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A General Solution of the Monge–Kantorovich Mass-Transfer Problem

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The solvability and the absence of a duality gap for the primal and the dual Monge–Kantorovich mass-transference programs for arbitrary Hausdorff topological spaces are established. © 1996 Academic Press, Inc.

Gaspar Monge gave for the first time in 1781 a mathematical formulation in terms of descriptive geometry of the ‘‘Cutting and Filling’’ problem which appeared in France at the end of the eighteenth century. Monge and his successors C. Dupin (1818), A. de St. Germain (1886), P. Appel (who won in 1885 the Bordin Price of the Academy of Sciences of Paris for a solution of this problem), and others studied this problem using techniques of descriptive and differential geometry very different from the actual techniques of modern Optimization Theory (see [27, 29]). Several years later L. V. Kantorovich (1942) posed the problem in terms of Measure Theory and this abstract approach is followed in the modern literature (see [1, 24, 25, 28]).

The Monge–Kantorovich mass-transference problem has attracted since its beginnings the attention of many specialists in various areas of mathematics such as differential geometry, functional analysis, infinite-dimen-

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sional linear programming, probability theory and mathematical statistics, information theory and cybernetics, statistical physics, theory of dynamical systems, and matrix theory. Moreover it has many applications in engineering, economy, logistics, theory of stochastic processes, and decision theory.

Along the years many results about the solvability of the primal and dual programs and about the absence of a duality gap for them have been stated under different assumptions about the spaces, the functions, and the measures. For instance, recently remarkable results have been obtained for completely regular spaces homeomorphic to some universally measurable subset of a compact space (certain functions and tight measures), polish spaces (and cost functions verifying certain subadditive condition), and compact spaces (continuous cost functions and (classical) Radon measures) (see [1, 17, 28]).

In [6] it is proved that the primal mass-transfer problem is solvable in the general situation of Hausdorff topological spaces, bounded lower semicontinuous cost functions, and Radon measures (in the sense of [30]). In the present paper it is established, also for arbitrary Hausdorff topological spaces and Radon measures (also in the sense of [30]), that the primal mass transfer problem is solvable for lower semicontinuous cost functions bounded below (Theorem 1), that the dual problem is solvable, and that there is no duality gap for the primal and the dual programs if the cost function is bounded and upper semicontinuous (Theorems 9 and 11), where the dual program is formulated in the set of the upper semicontinuous functions. In fact Theorem 8 states the absence of a duality gap under a weaker condition about the cost function c. Also in Section 4 a constructive proof is given of the solvability of the dual problem.

For bounded continuous cost functions it is proved in Theorem 12 that the restrictions $u|p_1(K)$ and $v|p_2(K)$ are continuous for every optimal solution (u, v) of the dual program, every compact subset $K \subset \operatorname{supp} \mu_0$, and every optimal solution μ_0 of the primal problem. If the Hausdorff topological spaces X and Y are uniformizable (in particular if they are compact), and the cost function is a bounded uniformly continuous one, then Proposition 13 states the existence of an optimal solution (u, v) of the dual program such that the functions u and v are uniformly continuous on X and Y, respectively.

In Section 3 some characterizations and properties of uniformly countably additive bounded subsets of measures are obtained and used to study the mass-transfer programs having an uniformly countably additive feasible set of the primal program. In this case it is proved that the primal program is also solvable for universally Lusin measurable cost functions bounded below, and that there is no duality gap for the primal and the dual programs also if the cost function is bounded and universally Lusin measurable. Example 22 shows that if $X = \mathbb{N}$ (or $Y = \mathbb{N}$) equipped with the discrete topology then the primal feasible set is uniformly countably additive.

1. PRELIMINARIES AND NOTATIONS

Let X and Y be two Hausdorff topological spaces and denote (following [30]) by $M_+^1(X)$ the space of all non-negative and finite Radon measures on X (and analogously for $Y, X \times Y, \dots$) equipped with the narrow topology, by p_1 and p_2 the natural projections from $X \times Y$ onto X and Y, respectively, and by $S_b(X)$ and $S_b(Y)$ the families of all bounded upper semicontinuous real functions defined on X and Y, respectively.

Consider from now on two Radon measures $\mu_1 \in M_+^1(X)$ and $\mu_2 \in M_+^1(Y)$ such that $\mu_1(X) = \mu_2(Y)$, a universally Borel measurable (i.e., μ-Borel measurable for every $\mu \in M_+^1(X \times Y)$) function c: $X \times Y \to \mathbb{R}$, and the following "primal" and "dual" mass transfer programs:

$$\left.\begin{aligned} &\min \int_{X\times Y} c\, d\mu \\ &\mu \in M_+^1(X \times Y) \\ &p_i(\mu) = \mu_i, \qquad i = 1, 2 \end{aligned}\right\} \tag{PP}$$

and

$$\begin{aligned} &\max \int_X u\, d\mu_1 + \int_Y v\, d\mu_2 \\ &(u, v) \in S_b(X) \times S_b(Y) \\ &u(x) + v(y) \le c(x, y) \\ &(x, y) \in X \times Y. \end{aligned} \tag{DP}$$

Henceforth $F(PP)$ and $F(DP)$ will denote as usual the feasible sets of the programs (PP) and (DP) (i.e., $F(PP) = \{\mu \in M_+^1(X \times Y)\colon\ p_i(\mu) = \mu_1,\ \ i = 1, 2\}$ and $F(DP) = \{(u, v) \in S_b(X) \times S_b(Y)\colon\ u(x) + v(y) \le c(x, y), (x, y) \in X \times Y\}$). Moreover, we will write occasionally $\mu(c)$, $\mu_1(u)$, and $\mu_2(v)$ instead of $\int_{X\times Y} c\, d\mu$, $\int_X u\, d\mu_1$, and $\int_Y v\, d\mu_2$, respectively, and $\mu_0 \in M_+^1(X \times Y)$ will denote throughout the paper an optimal solution of (PP). Also we will write $u + v \le c$ instead of "$u(x) + v(y) \le c(x, y)$ for every $(x, y) \in X \times Y$" and M will be $M = \sup\{|c(x, y)|\colon (x, y) \in X \times Y\}$. In what follows M will always be finite.

2. SOLVABILITY AND ABSENCE OF DUALITY GAP

Let us remark first that the primal problem (PP) is feasible since the product measure $\mu_1 \otimes \mu_2 \in M_+^1(X \times Y)$ exists and $(1/\mu_1(X))\mu_1 \otimes \mu_2 = (1/\mu_2(Y))\mu_1 \otimes \mu_2 \in F(PP)$ (clearly the case $\mu_1(X) = \mu_2(Y) = 0$ is trivial).

Moreover for every (PP)-feasible measure $\mu \in F(PP)$ and every (DP)-feasible couple $(u, v) \in F(DP)$ we have that

$$\int_X u\, d\mu_1 + \int_Y v\, d\mu_2 \leq \int_{X\times Y} c\, d\mu.$$

THEOREM 1. *If c is a lower semicontinuous function bounded below then either the primal problem (PP) is solvable or its value is $+\infty$.*

Proof. It follows from Proposition 2 and 3 of [30, pp. 371, 372] that $F(PP)$ is a closed subset of $M_+^1(X \times Y)$. Moreover, for every $\varepsilon > 0$ there exist two compact subsets $K_1 \subset X$ and $K_2 \subset Y$ such that $\mu_1(X - K_1) < \varepsilon/2$, $\mu_2(X - K_2) < \varepsilon/2$, and

$$\mu(X \times Y - K_1 \times K_2) \leq \mu_1(X - K_1) + \mu_2(X_2 - K_2) < \varepsilon$$

for every $\mu \in F(PP)$. Then it follows from Theorem 3 of [30, p. 379] that $F(PP)$ is a compact subset of $M_+^1(X \times Y)$ from where the result follows immediately since the operator $T_c(\mu) = \mu(c)$ $(\mu \in M_+^1(X \times Y))$ is lower semicontinuous on $M_+^1(X \times Y)$. Indeed, for every $\lambda \in \mathbb{R}$ and $\beta \in \{\mu \in M_+^1(X \times Y)\colon \mu(c) > \lambda\}$ it follows from the Lebesgue monotone convergence theorem that there exists $n \in \mathbb{N}$ such that $\beta(c_n) > \lambda$ where $c_n = \inf(c, n\chi_{X\times Y})$. Then c_n is a bounded lower semicontinuous function on $X \times Y$ and the mapping $\mu \to \mu(c_n)$ is lower semicontinuous on $M_+^1(X \times Y)$. Consequently there exists a neighborhood $V(\beta)$ of β such that $\beta \in V(\beta) \subset \{\mu \in M_+^1(X \times Y)\colon \mu(c_n) > \lambda\}$ and $\mu(c) \geq \mu(c_n) > \lambda$ for every $\mu \in V(\beta)$. Thus $V(\beta) \subset \{\mu \in M_+^1(X \times Y)\colon \mu(c) > \lambda\}$ and $\{\mu \in M_+^1(X \times Y)\colon \mu(c) > \lambda\}$ is an open subset of $M_+^1(X \times Y)$.

DEFINITION 2. A subset $H \subset M_+^1(X)$ is said to be uniformly inner regular with respect to the metrizable compact subsets (of X) if for every $\varepsilon > 0$ there exists a metrizable compact subset $K \subset X$ such that $\mu(X - K) < \varepsilon$ for every $\mu \in H$.

A measure $\mu \in M_+^1(X)$ is said to be a Radon measure of type $(\mathscr{K}_m)$ (see [12, 14]) if for every $\varepsilon > 0$ there exists a metrizable compact subset $K \subset X$ such that $\mu(X - K) < \varepsilon$.

We will say that $H \subset M_+^1(X)$ is bounded if $\{\mu(X)\colon \mu \in H\}$ is bounded.

PROPOSITION 3. *If a bounded subset $H \subset M_+^1(X)$ is uniformly inner regular with respect to the metrizable compact subsets (of X) then H is a metrizable relatively compact subset of $M_+^1(X)$.*

Proof. In fact there exists a sequence (K_n) of metrizable compact subsets of X such that $\mu(X - K_n) < 1/n$ for every $\mu \in H$ and every $n \in \mathbb{N}$. Then $L = \bigcup_n K_n$ is a Lusin space and therefore it follows from Theorem 7 of [30, p. 385] that $M_+^1(L)$ (equipped with the narrow topology, as always in this paper) is also a Lusin space. Hence by Theorem 3 and Corollary 2 of [30, pp. 379, 106] the adherence $\overline{H}_L$ of $H_L = \{\mu_L: \mu \in H\}$ (where μ_L denotes as usual the induced measure on L by μ), endowed with the narrow topology, is a metrizable compact space. Now the result follows immediately since $\mu(X - L) = 0$ for every $\mu \in H$.

PROPOSITION 4. *If μ_1 and μ_2 are Radon measures of type $(\mathscr{K}_m)$ then $F(PP)$ is a metrizable compact subset of $M_+^1(X \times Y)$.*

Proof. This follows immediately from Proposition 3 since $F(PP)$ is closed and in this case it is also uniformly inner regular with respect to the metrizable compact subsets of $X \times Y$ (see the proof of Theorem 1).

Remark 5. If c is a lower semicontinuous function bounded below, μ_1 and μ_2 are Radon measures of type $(\mathscr{K}_m)$, and a sequence $(\mu^n) \subset F(PP)$ verifies that

$$\lim_n \mu^n(c) = \inf\{\mu(c): \mu \in F(PP)\},$$

then it follows from Proposition 4 that there exists a subsequence (μ^{m_n}) convergent to some measure $\beta \in F(PP)$ and β is an optimal solution of (PP) since $\beta(c) = \inf\{\mu(c): \mu \in F(PP)\}$. Moreover, if the primal problem has a unique optimal solution then the initial sequence (μ^n) is convergent.

LEMMA 6. *Assume c to be bounded. Then the program obtained adding the constraints $|u| \leq M$ and $-2M \leq v \leq 0$ in (DP) is equivalent to the original one. In other words, (DP) is solvable if and only if the newly obtained program is solvable, and in this case both programs have the same optimal solutions.*

Proof. Let be $(u, v) \in F(DP)$. If $\alpha = \sup\{v(y): y \in Y\}$ then $(u', v') = (u + \alpha, v - \alpha) \in F(DP)$ and $\mu_1(u') + \mu_2(v') = \mu_1(u) + \mu_2(v)$. Moreover $\sup\{v'(y): y \in Y\} = 0$ and $u'(x) \leq M$ for every $x \in X$ since $u' + v' \leq c \leq M$. Considering now $u''(x) = \max\{u'(x), -M\}$ for every $x \in X$ and $v''(y) = \max\{v'(y), -2M\}$ for every $y \in Y$, we have that $(u'', v'') \in F(DP)$, $|u''| \leq M$, $-2M \leq v'' \leq 0$, and $\mu_1(u') + \mu_2(v') \leq \mu_1(u'') + \mu_2(v'')$, from where the result follows immediately.

LEMMA 7. *If X and Y are compact spaces and c is a bounded upper semicontinuous function on $X \times Y$ then (DP) is solvable and there is no duality gap for (PP) and (DP), i.e.,*

$$\sup\{ \mu_1(u) + \mu_2(v) : (u,v) \in F(DP)\} = \inf\{ \mu(c) : \mu \in F(PP)\}. \quad (1)$$

Proof. Let $(c_i)_{i \in I}$ be the set of all real continuous functions on $X \times Y$ such that $c \le c_i \le M$. Making $i \le j$ $(i, j \in I)$ when $c_j \le c_i$, $(I, \le)$ is a directed set. Therefore we have that $\lim_i c_i = c$ and $\lim_i \mu(c_i) = \mu(c)$ for every $\mu \in F(PP)$. It follows that $\lim_i m(c_i) = \inf_i m(c_i) = m(c)$, where $m(c_i) = \inf\{\mu(c_i): \mu \in F(PP)\}$ and $m(c) = \inf\{\mu(c): \mu \in F(PP)\}$.

Now it follows from Lemma 6 and [1, Theorem 5.2] that for every $i \in I$ there exists $u_i \in C(X)$ and $v_i \in C(Y)$ such that $u_i + v_i \le c_i$, $|u_i| \le M$, $-2M \le v_i \le 0$, and $\mu_1(u_i) + \mu_2(v_i) = m(c_i)$. Therefore,

$$\mu_1(u_i \chi_{B_1}) \mu_2(B_2) + \mu_1(B_1) \mu_2(v_i \chi_{B_2}) \le \int_{B_1 \times B_2} c_i \, d(\mu_1 \otimes \mu_2)$$

for all Borel subsets $B_1 \subset X$ and $B_2 \subset Y$.

Since it follows from the Alaoglu–Bourbaki theorem that $(u_i, v_i)_{i \in I}$ is a $\sigma(L^\infty(\mu_1) \times L^\infty(\mu_2), L^1(\mu_1) \times L^1(\mu_2))$-relatively compact subset of $L^1(\mu_1) \times L^1(\mu_2)$, it has an accumulation point (f, g) such that

$$\mu_1(f) + \mu_2(g) = \lim_i m(c_i) = m(c)$$

and

$$\mu_1(f\chi_{B_1}) \mu_2(B_2) + \mu_1(B_1) \mu_2(g \chi_{B_2}) \le \lim_i \int_{B_1 \times B_2} c_i \, d(\mu_1 \times \mu_2)$$

$$= \int_{B_1 \times B_2} c \, d(\mu_1 \otimes \mu_2). \quad (2)$$

Let us define now the functions

$$u(x) = \begin{cases} \inf\limits_{U \in V_x} \sup\left\{ \dfrac{\mu_1(f\chi_B)}{\mu_1(B)} : U \supset B \in \mathscr{B}(X),\ \mu_1(B) \ne 0 \right\} \\ \qquad\qquad \text{if } x \in \operatorname{supp} \mu_1 \\ -M \qquad \text{if } x \notin \operatorname{supp} \mu_1 \end{cases}$$

and

$$v(y) = \begin{cases} \inf_{V \in V_y} \sup \left\{ \dfrac{\mu_2(g\chi_B)}{\mu_2(B)} : V \supset B \in \mathscr{B}(Y),\ \mu_2(B) \neq 0 \right\} \\ \qquad\qquad \text{if } x \in \operatorname{supp} \mu_2 \\ -2M \qquad \text{if } x \notin \operatorname{supp} \mu_2, \end{cases}$$

where $\mathscr{B}(X)$ and $\mathscr{B}(Y)$ denote the families of the Borel subsets of X and Y, and V_x and V_y are fundamental systems of neighborhoods of x and y, respectively. Then $(u, v) \in F(DP)$ since $(u, v) \in S_b(X) \times S_b(Y)$ and it follows from (2) that $u + v \leq c$.

Let us see now that (u, v) is an optimal solution of (DP). In fact for every $n \in \mathbb{N}$ there exists a compact subset $K_n \subset X$ such that the restriction $u|K_n$ is continuous and $\mu_1(X - K_n) < 1/n$. For every $\varepsilon > 0$ and $x \in K_n$ there exists an open neighborhood U_x of x in K_n such that the oscillation of $u|K_n$ in U_x is less than ε and $\mu_1(f\chi_B) \leq \mu_1(B)(u(x) + \varepsilon)$ for every $B \in \mathscr{B}(X)$, $B \subset U_x$. Since K_n is compact there exists a finite subset $\{x_1, \ldots, x_n\} \subset K_n$ such that $K_n = \bigcup_{i=1}^n U_{x_i}$. Making $U_i' = U_{x_i} - \bigcup_{j=1}^{i-1} U_{x_j} (1 \leq j \leq n)$ we have that

$$\begin{aligned} \mu_1(f\chi_{k_n}) &= \sum_{i=1}^n \mu_1(f\chi_{U_i'}) \\ &\leq \sum_{i=1}^n \mu_1(U_i')[u(x_i) + \varepsilon] \\ &\leq \int_{K_n} u\,d\mu_1 + 2\varepsilon\mu_1(K_n). \end{aligned}$$

Thus

$$\mu_1(f\chi_{k_n}) \leq \int_{K_n} u\,d\mu_1$$

and

$$\mu_1(f) \leq \int_X u\,d\mu_1$$

since $\lim_n \mu_1(X - K_n) = 0$.

In a similar way we obtain that

$$\mu_2(g) \leq \int_Y v\,d\mu_2.$$

Therefore,

$$m(c) = \mu_1(f) + \mu_2(g) \leq \int_X u\,d\mu_1 + \int_Y v\,d\mu_2 \leq m(c)$$

hence (u, v) is an optimal solution and satisfies (1).

THEOREM 8. *Let* $c\colon X \times Y \to \mathbb{R}$ *be a bounded function verifying that for every* $\varepsilon > 0$ *there exists a compact subset* $K \subset X \times Y$ *such that* $c|K$ *is an upper semicontinuous function and* $\mu(X \times Y - K) < \varepsilon$ *for every* $\mu \in F(PP)$. *Then there is no duality gap for* (PP) *and* (DP).

Proof. Let be $m = \inf\{\mu(c)\colon \mu \in F(PP)\}$ and $\mu' = (1/\mu_1(X))\mu_1 \otimes \mu_2$. Then for every $n \in \mathbb{N}$ there exists a compact subset $K_n \subset X \times Y$ such that the function $c|K_n$ is upper semicontinuous and $\mu(X \times Y - K_n) < 1/n$ for every $\mu \in F(PP)$. If $K_n^i = p_i(K_n)$, $\mu_i^n = p_i(\mu'_{K_n^1 \times K_n^2})$ $(i = 1, 2)$ and

$$c_n(x, y) = \begin{cases} c(x, y) & \text{if } (x, y) \in K_n \\ -M & \text{if } (x, y) \in K_n^1 \times K_n^2 - K_N \end{cases} \qquad (n \in \mathbb{N}),$$

then it follows from Lemmas 6 and 7 that there exist $\mu_0^n \in M_+^1(K_n^1 \times K_n^2)$, $u_n \in S_b(K_n^1)$, and $v_n \in S_b(K_n^2)$ such that $p_i(\mu_0^n) = \mu_i^n$ $(i = 1, 2)$, $|u_n| \le M$, $-2M \le v_n \le 0$, $u_n + v_n \le c_n$, and $\mu_1^n(u_n) + \mu_2^n(v_n) \ge \mu_0^n(c_n) - M/n$. If we extend now the functions u_n and v_n to X and Y making $u_n = -M$ on $X - K_n^1$ and $v_n = -2M$ on $Y - K_n^2$, and we make $\mu_n = \mu' - \mu'_{K_n^1 \times K_n^2} + \mu_0^n$ (where we denote also by $\mu'_{K^1 \times K^2}$ and μ_0^n the measures on $X \times Y$ which are the images under the canonical injection $K_n^1 \times K_n^2 \to X \times Y$ of the measures $\mu'_{K_n^1 \times K_n^2}$ and μ_0^n), then $(u_n, v_n) \in F(DP)$, $\mu_n \in F(PP)$ because $p_i(\mu_n) = p_i(\mu') = \mu_i$ $(i = 1, 2)$,

$$\mu_0^n\big(K_n^1 \times K_n^2 - K_n\big) \le \mu_n(X \times Y - K_n) < \frac{1}{n},$$

$$\int_{K_n^1 \times K_n^2} c\, d\mu_0^n = \mu_n(c) - \int_{X \times Y} c\, d\big(\mu' - \mu'_{K_n^1 \times K_n^2}\big) \ge m - \frac{M}{n},$$

$$\big|\mu_1(u_n) - \mu_1^n(u_n)\big| \le M\mu_1\big(X - K_n^1\big) + \left|1 - \frac{\mu_2\big(K_n^2\big)}{\mu_2(Y)}\right| M\mu_2(Y) \le 2\frac{M}{n}$$

and analogously $|\mu_2(u_n) - \mu_2^n(u_n)| \le 4(M/n)$. Therefore,

$$\begin{aligned} \mu_1(u_n) + \mu_2(v_n) &\ge \mu_1^n(u_n) + \mu_2^n(v_n) - 6\frac{M}{n} \\ &\ge \mu_0^n(c_n) - 7\frac{M}{n} \\ &\ge \int_{K_n^1 \times K_n^2} c\, d\mu_0^n - 2M\mu_0^n\big(K_n^1 \times K_n^2 - K_n\big) - 7\frac{M}{n} \\ &\ge m - 10\frac{M}{n}, \end{aligned}$$

and the result follows immediately.

This has the following remarkable consequence:

THEOREM 9. *If c is a bounded upper semicontinuous function then there is no duality gap for (PP) and (DP).*

Remark 10. The family of all real bounded functions g defined on $X \times Y$ verifying that for every $\varepsilon > 0$ there exists a compact subset $K \subset X \times Y$ such that the function $g|K$ is continuous and $\mu(X \times Y - K) < \varepsilon$ for every $\mu \in F(PP)$ (so these functions g satisfy the condition of Theorem 8), is an algebra, a lattice, and a Banach space (endowed with the usual supremum norm) which contains the bounded continuous real functions on $X \times Y$ and the functions $f(x, y) = f_1(x)$ and $f(x, y) = f_2(y)$ ($(x, y) \in X \times Y$) when f_i is a bounded μ_i-Lusin measurable function ($i = 1, 2$).

THEOREM 11. *If c is a bounded upper semicontinuous function then the program (DP) is solvable.*

Proof. Consider the set $H = \{(u, v) \in F(DP): |u| \leq M, -2M \leq v \leq 0\}$. Then it follows from the Alaoglu−Bourbaki theorem that H is a $\sigma - (L^\infty(\mu_1) \times L^\infty(\mu_2), L^1(\mu_1) \times L^1(\mu_2))$ relatively compact subset of $L^\infty(\mu_1) \times L^\infty(\mu_2)$ and therefore it follows from Lemma 6 and Theorem 9 that there exists $(f, g) \in \overline{H}^\sigma$ such that $\mu_1(f) + \mu_2(g) = m$ where $m = \inf\{\mu(c): \mu \in F(PP)\}$. Moreover

$$\mu_1(f\chi_{B'})\mu_2(B'') + \mu_1(B')\mu_2(g\chi_{B''}) \leq \int_{B' \times B''} c\, d(\mu_1 \otimes \mu_2)$$

for all Borel subsets $B' \in \mathscr{B}(X)$ and $B'' \in B(Y)$ and then if we define (as in the proof of Lemma 7)

$$u(x) = \begin{cases} \inf_{U \in V_x} \sup\left\{\dfrac{\mu_1(f\chi_B)}{\mu_1(B)} : U \supset B \in \mathscr{B}(X),\ \mu_1(B) \neq 0\right\} \\ \qquad\qquad \text{if } x \in \operatorname{supp} \mu_1 \\ -M \qquad \text{if } x \in X - \operatorname{supp} \mu_1 \end{cases}$$

and

$$v(y) = \begin{cases} \inf_{V \in V_y} \sup\left\{\dfrac{\mu_2(g\chi_B)}{\mu_2(B)} : V \supset B \in \mathscr{B}(X),\ \mu_2(B) \neq 0\right\} \\ \qquad\qquad \text{if } y \in \operatorname{supp} \mu_2 \\ -2M \qquad \text{if } y \in Y - \operatorname{supp} \mu_2, \end{cases}$$

where V_x and V_y are fundamental systems of neighborhoods of x and y, respectively, then $(u, v) \in F(DP)$, and proceeding as in the proof of Lemma 7 it can be proved that (u, v) is an optimal solution of the program (DP) since μ_1 and μ_2 are Radon measures.

THEOREM 12. *If c is a bounded continuous function and (u, v) is an optimal solution of (DP) then $u(x) + v(y) = c(x, y)$ for every $(x, y) \in S =$ $\operatorname{supp} \mu_0$ and the functions $u|p_1(K)$ and $v|p_2(K)$ are continuous for every compact subset $K \subset S$.*

Proof. Let (u, v) be an optimal solution of the program (DP). Then $c - u - v$ is a non-negative lower semicontinuous function and $\mu_0(c - u - v) = 0$ and so $u(x) + v(y) = c(x, y)$ for every $(x, y) \in S$.

Moreover for every compact subset $K \subset S$ and every net $(x_i)_{i \in I} \subset p_1(K)$ convergent to some $x \in p_1(K)$ there exists $(y_i)_{i \in I} \subset p_2(K)$ such that $(x_i, y_i)_{i \in I} \subset K$. Let $(x, y) \in K$ be an accumulation point of the net $(x_i, y_i)_{i \in I}$. Then there exists an ultrafilter $\mathscr{U}$ in I, finer than the filter of sections of I, such that $\lim_{\mathscr{U}} y_i = y$ and therefore

$$u(x) + v(y) = c(x, y) = \lim_{\mathscr{U}} c(x_i, y_i) = \lim_{\mathscr{U}} u(x_i) + \lim_{\mathscr{U}} v(y_i).$$

Since

$$u(x) \geq \lim_{\mathscr{U}} u(x_i) \qquad \text{and} \qquad v(y) \geq \lim_{\mathscr{U}} v(y_i),$$

we have $u(x) = \lim_{\mathscr{U}} u(x_i)$ and the function $u|p_1(K)$ is continuous. In a similar way it is proved that $v|p_2(K)$ is continuous.

Let us remark that with the notation and conditions of Theorem 12, since $u = c - v$ on S, u is upper semicontinuous and $c - v$ is lower semicontinuous, we have that the functions $(x, y) \to u(x)$ and $(x, y) \to v(y)$ are continuous on S.

PROPOSITION 13. *If the spaces X and Y are uniformizable and c is a bouneed uniformly continuous function then there is an optimal solution (u, v) of (DP) such that the functions u and v are uniformly continuous.*

Proof. Let (u', v') be a (DP)-optimal solution (whose existence follows from Theorem 11) and consider the functions

$$u(x) = \inf\{c(x, y) - v'(y) : y \in Y\} \qquad (x \in X)$$

and

$$v(y) = \inf\{c(x, y) - u(x) : x \in X\} \qquad (y \in Y).$$

Then

$$u(x) = \inf\{c(x, y) - v(y) \colon y \in Y\} \qquad (x \in X),$$

$u' \le u$, $v' \le v$, and $u + v \le c$.

For every $\varepsilon > 0$ there exists a symmetrical element $N = N^{-1}$ of the uniformity of X such that $c(x, y) \le c(x', y) + \varepsilon$ for every $(x, x') \in N$ and $y \in Y$. Then for every $(x, x') \in N$ there exists $y' \in Y$ such that $c(x', y') - v(y') \le u(x') + \varepsilon$ and so

$$u(x) + v(y') \le c(x, y') \le c(x', y') + \varepsilon \le u(x') + v(y') + 2\varepsilon.$$

Therefore $u(x) - u(x') \le 2\varepsilon$ and $u(x') - u(x) \le 2\varepsilon$ for every $(x, x') \in N$ and the function u is uniformly continuous. In a similar way it is proved that the function v is also uniformly continuous. Now the statement is already proved since clearly

$$\int_X u' \, d\mu_1 + \int_Y v' \, d\mu_2 \le \int_X u \, d\mu_1 + \int_Y v \, d\mu_2.$$

An interesting particular case of the last proposition is the following (extension of [1, Theorem 5.2(a)]):

PROPOSITION 14. *If X and Y are compact Hausdorff spaces and c is a continuous function then there is an optimal solution (u, v) of (DP) such that the functions u and v are continuous.*

3. MASS-TRANSFER PROGRAMS WITH A UNIFORMLY COUNTABLY ADDITIVE PRIMAL FEASIBLE SET

This section studies the solvability and the absence of a duality gap in the particular case when $F(PP)$ is a uniformly countably additive set. Also some characterizations and properties of the uniformly countably additive bounded subsets of $M_+^1(X)$ are presented.

PROPOSITION 15. *Let $A \subset M_+^1(X)$ be a bounded set (i.e., $\{\mu(X) \colon \mu \in A\}$ is bounded). The following assertions are equivalent:*[1]

(15.1) *A is a uniformly countably additive.*

(15.2) *For every $\varepsilon > 0$ and every universally Lusin measurable bounded function f on X there exists a compact subset $K \subset X$ such that the function $f|K$ is continuous and $\mu(X - K) < \varepsilon$ for every $\mu \in A$.*

[1] The equivalence between (15.1) and (15.3) is proved for a compact space X, in [9, Lemma 3, p. 157].

(15.3) *A is uniformly inner regular, i.e., for every Borel subset $B \subset X$ and every $\varepsilon > 0$ there exists a compact subset $K \subset B$ such that $\mu(B - K) < \varepsilon$ for every $\mu \in A$.*

Proof. If the set A is uniformly countably additive then there exists a Radon measure β on X such that A is uniformly β-continuous (i.e., for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\mu(B) < \varepsilon$ for every Borel subset $B \subset X$ such that $\beta(B) < \delta$) (see Theorem 4 of [9, p. 11]). Hence for every universally Lusin measurable bounded function f on X there exists a compact subset $K \subset X$ such that $f|K$ is continuous and $\beta(X - K) < \delta$. So $\mu(X - K) < \varepsilon$ for every $\mu \in A$ and (15.2) holds.

let us suppose now that (15.2) holds. Then for every Borel subset $B \subset X$ and every $\varepsilon > 0$ there exists a compact subset $K \subset X$ such that $\chi_B|K$ is continuous and $\mu(X - K) < \varepsilon$ for every $\mu \in A$. Therefore $B \cap K$ is compact and $\mu(B - (B \cap K)) \leq \mu(X - K) < \varepsilon$ for every $\mu \in A$, and (15.3) holds.

Finally suppose that (15.3) is verified and let us prove that (15.1) holds. In fact if the set A is not uniformly countably additive then there exists $\varepsilon > 0$, a non-increasing sequence of Borel subsets $X \supset B_n \downarrow \varnothing$ and a sequence $(\mu_n) \subset A$ such that $\mu_n(B_n) > \varepsilon$ for every $n \in \mathbb{N}$.

It follows now from (15.3) that for every $n \in \mathbb{N}$ there exists two compact subsets $K'_n \subset B_n$ and $K''_n \subset X - B_n$ such that $\mu(B_n - K'_n) < \varepsilon/2^{n+1}$ and $\mu(B_n^c - K''_n) < \varepsilon/2^{n+1}$ for every $\mu \in A$. If $K_n = K'_n \cup K''_n$ we have that $\mu(X - K_n) < \varepsilon/2^n$ for every $\mu \in A$ and the function $\chi_{B_n}|K_n$ is continuous. Then $K = \bigcap_n K_n$ is a compact subset of X such that $\mu_n(X - K) < \varepsilon$ for every $n \in \mathbb{N}$ and $(B_n \cap K)$ is a non-increasing sequence of compact subsets such that $\bigcap_n (B_n \cap K) = \varnothing$. Therefore there exists $n_0 \in \mathbb{N}$ such that $B_{n_0} \cap K = \varnothing$ and

$$\varepsilon > \mu_{n_0}(X - K) \geq \mu_{n_0}\big(B_{n_0} - K\big) = \mu_{n_0}\big(B_{n_0}\big) > \varepsilon,$$

which is a contradiction.

THEOREM 16. *If the primal feasible set $F(PP)$ is uniformly countably additive and c is a universally Lusin measurable bounded function then there is no duality gap for (PP) and (DP).*

Proof. This is an immediate consequence of Theorem 8 and Proposition 15.

PROPOSITION 17. *Let Σ be a σ-algebra of subsets of a set Ω. A bounded set A of non-negative countably additive measures on (Ω, Σ) is uniformly* *countably additive*[2] *if and only if the measure*

$$\mu': \mu'(E) = \lim_{\mathscr{U}} \mu(E) \qquad (E \in \Sigma)$$

is countably additive for every ultrafilter $\mathscr{U}$ *on* A.

Proof. Suppose first that the set A is uniformly countable additive. Then from [9, Theorem 4, p. 11] follows the existence of a non-negative (real-valued) countably additive measure β on Σ such that A is uniformly β-continuous. This implies that μ' is countably additive.

Let us prove now that the condition is also sufficient. Suppose that for every ultrafilter $\mathscr{U}$ on A the measure μ' is countable additive and that A is not uniformly countably additive. Then there exist $\varepsilon > 0$, a non-increasing sequence of measurable sets $E_n \downarrow \varnothing$ and a sequence of measures $(\mu_n) \subset A$ such that $\mu_n(E_n) > \varepsilon$ for every $n \in \mathbb{N}$. If $\mathscr{U}$ is a (non-trivial) ultrafilter on $\mathbb{N}$ and

$$\mu'(E) = \lim_{n, \mathscr{U}} \mu_n(E) \qquad (E \in \Sigma),$$

we have that

$$\mu'(E_n) = \lim_{k, \mathscr{U}} \mu_k(E_n) \geq \varepsilon$$

for every $n \in \mathbb{N}$ and then, since the measure μ' is countable additive, it follows that

$$0 = \lim_{n} \mu'(E_n) \geq \varepsilon > 0,$$

which is a contradiction.

PROPOSITION 18. *If* $F(PP)$ *is uniformly countably additive and* $(\mu_i)_{i \in I} \subset F(PP)$ *is a net which converges (in the narrow topology) to some measure* $\mu \in F(PP)$, *then*

$$\mu(c) = \lim_{i} \mu_i(c)$$

for every universally Lusin measurable bounded function c on $X \times Y$.

Proof. Let $\mathscr{U}$ be an ultrafilter on I finer than the filter of sections of I. It follows from [9, Theorem 4, p. 11] that there exists a non-negative countably additive measure β which in this case can moreover be taken to be a Radon measure, such that $F(PP)$ is uniformly β-continuous. Then it

[2] Recall that a subset $\mathrm{ca}(\Omega, \Sigma)$ is relatively sequentially compact if and only if it is uniformly bounded and uniformly countably additive (see [10, p. 306]).

follows from the proof of Proposition 17 that

$$\mu'(E) = \lim_{i,\mathscr{U}} \mu_i(E) \qquad (E \in \mathscr{B}(X \times Y))$$

is a Radon measure on $X \times Y$, and since $M^1_+(X \times Y)$ is Hausdorff we have that $\mu' = \mu$ and therefore $\mu(E) = \lim_i \mu_i(E)$ for every Borel subset $E \subset X \times Y$, from where the result follows immediately.

This has the following remarkable consequence:

THEOREM 19. *If c is a universally Lusin measurable bounded function and $F(PP)$ is uniformly countably additive, then the primal program (PP) is solvable.*

PROPOSITION 20. *The primal feasible set $F(PP)$ $(\subset M^1_+(X \times Y))$ is uniformly countably additive if and only if for every compact subset $K \subset X \times Y$ the mapping $\mu \to \mu_K$, from $F(PP)$ into $C(K)$, is continuous (for the corresponding narrow topologies).*

Proof. If the primal feasible set $F(PP)$ is uniformly countably additive then it follows from [30, Theorem 3, p. 379] that the measure $\mu' = \lim_{\mathscr{U}} \mu \in F(PP)$ exists for every ultrafilter $\mathscr{U}$ on $F(PP)$ and so Proposition 18 assures that

$$\mu'(E) = \lim_{\mathscr{U}} \mu(E)$$

for every Borel subset $E \subset X \times Y$. Therefore

$$\mu'_K(E) = \lim_{\mathscr{U}} \mu_K(E) \qquad (E \in \mathscr{B}(K)),$$

$\{\mu_K\colon \mu \in \mathscr{U}\}$ converges to μ'_K and the mapping $\mu \to \mu_K$ is continuous.

Suppose now that this mapping is continuous from $F(PP)$ into $C(K)$ for every compact subset $K \subset X$. Let $\mathscr{U}$ be an ultrafilter on $F(PP)$ which converges to $\mu' \in F(PP)$. If $E \in \mathscr{B}(X \times Y)$ then for every $\varepsilon > 0$ there exists a compact subset $K \subset E$ such that $\mu'(E - K) < \infty$ and

$$\underline{\lim}_{\mathscr{U}} \mu(E) \geq \underline{\lim}_{\mathscr{U}} \mu(K) = \lim_{\mathscr{U}} \mu_K(K) = \mu'_K(K) = \mu'(K) > \mu'(E) - \varepsilon.$$

Therefore

$$\underline{\lim}_{\mathscr{U}} \mu(E) \geq \mu'(E) \qquad \text{and} \qquad \underline{\lim}_{\mathscr{U}} \mu(X \times Y - E) \geq \mu'(X \times Y - E),$$

which implies that $\lim_{\mathscr{U}} \mu(E) = \mu'(E)$ since $\mu(X \times Y) = \mu_1(X) = \mu_2(Y)$ for every $\mu \in F(PP)$. It follows immediately from Proposition 17 that the set $F(PP)$ is uniformly countably additive.

Remark 21. It follows from the proof of Proposition 20 that if the limit

$$\lim_{\mathscr{U}} \mu(K) = \mu'(K)$$

exists for every compact subset $K \subset X \times Y$ and every ultrafilter $\mathscr{U}$ on $F(PP)$ which converges to μ', then $F(PP)$ is uniformly countably additive.

Let us see now an example of a mass-transfer program with a uniformly countably additive primal feasible set, and also an example of a mass-transfer program without this property.

EXAMPLE 22. If $Y = \mathbb{N}$ equipped with the discrete topology then $F(PP)$ is uniformly countably additive.

In fact, let f be a universally Lusin measurable bounded real function defined on $X \times Y$. Then for every $\varepsilon > 0$ and $n \in \mathbb{N}$ there exists a compact subset $K'_n \subset X$ such that $f(\cdot, n)|K'_n$ is continuous and $\mu_1(X - K'_n) < \varepsilon/(2^{n+1}\mu_2(Y))$. If $K' = \bigcap_n K'_n$ then $\mu_1(X - K') < \varepsilon/(2\mu_2(Y))$ and $f(\cdot, n)|K'$ is continuous. Let $K'' \subset Y$ be a compact subset such that

$$\mu_2(Y - K'') < \frac{\varepsilon}{2\mu_1(X)},$$

then $f|K' \times K''$ is continuous and

$$\mu(X \times Y - K' \times K'') \leq \mu_1(X - K')\mu_2(Y) + \mu_1(X)\mu_2(Y - K'') < \varepsilon$$

for every $\mu \in F(PP)$. It follows from Proposition 15 that $F(PP)$ is a uniformly countably additive set.

EXAMPLE 23. If $X = Y = [0,1]$ and μ_1 and μ_2 are the Lebesgue measure on $[0,1]$ then $F(PP)$ is not uniformly countably additive.

Let E_0 be the diagonal (we always take only the diagonals parallel to the line $y = x$) of the square $[0,1] \times [0,1]$, E'_1 the union of the diagonals of all the squares of the partition of $X \times Y$ in squares of length side $1/2$ (i.e., $\{[0,1/2] \times [0,1/2], [0,1/2] \times [1/2,1], [1/2,1] \times [0,1/2], [1/2,1] \times [1/2,1]\}$), and E'_n the union of the diagonals of all the squares of the partition of $X \times Y$ in squares of length side $1/2^n$. Consider now $E_1 = E'_1 - E_0$, $E_n = E'_n - \bigcup_{i=1}^{n-1} E'_i$ $(n > 1)$ and the Radon measures μ_n $(n \in \mathbb{N})$ on $X \times Y$ such that $\mu_n(B) = (1/(2^{n-1}\sqrt{2}))\lambda(B \cap E_n)$ for every Borel subset $B \subset X \times Y$, where

$$\lambda(B \cap E_n) = \sum_{m=1}^{2^{2n-1}} \lambda_m^n(B \cap D_m^n),$$

$\{D_m^n\}$ are the diagonals of the squares of length side $1/2^n$ such that $E_n = \bigcup_{m=1}^{2^{2n-1}} D_m^n$, furthermore $\lambda_m^n = \varphi_m^n \rho$, where ρ denotes the Lebesgue measure on $[0, \sqrt{2}/2^n]$ and

$$\varphi_m^n : \left[0, \frac{\sqrt{2}}{2^n}\right] \to D_m^n$$

is an isometric homeomorphism.

Then $(\mu_n) \subset F(PP)$, (E_n) is a pairwise disjoint sequence of Borel subsets,

$$\mu_n\left(\bigcup_{i=1}^{+\infty} E_i\right) = \mu_n(E_n) = 1$$

for every $n \in \mathbb{N}$ and therefore $F(PP)$ is not uniformly countably additive.

4. A CONSTRUCTIVE PROOF OF THE SOLVABILITY OF THE DUAL PROGRAM (DP)

Assume that the spaces X and Y are uniformizable and let c be an uniformly continuous bounded real function defined on $X \times Y$. Proceeding as in the proof of Theorem 8 we can find two sequences $(u_n) \subset S_b(X)$ and $(v_n) \subset S_b(Y)$ such that $|u_n| \le M$, $-2M \le v_n \le 0$, $u_n + v_n \le c$ $(n \in \mathbb{N})$ and

$$\lim_n [\mu_1(u_n) + \mu_2(v_n)] = \mu_0(c).$$

Therefore

$$\lim_n \int_{X\times Y} (c - u_n - v_n)\, d\mu_0 = 0,$$

thus $(u_n + v_n)$ converges to c in μ_0-measure and so there exists a μ_0-measurable subset $S_0 \subset S = \operatorname{supp} \mu_0$ such that $\mu_0(S - S_0) = 0$, and a sequence $(u_{n_k} + v_{n_k})$ converging pointwise to c on S_0. Moreover, for every $\varepsilon > 0$ there exists a symmetrical element $N = N^{-1}$ of the uniformity of X such that

$$c(x, y') < c(x', y') + \varepsilon$$

for every $(x, x') \in N$, $y' \in Y$ and

$$\overline{\lim_k} \left(u_{n_k}(x) + v_{n_k}(y)\right) \le c(x, y') < c(x', y') + \varepsilon$$

$$= \lim_k \left(u_{n_k}(x') + v_{n_k}(y')\right) + \varepsilon$$

for every $(x', y') \in S_0$, $(x, x') \in N$. Therefore

$$\overline{\lim_k}\left[u_{n_k}(x) - u_{n_k}(x')\right] < \varepsilon$$

for every $(x, x') \in N$ such that $x' \in p_1(S_0)$. In particular

$$\overline{\lim_k}\left|u_{n_k}(x) - u_{n_k}(x')\right| < \varepsilon$$

for every $x, x' \in p_1(S_0)$ such that $(x, x') \in N$. If $\mathscr{U}$ is an ultrafilter on $\mathbb{N}$, finer than the Fréchet Filter, and $u(x) = \lim_{k,\mathscr{U}} u_{n_k}(x)$ for every $x \in p_1(S_0)$ then u is a uniformly continuous function on $p_1(S_0)$. In a similar way it is proved that the function $v(y) = \lim_{k,\mathscr{U}} v_n(y)$ $(y \in p_2(S_0))$ is uniformly continuous on $p_2(S_0)$. Moreover $u(x) + v(y) \leq c(x, y)$ for every $(x, y) \in p_1(S_0) \times p_2(S_0)$ and $u(x) + v(y) = c(x, y)$ if $(x, y) \in S_0$.

Let us suppose now that $p_1(S_0)$ is separable (this happens in particular if μ_1 is a Radon measure of type $(\mathscr{K}_m)$ since $\overline{p_1(S_0)} = \operatorname{supp} \mu_1$ and $\mu_1(\operatorname{supp} \mu_1 - p_1(S_0)) \leq \mu_0(X - S_0) = 0$) and let D be a countable subset of $p_1(S_0)$ such that $\overline{D} = p_1(S_0)$, then by means of the diagonal method a pointwise convergent subsequence $(u_{n'_k})$ can be constructed on D, and we have that

$$\begin{aligned}\overline{\lim_{h,k}}\left|u_{n'_k}(x) - u_{n'_h}(x)\right| &\leq \overline{\lim_k}\left|u_{n'_k}(x) - u_{n'_k}(x')\right| + \overline{\lim_h}\left|u_{n'_h}(x) - u_{h'_h}(x')\right| \\ &< 2\varepsilon\end{aligned}$$

for every $x \in p_1(S_0)$ and $x' \in D$ with $(x, x') \in N$. Therefore the limit

$$u(x) = \lim_k u_{n'_k}(x)$$

exists for every $x \in p_1(S_0)$. Moreover, for every $y \in p_2(S_0)$ there is an $x \in p_1(S_0)$ such that $(x, y) \in S_0$ and then the following limit exists:

$$v(y) = \lim_k v_{n'_k}(y) = \lim_k\left[u_{n'_k}(x) + v_{n'_k}(y)\right] - \lim_k u_{n'_k}(x).$$

Clearly $u(x) + v(y) \leq c(x, y)$ for every $(x, y) \in p_1(S_0) \times p_2(S_0)$ and $u(x) + v(y) = c(x, y)$ for every $(x, y) \in S_0$.

If we consider the functions

$$u^*(x) = \inf\{c(x, y) - v(y) : y \in p_2(S_0)\} \qquad (x \in X)$$

and

$$v^*(y) = \inf\{c(x, y) - u^*(x) : x \in X\} \qquad (y \in Y),$$

then it follows from the proof of Proposition 13 that (u^*, v^*) is an optimal solution of (DP) and the functions u^* and v^* are uniformly continuous on X and Y, respectively.

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