1 & \text{if } i \text{ is odd} \\ 2 & \text{if } i \equiv 2 \pmod 4 \\ 1 & \text{if } i \equiv 0 \pmod 4. \end{cases}$$

Then, the following cases define the coloring of the remaining edges of R_1.

Subcase 2.1: $t \equiv 3 \pmod 4$.

$$\text{For } 1 \le i \le t-1, \qquad f(s_i) = \begin{cases} g(C)-i-1 & \text{if } i \text{ is odd} \\ 2 & \text{if } i \equiv 2 \pmod 4 \\ 1 & \text{if } i \equiv 0 \pmod 4. \end{cases}$$

Subcase 2.2: $t \equiv 0 \pmod 4$. We use the same coloring scheme as in Subcase 2.1 except that $f(s_{t-1}) = g(C) - t + 1$.

Case 3: $q \equiv 2(\text{mod } 4)$ and $t \equiv 0, 1, 2(\text{mod } 4)$. A mapping f of $E(C) \setminus \{e_q, e_{q-1}\}$ is defined as follows:

$$\text{For } 1 \le i \le q-2, \qquad f(e_i) = \begin{cases} a+n-i+1 & \text{if } i \text{ is odd} \\ 1 & \text{if } i \equiv 2(\text{mod } 4) \\ 2 & \text{if } i \equiv 0(\text{mod } 4). \end{cases}$$

Then, the following cases define the coloring of the remaining edges of R_1.

Subcase 3.1: $t \equiv 0(\text{mod } 4)$. $f(e_q) = 2, f(e_{q-1}) = n+a-q+2$ and

$$\text{For } 1 \le i \le t-1, \qquad f(s_i) = \begin{cases} g(C)-i & \text{if } i \text{ is even} \\ 1 & \text{if } i \equiv 1(\text{mod } 4) \\ 2 & \text{if } i \equiv 3(\text{mod } 4). \end{cases}$$

Subcase 3.2: $t \equiv 1(\text{mod } 4)$. $f(e_q) = 2, f(e_{q-1}) = a+n-q+1, f(s_1) = g(C)+1$ and

$$\text{For } 2 \le i \le t-1, \qquad f(s_i) = \begin{cases} g(C)-i+1 & \text{if } i \text{ is odd} \\ 1 & \text{if } i \equiv 2(\text{mod } 4) \\ 2 & \text{if } i \equiv 0(\text{mod } 4). \end{cases}$$

Subcase 3.3: $t \equiv 2(\text{mod } 4)$. We use the same coloring scheme as in Subcase 3.2 except that $f(s_{t-1}) = g(C)-t+3$.

Case 4: $q \equiv 2(\text{mod } 4)$ and $t \equiv 3(\text{mod } 4)$. A mapping f of $E(R_1)$ is defined as follows:

$$\text{For } 2 \le i \le q-2, \qquad f(e_i) = \begin{cases} a+n-i-2 & \text{if } i \text{ is odd} \\ 2 & \text{if } i \equiv 2(\text{mod } 4) \\ 1 & \text{if } i \equiv 0(\text{mod } 4) \end{cases}$$

$f(e_1) = a+n, f(e_{q-1}) = a+n-4, f(e_q) = 1, f(s_1) = a+n-2$ and

$$\text{For } 2 \le i \le t-1, \qquad f(s_i) = \begin{cases} g(C)-i+1 & \text{if } i \text{ is odd} \\ 1 & \text{if } i \equiv 0(\text{mod } 4) \\ 2 & \text{if } i \equiv 2(\text{mod } 4). \end{cases}$$

Case 5: $q \equiv 3(\text{mod } 4)$ and $t \equiv 0(\text{mod } 4)$. A mapping f of $E(R_1)$ is defined as follows:

$$\text{For } 1 \le i \le q-1, \qquad f(e_i) = \begin{cases} a+n-i+1 & \text{if } i \text{ is odd} \\ 2 & \text{if } i \equiv 2(\text{mod } 4) \\ 1 & \text{if } i \equiv 0(\text{mod } 4). \end{cases}$$

$f(e_q) = 1, f(s_{t-1}) = g(C)-t+1$ and

$$\text{For } 1 \le i \le t-2, \qquad f(s_i) = \begin{cases} g(C)-i-1 & \text{if } i \text{ is odd} \\ 1 & \text{if } i \equiv 0(\text{mod } 4) \\ 2 & \text{if } i \equiv 2(\text{mod } 4). \end{cases}$$

Case 6: $q \equiv 3(\text{mod } 4)$ and $t \equiv 1(\text{mod } 4)$. A mapping f of $E(R_1)$ is defined as follows:

$$\text{For } 1 \le i \le q-1, \qquad f(e_i) = \begin{cases} a+n-i+2 & \text{if } i \text{ is even} \\ 2 & \text{if } i \equiv 1(\text{mod } 4) \\ 1 & \text{if } i \equiv 3(\text{mod } 4) \end{cases}$$

$f(e_q) = a+n-q+2, f(s_{t-1}) = g(C)-t+2$ and

$$\text{For } 1 \le i \le t-2, \qquad f(s_i) = \begin{cases} g(C)-i & \text{if } i \text{ is even} \\ 1 & \text{if } i \equiv 1(\text{mod } 4) \\ 2 & \text{if } i \equiv 3(\text{mod } 4). \end{cases}$$

Case 7: $q \equiv 3(\text{mod } 4)$ and $t \equiv 2, 3(\text{mod } 4)$. A mapping f of $E(C)$ is defined as follows: $f(e_q) = n+a-q+1$ and

$$\text{For } 1 \le i \le q-1, \qquad f(e_i) = \begin{cases} a+n-i+2 & \text{if } i \text{ is even} \\ 1 & \text{if } i \equiv 1(\text{mod } 4) \\ 2 & \text{if } i \equiv 3(\text{mod } 4). \end{cases}$$

Then, the following cases define the coloring of the remaining edges of R_1.

Subcase 7.1: $t \equiv 2(\text{mod } 4)$

$$\text{For } 1 \le i \le t-1, \qquad f(s_i) = \begin{cases} g(C)-i+1 & \text{if } i \text{ is even} \\ 1 & \text{if } i \equiv 3(\text{mod } 4) \\ 2 & \text{if } i \equiv 1(\text{mod } 4). \end{cases}$$

Subcase 7.2: $t \equiv 3(\text{mod } 4)$. We use the same coloring scheme as in Subcase 7.1 except that $f(s_{t-1}) = g(C)-t+3$.

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