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Generating sequences and Poincaré series for a finite set of plane divisorial valuations ☆

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Abstract

Let V be a finite set of divisorial valuations centered at a 2-dimensional regular local ring R. In this paper we study its structure by means of the semigroup of values, S_V, and the multi-index graded algebra defined by V, $\mathrm{gr}_V R$. We prove that S_V is finitely generated and we compute its minimal set of generators following the study of reduced curve singularities. Moreover, we prove a unique decomposition theorem for the elements of the semigroup. The comparison between valuations in V, the approximation of a reduced plane curve singularity C by families of sets $V^{(k)}$ of divisorial valuations, and the relationship between the value semigroup of C and the semigroups of the sets $V^{(k)}$, allow us to obtain the (finite) minimal generating sequences for C as well as for V.

We also analyze the structure of the homogeneous components of $\mathrm{gr}_V R$. The study of their dimensions allows us to relate the Poincaré series for V and for a general curve C of V. Since the last series coincides with the Alexander polynomial of the singularity, we can deduce a formula of A'Campo type for the Poincaré series of V. Moreover, the Poincaré series of C could be seen as the limit of the series of $V^{(k)}$, $k \geqslant 0$.

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0. Introduction

This paper deals with the structure of a finite number of divisorial valuations centered at a regular local ring of dimension two. In singularity theory there are many problems that involve finitely many interrelated exceptional divisors (and so, their corresponding divisorial valuations), which cannot be analyzed independently without losing some information. Classification of sandwiched singularities, minimal resolutions and the Nash problem are examples of this situation. The study of plane curve singularities constitutes a similar situation and the treatment of a branch is rather different of the one of the whole curve (see [5] and [6]). Problems as uniformization and monomialization of valuations, studied historically by Zariski and Abhyankar, are also object of recent activity (see e.g. [17]), providing another motivation to our study.

This paper is inspired in two sources. Firstly, the results for the case of a single divisorial valuation by Spivakovsky [16], where minimal generating sequences were computed (see also [13]), and Galindo, who computes the Poincaré series [12]. The second one is the set of results [4–6] obtained by Campillo, Delgado and Gusein-Zade for a plane curve singularity with several branches where generation of the semigroup, zeta function and Poincaré polynomial are considered.

Throughout this paper, we will assume that $(R, \mathtt{m})$ is a local, regular, complete and 2-dimensional ring and that it has an algebraically closed coefficient field. Replacing curves defined by elements in R by analytically reduced curves defined by elements in the completion $\hat{R}$ of R, and considering the valuations defined by their branches in the ring R (see e.g. [10]), theorems stated in the paper remain true without the assumption of completeness. However, we will consider the complete case because it simplifies the proofs and gives a more clear intuition.

Each irreducible component E_α of the exceptional divisor E of a modification π of $\mathtt{Spec}\, R$ defines a valuation of the fraction field of R centered at R, named divisorial and denoted by ν_α. An irreducible element in R such that the strict transform by π of the corresponding curve is smooth and intersect transversely E_α at a smooth point of E is, generically, denoted by Q_α and plays an important role in this paper. Consider a finite set $V = \{\nu_1, \ldots, \nu_r\}$ of $r \geqslant 1$ divisorial valuations associated to exceptional components $E_{\alpha(i)}$ of E where $\pi : (X, E) \to (\mathtt{Spec}\, R, \mathtt{m})$ is the minimal modification such that $E_{\alpha(i)} \subset E$ for $1 \leqslant i \leqslant r$. We define the semigroup of values of V as the subsemigroup of $\mathbb{Z}^r_{\geqslant 0}$ given by $S_V := \{\underline{\nu}(f) := (\nu_1(f), \ldots, \nu_r(f)) \mid f \in R \setminus \{0\}\}$. A general curve of V is a reduced plane curve with r branches each one defined by an equation $Q_{\alpha(i)} = 0$. When $r = 1$, S_V coincides with the semigroup of values of any general curve of V [18]. The valuation ideals $J(\underline{m}) = \{g \in R \mid \underline{\nu}(g) \geqslant \underline{m}\}$ define a multi-index filtration of the ring R which gives rise to a graded algebra $\mathrm{gr}_V\, R = \bigoplus_{\underline{m} \in \mathbb{Z}^r_{\geqslant 0}} J(\underline{m})/J(\underline{m} + \underline{e})$, $\underline{e} = (1, \ldots, 1)$. This paper analyzes both objects, the semigroup and the graded algebra, for a set valuations V looking for its essential arithmetical and algebraic properties.

A description of the semigroup S_V is given in Section 2. In Theorem 1, we give the minimal set of generators of S_V, proving that it is finitely generated, unlike the case of a reduced plane curve singularity (see [7] and [4]). The set $\{B^i := \underline{\nu}(Q_{\alpha(i)}) \mid i = 1, \ldots, r\}$ plays a very special role in S_V. Proposition 6 shows that the projectivization of the vector space $D(B^i) = J(B^i)/J(B^i + \underline{e})$ is canonically isomorphic to the exceptional divisor $E_{\alpha(i)}$ which defines the valuation ν_i. In particular, $D(B^i)$ is bidimensional. The study of the dimension $d_i(\underline{m})$ of the spaces $D_i(\underline{m}) = J(\underline{m})/J(\underline{m} + e_i)$ allows to prove Theorem 3, which gives a unique decomposition for the elements in S_V in terms of the set $\{B^1, \dots, B^r\}$. In particular, we give another proof of the fact that if V consists of all the divisors of a modification, then S_V is a free semigroup generated by $B^1, \dots, B^r$.

In Section 3 we describe a generating sequence for a finite set V of divisorial valuations and for a reduced curve with several branches. Denote by $\mathcal{E}$ the set of end divisors of the minimal resolution of V, i.e., the exceptional components E_α such that $E \setminus E_\alpha$ is connected, and set $\Lambda_{\mathcal{E}} = \{Q_\rho \mid E_\rho \in \mathcal{E}\}$. The main result of this section, Theorem 5, states that $\Lambda_{\mathcal{E}}$ is a minimal generating sequence of V, that is, any valuation ideal is generated by monomials in the set $\Lambda_{\mathcal{E}}$. After a result of Campillo and Galindo [2], this is equivalent to the fact that $\mathrm{gr}_V R$ is the $R/\mathtt{m}$-algebra generated by the classes of the elements in $\Lambda_{\mathcal{E}}$.

A similar result is true for the set W of valuations defined by the branches of a reduced plane curve C, however we must change the set $\Lambda_{\mathcal{E}}$ by another one $\Lambda_{\overline{\mathcal{E}}}$ which is also finite (see, again, Theorem 5). The key to understand it, is that W can be regarded as a limit of families of divisorial valuations $V^{(k)}$, $k \geqslant 0$, and so $\Lambda_{\overline{\mathcal{E}}}$ as the limit of the sequence $\Lambda_{\mathcal{E}^{(k)}}$ given by $V^{(k)}$. The number of classes in $\mathrm{gr}_V R$ produced by each element in $\Lambda_{\mathcal{E}}$ is finite, however some elements in $\Lambda_{\overline{\mathcal{E}}}$ give infinitely many different classes in the corresponding algebra (see the last remark of Section 3). This fact explains the apparent contradiction between the infinite generation of the semigroup of a plane curve singularity and the existence of a finite generating sequence.

It is worthwhile to mention that the so called multipliers ideals of ideals in the ring R, can be regarded as ideals $J(\underline{m})$ for concrete sets, V, and elements $\underline{m}$ [14]. Notice that, in our case, these ideals are exactly the complete ones [15].

The dimensions $d(\underline{m}) = \dim J(\underline{m})/J(\underline{m} + \underline{e})$ of the homogeneous pieces of the graded ring $\mathrm{gr}_V R$ can be collected in the Laurent series $L_V(t_1, \dots, t_r) = \sum_{\underline{m} \in \mathbb{Z}^r} d(\underline{m}) \underline{t}^{\underline{m}}$ (note that the sum extends to $\mathbb{Z}^r$). Following [6] and [8], the Poincaré series of V is defined as the formal series with integral coefficients

$$P_V(t_1, \dots, t_r) = \frac{L_V(t_1, \dots, t_r) \cdot \prod_{i=1}^r (t_i - 1)}{t_1 t_2 \cdots t_r - 1}.$$

As an application of the results and techniques developed in the previous sections, Section 4 is devoted to the computation of the Poincaré series P_V. So, in Theorem 6 we state the relation between the Poincaré series of V and the Poincaré polynomial P_C of any general curve C of V. This polynomial coincides with the Alexander polynomial of the link of the singularity [6]. In the complex case, the above expression leads to an explicit formula for P_V in terms of the topology of the exceptional divisor, very similar to the formula of A'Campo (see [1]) for the zeta function (extended by Eisenbud and Neumann in [11] for the Alexander polynomial):

$$P_V(t_1, \dots, t_r) = \prod_{E_\alpha \subset E} \left(1 - \underline{t}^{\underline{\nu}^\alpha}\right)^{-\chi(\overset{\bullet}{E}_\alpha)}$$

where $\chi(\overset{\bullet}{E}_\alpha)$ is the Euler characteristic of the smooth part $\overset{\bullet}{E}_\alpha$ of $E_\alpha \subset E$. This formula was conjectured by the authors some time ago, but the first complete proof has been given in [8] by using a very different approach: the integration on infinite-dimensional spaces with respect to the Euler characteristic.

To prove our results, we develop two different kind of techniques which in our opinion have interest by themselves. The main steps of the first one are included in Section 1. There, we consider pairs of elements in R with the same value by a valuation ν_α associated to a component E_α

of the exceptional divisor E of a modification and we find the relation between the initial forms of these elements with respect to ν_α as well as their values for the valuations corresponding to other components of E. Such study is given in terms of the topology of the exceptional divisor. In the proofs we systematically use the geometry of pencils of plane curves. In particular, the fact that the divisor E_α be dicritical for the pencil $\{\lambda f + \mu g\}$ if and only if the initial forms of the functions f and g with respect to ν_α are linearly independent, explains the deep relationship between both concepts.

The second technique, specially used in Section 4, is the cited approximation of curves by divisorial valuations. We can see it as a way to go from results related to a curve C to similar results for the corresponding sets $V^{(k)}$ of divisorial valuations, and vice versa. Corollary 2, which presents the Poincaré polynomial of a general curve C of V as the limit of the Poincaré series of sets of divisorial valuations $V^{(k)}$, gives a good example of this philosophy.

1. Divisorial valuations

Let $(R, \mathfrak{m})$ be a local, regular, complete and 2-dimensional ring with an algebraically closed coefficient field K. For us, a curve will be a subscheme of $\operatorname{Spec} R$, C_f, defined by some element $f \in \mathfrak{m}$. A **divisorial valuation** ν is a discrete valuation of the fraction field of R, centered at R (i.e., $R \cap \mathfrak{m}_\nu = \mathfrak{m}$, where $(R_\nu, \mathfrak{m}_\nu)$ is the valuation ring of ν), with rank 1 and transcendence degree 1.

Given a **modification**, that is, a finite sequence of point blowing-ups, $\pi : X \to \operatorname{Spec} R$, there is a divisorial valuation, ν_α, associated to each irreducible component E_α of the total exceptional divisor E of π, namely, for $f \in R$, $\nu_\alpha(f)$ is the vanishing order of the function $f \circ \pi : X \to K$ along the divisor E_α. We will say that ν_α is the E_α-valuation.

Assume that π is given by the sequence

$$\pi : X = X_{N+1} \xrightarrow{\pi_{N+1}} X_N \longrightarrow \cdots \longrightarrow X_1 \xrightarrow{\pi_1} X_0 = \operatorname{Spec} R,$$

and denote, for $0 \leqslant i \leqslant N$, by P_i the center of π_{i+1} in X_i ($P_0 = \mathfrak{m}$), by $(R_i, \mathfrak{m}_i)$ the local ring of X_i at P_i, and by E_{i+1} the exceptional divisor of π_i. Then, for $1 \leqslant \alpha \leqslant N+1$, ν_α is the $\mathfrak{m}_{\alpha-1}$-adic valuation. Given $\nu = \nu_\alpha$, we will sometimes denote P_ν and $E_{\alpha(\nu)}$ instead of P_α and E_α.

In fact, divisorial valuations correspond 1–1 to finite sequences of point blowing-ups, by associating to ν its **minimal resolution**, defined as follows: π_{i+1} is the blowing-up of X_i at P_i, $P_0 = \mathfrak{m}$, and for $i \geqslant 1$ P_i is the unique point in the exceptional divisor of π_i, E_i, such that R_ν dominates the local ring of X_i at P_i. In this way, ν is the divisorial valuation associated to E_{N+1}.

Given the E_ν divisorial valuation ν, denote by $\mathcal{C}_\nu$ the set of all irreducible curves in $\operatorname{Spec} R$ whose strict transform by the minimal resolution π of ν is smooth and meets E_ν transversely at a nonsingular point of the total exceptional divisor of π. An element $f \in \mathfrak{m}$ is said to be a **general element** of ν if $C_f \in \mathcal{C}_\nu$. In [16] it is proved that for $f \in R$,

$$\begin{aligned} \nu(f) &= \min\bigl\{(f, g) \bigm| g \in \mathcal{C}_\nu\bigr\} \\ &= (f, g) \quad \text{if } \widetilde{C}_f \cap \widetilde{C}_g = \emptyset \text{ and } g \in \mathcal{C}_\nu, \end{aligned} \tag{1}$$

where (f, g) stands for the intersection multiplicity (C_f, C_g) between the curves C_f and C_g and $\widetilde{C}_f, \widetilde{C}_g$ for the strict transforms by π of the curves C_f and C_g. The minimal resolution $\pi : X \to \operatorname{Spec} R$ of the divisorial valuation ν is an embedded resolution of C_f for $f \in \mathcal{C}_\nu$, in general not the minimal one.

Conversely, let C_f be any irreducible curve in $\mathtt{Spec}\, R$, and take the associated (infinity) sequence of blowing-ups with centers at the infinitely near points of f,

$$\cdots \longrightarrow X_{i+1} \xrightarrow{\pi_{i+1}} X_i \longrightarrow \cdots \longrightarrow X_1 \xrightarrow{\pi_1} X_0 = \mathtt{Spec}\, R. \tag{2}$$

For each $i \geqslant 0$ set ν_i the E_{i+1} divisorial valuation. Since the curve C_f is determined by the sequence (2), we can think of the sequence of valuations $\{\nu_i\}_{i\geqslant 0}$ as an approaching of C_f. Indeed, for any $g \in R$, nonzero in the ring $R/(f)$, $\nu_i(g) = (f, g)$ for $i \gg 0$. So, the study of divisorial valuations and of irreducible curves is closely related (see for example [16]).

Let $\pi : (X, E) \to (\mathtt{Spec}\, R, \mathtt{m})$ be a modification. The **dual graph** $\mathcal{G}(\pi)$ of π is the dual figure of the exceptional divisor E; that is, it is a graph with a vertex α for each irreducible component E_α of E and where two vertices are adjacent if and only if their corresponding exceptional divisors intersect.

The graph $\mathcal{G}(\pi)$ is a tree. We will denote by $\mathbf{1}$ the vertex corresponding to the first exceptional divisor and by $[\beta, \alpha]$ the path joining β and α. Along this paper, for a vertex α in $\mathcal{G}(\pi)$, Q_α will stand for any irreducible element of $\mathtt{m}$ such that the strict transform of the curve C_{Q_α} on X is smooth and meets E_α transversely at a nonsingular point. C_{Q_α} gives in particular a general element of the E_α-valuation.

A **dead end** (respectively, **star vertex**) of the graph $\mathcal{G}(\pi)$ is a vertex which is adjacent to a unique (respectively, to at least three) vertices. The set of dead ends will be denoted by $\mathcal{E}$. Given a dead end $\rho \neq \mathbf{1}$, st_ρ will denote the nearest to ρ star vertex of $\mathcal{G}(\pi)$.

In this paper we will make use of the concept of **pencil** of elements in R. Recall that if we consider the pencil $L = \{\lambda f + \mu g \mid \lambda, \mu \in K\}$, relative to two elements $f, g \in R$, a component E_α of the exceptional divisor E of a modification π is said to be **dicritical** for L if the E_α-valuation, ν_α, is constant on L. This condition is equivalent to say that the lifting $\widetilde{\varphi} = \varphi \circ \pi$ of the rational function $\varphi = f/g$ to X restricts to a surjective (that is, nonconstant) morphism from E_α onto $\mathbb{P}^1_K$. In the sequel, we will identify $\mathbb{P}^1_K$ with $K \cup \{\infty\}$. The fibers of L are studied in [9] in the analytic complex case, and we will use those results because they can be easily extended to our context.

In particular from Theorems 1, 2 and 3 in [9] we can deduce the following:

Let $\pi : (X, E) \to \mathtt{Spec}\, R$ be a modification and α a vertex of $\mathcal{G}(\pi)$. For a subset A of $\mathcal{G}(\pi)$ denote $E_A = \bigcup_{\beta \in A} E_\beta$. Assume that $\widetilde{\varphi}$ is constant in E_α, $\widetilde{\varphi}|_{E_\alpha} \equiv c \in \mathbb{P}^1$. Then the strict transform $\widetilde{C}_{f-cg}$ of C_{f-cg} intersects E_A, A being the maximal connected subset of $\mathcal{G}(\pi)$ such that $\alpha \in A$, and $\widetilde{\varphi} \equiv c$ along E_A.

On the other hand, assume that E_α is dicritical and $P \in E_\alpha$ is such that $\widetilde{\varphi}(P) = c$. If P is a smooth point of E then $\widetilde{C}_{f-cg}$ intersects E_α in P, and if P is singular and Δ is the connected component of $\mathcal{G}(\pi) \setminus \{\alpha\}$ such that $E_\Delta \cap E_\alpha = \{P\}$ then $\widetilde{C}_{f-cg}$ intersects E_Δ.

Next results are stated for a modification $\pi : (X, E) \to \mathtt{Spec}\, R$ and a vertex $\alpha \in \mathcal{G}(\pi)$.

Lemma 1. *Let $h \in R$ be such that $\nu_\alpha(h) = \nu_\alpha(Q_\alpha)$ and assume that $\widetilde{C}_h \cap \widetilde{C}_{Q_\alpha} = \emptyset$. Then E_α is the unique dicritical divisor of the pencil $L = \{\lambda Q_\alpha + \mu h \mid \lambda, \mu \in K\}$, and the lifting $\widetilde{\varphi}$ of the rational function $\varphi = Q_\alpha / h$ to X restricts to an isomorphism in E_α.*

Proof. Since $\nu_\alpha(Q_\alpha) = \nu_\alpha(h)$, $\widetilde{\varphi}$ is defined in every point of E_α, and $\widetilde{\varphi}_\alpha := \widetilde{\varphi}|_{E_\alpha} \not\equiv 0, \infty$. Since moreover $\widetilde{C}_h \cap \widetilde{C}_{Q_\alpha} = \emptyset$, $\widetilde{\varphi}_\alpha(\widetilde{C}_{Q_\alpha} \cap E_\alpha) = 0$, so E_α is a dicritical component for L. In fact, E_α is the unique dicritical component for L, because the existence of another one would contradict the irreducibility of the fiber Q_α.

Let $P \in E_\alpha$ be such that $\widetilde{\varphi}_\alpha(P) = 0$. If there were a connected component Δ of $\mathcal{G}(\pi) \setminus \{\alpha\}$ such that $P \in E_\Delta$, then $\widetilde{C}_{Q_\alpha}$ would intersect E_Δ, which is impossible by the election of Q_α. Hence, P is a smooth point of E, $P = \widetilde{C}_{Q_\alpha} \cap E_\alpha$. Moreover, from Theorem 3 of [9], P is not a critical point of $\widetilde{\varphi}_\alpha$, so $\widetilde{\varphi}_\alpha$ has degree 1, i.e., it is an isomorphism. □

The next result is a generalization of Lemma 4 in [3].

Proposition 1. *Let $h \in R$ be such that $\nu_\alpha(h) = \nu_\alpha(Q_\alpha)$ and such that the strict transform $\widetilde{C}_h$ of C_h on X does not intersect E_α. Then there exists a unique connected component Δ of $\mathcal{G}(\pi) \setminus \{\alpha\}$ such that $\widetilde{C}_h \cap E_\Delta \neq \emptyset$. Moreover, $\nu_\gamma(h) = \nu_\gamma(Q_\alpha)$ if $\gamma \in \mathcal{G}(\pi) \setminus \Delta$, and $\nu_\gamma(h) > \nu_\gamma(Q_\alpha)$ otherwise.*

Proof. Keep the notations of Lemma 1. Let Δ be a connected component of $\mathcal{G}(\pi) \setminus \{\alpha\}$ such that $\widetilde{C}_h \cap E_\Delta \neq \emptyset$. By Lemma 1, there are not dicritical divisors of L in Δ, and since $\widetilde{\varphi}(P) = \infty$ for any $P \in \widetilde{C}_h \cap E_\Delta$, then $\widetilde{\varphi}|_{E_\Delta} \equiv \infty$, which implies $\nu_\gamma(h) > \nu_\gamma(Q_\alpha)$ for $\gamma \in \Delta$. As $\widetilde{\varphi}_\alpha$ is an isomorphism, $E_\alpha \cap E_\Delta$ is the unique point $Q \in E_\alpha$ such that $\widetilde{\varphi}(Q) = \infty$, hence Δ is the unique connected component of $\mathcal{G}(\pi) \setminus \{\alpha\}$ such that $\widetilde{C}_h \cap E_\Delta \neq \emptyset$ and moreover we deduce that $\nu_\gamma(Q_\alpha) = \nu_\gamma(h)$ if $\gamma \notin \Delta$. □

A close result holds when we change Q_α by whatever element $f \in R$:

Proposition 2. *Let h and f be elements in R such that $\nu_\alpha(h) = \nu_\alpha(f)$ and assume that there exists a connected component Δ of $\mathcal{G}(\pi) \setminus \{\alpha\}$ such that E_Δ contains $\widetilde{C}_h \cap E$ and $\widetilde{C}_f \cap E$. Then $\nu_\gamma(f) = \nu_\gamma(h)$ for each $\gamma \notin \Delta$ and there exists $c \in K$, $c \neq 0$, such that $\nu_\alpha(f - ch) > \nu_\alpha(f)$.*

Proof. We can assume that π is an embedded resolution of the curve C_{fh}, since the additional blowing-ups we need for it do not modify the connected subset $T = \mathcal{G}(\pi) \setminus \Delta$.

If $\nu_\gamma(f) > \nu_\gamma(h)$ for some $\gamma \in T$, we deduce that $\widetilde{C}_f$ intersects E_A, where A is the maximal connected subset of $\mathcal{G}(\pi) \setminus \{\alpha\}$ such that $\gamma \in A$ and $\nu_\beta(f) > \nu_\beta(h)$ for every $\beta \in A$. In particular, as $A \subset T$, $\widetilde{C}_f$ intersects E_T, which contradicts the hypothesis. Thus, $\nu_\gamma(f) = \nu_\gamma(h)$ for every $\gamma \in T$ and the lifting $\widetilde{\varphi}$ of the rational function $\varphi = f/h$ to the space X is defined at every point of E_T. Moreover, $\widetilde{\varphi}|_{E_\beta} : E_\beta \to \mathbb{P}^1$ cannot be surjective for $\beta \in T$, since $\widetilde{C}_h \cap E_T = \emptyset$ and $\widetilde{C}_f \cap E_T = \emptyset$. Therefore, there exists $c \in \mathbb{P}^1_K$, $c \neq 0, \infty$, such that $\widetilde{\varphi}|_{E_T} \equiv c$. Then the lifting of $(f - ch)/h$ vanishes on E_T and in particular $\nu_\alpha(f - ch) > \nu_\alpha(h)$ (in fact $\nu_\gamma(f - ch) > \nu_\gamma(h)$ for every $\gamma \notin \Delta$). □

Remark. Let $f, h \in R$ such that $\nu_\alpha(f) = \nu_\alpha(h)$ and assume that there exists $c \in \mathbb{P}^1$, $c \neq 0, \infty$, such that $\nu_\alpha(f - ch) > \nu_\alpha(f)$. Then, the lifting of the rational function $\varphi = f/h$ is constant and equal to c along E_α. As a consequence, the strict transforms of C_f and C_h intersect the same points of E_α and the same connected components of $\mathcal{G}(\pi) \setminus \{\alpha\}$ (otherwise the corresponding point of intersection in E_α must be a zero or a pole of φ).

On the other hand, let $f \in R$ be such that $\widetilde{C}_f \cap E = P \in E_\alpha$ is a smooth point of E, set $r = (E_\alpha, \widetilde{C}_f)$ and pick Q_α by $P \in E_\alpha$. Then $\nu_\alpha(Q_\alpha^r) = \nu_\alpha(f)$ and after some additional blowing-ups we could apply the above proposition, proving the existence of $c \neq 0, \infty$ such that $\nu_\alpha(f - cQ_\alpha^r) > \nu_\alpha(f)$.

Now we recall some known facts about curve singularities and divisorial valuations.

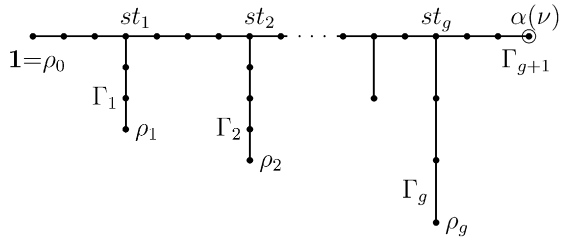

Fig. 1. The dual graph of a divisorial valuation.

Let ν be a divisorial valuation, $f \in R$ a general element of ν defining a curve C_f and v the discrete valuation of the fraction field of $R/(f)$ given by its integral closure. Let h be an element of R such that the strict transform $\widetilde{C}_h$ of C_h by the minimal resolution of ν does not intersect the strict transform $\widetilde{C}_f$ of C_f. Then equality (1) implies that $\nu(h) = v(h) = (f, h)$. If $\widetilde{C}_f \cap \widetilde{C}_h \neq \emptyset$ one can use a generic element f' for which $\widetilde{C}_{f'} \cap \widetilde{C}_h = \emptyset$ and $\nu(h) = v'(h)$, where v' is the valuation corresponding to f'.

The dual graph of the minimal resolution of a valuation ν looks like that of Fig. 1, where $\alpha(\nu)$ is the vertex corresponding to the divisor $E_{N+1} = E_{\alpha(\nu)}$ defining the valuation ν, st_i stands for the star vertex of the dead end ρ_i and Γ_i denotes the path from st_{i-1} to ρ_i.

If C_f is general for ν (i.e., f is a general element of ν), then the dead ends of $\mathcal{G}(\pi)$, $\rho_0, \ldots, \rho_g$, are also dead ends for the dual graph of C_f, which is the dual graph of the minimal embedded resolution of C_f together with an arrow attached to the vertex, $\alpha(f)$, corresponding to the component intersected by $\widetilde{C}_f$. We will denote $Q_i := Q_{\rho_i}$ and we set $\bar{\beta}_i = v(Q_i)$ $(0 \leqslant i \leqslant g)$, values which are usually called **maximal contact values** of the curve singularity C_f. It is known that the set $\{\bar{\beta}_0, \ldots, \bar{\beta}_g\}$ and the Puiseux pairs of C_f, and hence the equisingularity type of C_f, are equivalent data (e.g. $\bar{\beta}_0$ is the multiplicity $m(f)$ of C_f at the origin). Moreover, $\{\bar{\beta}_0, \ldots, \bar{\beta}_g\}$ is a minimal set of generators of the semigroup of values $S_{C_f} := \{v(h) \mid h \in R/(f)^*\}$ of C_f, $R/(f)^*$ denoting the nonzero elements of the ring $R/(f)$.

For the divisorial valuation ν we have $\nu(Q_i) = \bar{\beta}_i = v(Q_i)$ and so for the semigroup of values of ν, $S_\nu := \{\nu(h) \mid h \in R \setminus \{0\}\}$, one has $S_\nu = \langle \bar{\beta}_0, \ldots, \bar{\beta}_g \rangle = S_{C_f}$. Thus, arithmetical properties of v are also true for the valuation ν (in [18], the reader can see proofs for the main properties which we will use later in this context). For the sake of completeness we will denote $\bar{\beta}_{g+1} = \nu(Q_{\alpha(\nu)})$. It holds that $\bar{\beta}_{g+1} = e_{g-1}\bar{\beta}_g + c$, where e_{g-1} is the smallest positive integer such that $e_{g-1}\bar{\beta}_g \in \langle \bar{\beta}_0, \ldots, \bar{\beta}_{g-1} \rangle$ and $c \geqslant 0$ is the number of blowing-ups needed to create $E_{\alpha(\nu)}$ after the divisor corresponding to st_g was obtained. Thus, $\bar{\beta}_{g+1}$ gives an additional datum to the semigroup of values S_ν which permits to recover the dual graph of the divisorial valuation ν (see [16]). The element $\bar{\beta}_{g+1}$ has an expression $\bar{\beta}_{g+1} = \sum_{j=0}^{g} \lambda_j \bar{\beta}_j$ with $\lambda_j \geqslant 0$ for $0 \leqslant j \leqslant g$, which is unique if we add some restrictions to the coefficients λ_j. The case $c = 0$ corresponds to $\alpha(\nu) = st_g$, or equivalently, to the case in which $\mathcal{G}(\pi) \setminus \{\alpha(\nu)\}$ has two connected components, and in this case $\lambda_g = 0$, thus, $\bar{\beta}_{g+1} = e_{g-1}\bar{\beta}_g = \sum_{j=0}^{g-1} \lambda_j \bar{\beta}_j$.

For simplicity, we will often use the term "monomial" to indicate a monomial in the set $\{Q_\rho \mid \rho \in \mathcal{E}\}$, that is, a finite product of the type $\prod_{\rho \in \mathcal{E}} Q_\rho^{\lambda_\rho}$ with $\lambda_\rho \in \mathbb{Z}_{\geqslant 0}$.

Proposition 3. *Let $\pi : X \to \mathtt{Spec}\, R$ be a modification. Pick $\alpha \in \mathcal{G}(\pi)$ and let Δ be a connected component of $\mathcal{G}(\pi) \setminus \{\alpha\}$. Then, there exists a monomial $q_\Delta = \prod_{\rho \in \mathcal{E} \cap \Delta} Q_\rho^{\lambda_\rho}$ such that $\nu_\gamma(q_\Delta) = \nu_\gamma(Q_\alpha)$ if $\gamma \in \mathcal{G}(\pi) \setminus \Delta$ and $\nu_\gamma(q_\Delta) > \nu_\gamma(Q_\alpha)$ otherwise.*

Proof. We only need to find a monomial $q_\Delta = \prod_{\rho\in\mathcal{E}\cap\Delta} Q_\rho^{\lambda_\rho}$ such that $\nu_\alpha(q_\Delta) = \nu_\alpha(Q_\alpha)$, because it would satisfy $\widetilde{C}_{q_\Delta} \cap E_\Delta \neq \emptyset$, and then, by Proposition 1, it solves our problem.

Firstly, let us assume that $\pi : X \to \operatorname{Spec} R$ is the minimal resolution of ν_α. With the above notations, $\bar\beta_{g+1} = \nu_\alpha(Q_\alpha) = \sum_{i=0}^{g} \lambda_i \bar\beta_i$ and we have two possibilities depending whether $\mathcal{G}(\pi) \setminus \{\alpha\}$ is connected or not. In the first case, the decomposition of $\bar\beta_{g+1}$ provides the monomial $q_\Delta = \prod_{i=0}^{g} Q_{\rho_i}^{\lambda_i}$. Otherwise $\mathcal{G}(\pi) \setminus \{\alpha\}$ has two connected components; then, if $[\mathbf{1}] \in \Delta$, we have $\{\rho_0, \ldots, \rho_{g-1}\} = \mathcal{E} \cap \Delta$ and the monomial is $q_\Delta = \prod_{i=0}^{g-1} Q_{\rho_i}^{\lambda_i}$ (recall that in this case $\lambda_g = 0$), and if $[\mathbf{1}] \notin \Delta$ we have $\{\rho_g\} = \mathcal{E} \cap \Delta$ and the monomial is $q_\Delta = Q_g^{e_{g-1}}$.

In general, let us denote by $\pi' : (Y, F) \to \operatorname{Spec} R$ the minimal resolution of ν_α and let $\sigma : X \to Y$ be the composition of the sequence of point blowing-ups which produces X starting from Y. We claim that if Ω is any connected component of $\mathcal{G}(\pi) \setminus \mathcal{G}(\pi')$ such that $\sigma(E_\Omega) = P \in E_\beta$ is a smooth point of F, then there exists a dead end $\rho \in \mathcal{E} \cap \Omega$ such that $\nu_\gamma(Q_\rho) = \nu_\gamma(Q_\beta)$ for any $\gamma \notin \Omega$. Indeed, it suffices to choose ρ as an element of $\mathcal{E} \cap \Omega$ making minimal the number of blowing-ups needed to obtain it, since for this ρ, the strict transform of Q_ρ by π' is smooth and transversal to F at P.

Now, if $\sigma(E_\Delta)$ is a smooth point $P \in E_\alpha$ of F, the above construction applied to Δ gives $\rho \in \mathcal{E} \cap \Delta$ such that $\nu_\alpha(Q_\rho) = \nu_\alpha(Q_\alpha)$, so we can choose $q_\Delta = Q_\rho$.

Otherwise, $\sigma(E_\Delta) \subset \overline{F \setminus E_\alpha}$. In this case, if some dead end ρ' of $\mathcal{G}(\pi')$ is not a dead end of $\mathcal{G}(\pi)$, then there exists a connected component Ω of $\mathcal{G}(\pi) \setminus \mathcal{G}(\pi')$ such that $\sigma(E_\Omega) = P \in E_{\rho'}$, P a smooth point of F, and our claim gives a dead end ρ of $\mathcal{G}(\pi)$ such that $\nu_\gamma(Q_\rho) = \nu_\gamma(Q_{\rho'})$ (and $\rho \in \Delta$ if $\rho' \in \Delta$). Hence, if $\{\rho_0', \ldots, \rho_g'\}$ are the dead ends of $\mathcal{G}(\pi')$, we can find $\{\rho_0, \ldots, \rho_g\}$ in $\mathcal{E} \cap \Delta$ such that $\nu_\gamma(Q_{\rho_i}) = \nu_\gamma(Q_{\rho_i'}) = \bar\beta_i$ for $0 \leqslant i \leqslant g$, and the monomial is given as in the case in which π is the minimal resolution. $\square$

To end this section, assume that $h \in R$ is irreducible and $\pi : X \to \operatorname{Spec} R$ a modification such that the strict transform of the curve C_h by π only meets one irreducible component, that we will denote $E_{\alpha(h)}$, of the exceptional divisor of π.

Proposition 4. *For any vertex $\beta \in \mathcal{G}(\pi)$, there exists a monomial $q = \prod_{\rho\in\mathcal{E}} Q_\rho^{\lambda_\rho}$ such that $\nu_\beta(q) = \nu_\beta(h)$ and $\nu_\gamma(q) \geqslant \nu_\gamma(h)$ for every $\gamma \neq \beta$. Moreover, if $\beta \neq \alpha(h)$, then the vertices ρ such that $\lambda_\rho \neq 0$ belong to the connected component of $\alpha(h)$ in $\mathcal{G}(\pi) \setminus \{\beta\}$.*

Proof. We can choose $Q_{\alpha(h)}$ through $P = E_{\alpha(h)} \cap \widetilde{C}_h$. Setting $r = (E_{\alpha(h)}, \widetilde{C}_h)$ we have $\nu_\beta(Q_{\alpha(h)}^r) = \nu_\beta(h)$ for any $\beta \in \mathcal{G}(\pi)$ (see the remark after Proposition 2). So, it suffices to obtain q for the case $Q_{\alpha(h)}$, since then q^r would solve the problem for h.

Now, the monomial q_Δ given in Proposition 3 for any connected component Δ of $\mathcal{G}(\pi) \setminus \{\alpha(h)\}$ such that $\beta \notin \Delta$, if it exists, satisfies the requirements of the proposition. Moreover, if $\beta \neq \alpha(h)$, then $\beta \notin \Delta \cup \{\alpha(h)\}$ and this set is a connected subset of $\mathcal{G}(\pi) \setminus \{\beta\}$, thus $\Delta \cup \{\alpha(h)\}$ is contained in a connected component of $\mathcal{G}(\pi) \setminus \{\beta\}$.

Otherwise, that is β belongs to every connected component of $\mathcal{G}(\pi) \setminus \{\alpha(h)\}$, $\alpha(h)$ must be a dead end and we can take $q = Q_{\alpha(h)}$. $\square$

2. Semigroup of values

Let $V = \{\nu_1, \ldots, \nu_r\}$ be a finite set of $r \geqslant 1$ divisorial valuations and denote by $\mathbb{Z}_{\geqslant 0}$ the set of nonnegative integers. The **semigroup of values** of V is the additive subsemigroup S_V of $\mathbb{Z}_{\geqslant 0}^r$

given by

$$S_V = \{\underline{\nu}(h) := (\nu_1(h), \dots, \nu_r(h)) \mid h \in R \setminus \{0\}\}.$$

The **minimal resolution** of V is a modification $\pi : (X, E) \to (\mathtt{Spec}\, R, \mathfrak{m})$ such that, for each $i \in \{1, \dots, r\}$, ν_i is the $E_{\alpha(i)}$-valuation for an irreducible component $E_{\alpha(i)}$ of the exceptional divisor E, and π is minimal with this property. It is clear that a minimal resolution of V can be recursively obtained by blowing-up $\mathtt{Spec}\, R$ at $\mathfrak{m}$ and any new obtained space X_i at the closed centers of the valuations in V. The dual graph of V is the dual graph of π with the vertices $\alpha(i)$ highlighted (for example, using a different draw for the point, see Fig. 1).

Let $C = \bigcup_{i=1}^r C_i$ be a reduced curve, with components $C_1, \dots, C_r$, defined by an element $f \in R$, and denote by $R/(f)^*$ the set of nonzero divisors of the ring $R/(f)$. The **semigroup of values** S_C of C is the additive subsemigroup of $\mathbb{Z}^r_{\geqslant 0}$ given by

$$S_C := \{\underline{v}(g) = (v_1(g), \dots, v_r(g)) \mid g \in R/(f)^*\},$$

where each v_i is the valuation corresponding to C_i. Sometimes we will consider "the value" $\underline{v}(h)$ (not in S_C) of zero divisors of $R/(f)$, understanding $\nu_i(h) = \infty$ for h in the ideal of R defining C_i, and $n < \infty$ for any $n \in \mathbb{Z}_{\geqslant 0}$.

The dual graph of C is the dual graph of its minimal embedded resolution, attaching an arrow, for each irreducible component C_i of C, to the exceptional component which meets the strict transform on X of C_i. The equisingularity type of C (i.e., the set of Puiseux pairs for each branch C_i of C together with the intersection multiplicities between pairs of branches) and its dual graph, labeling each vertex α with the minimal number of blowing-ups needed to create E_α, $w(\alpha)$, are equivalent data.

Let $\mathcal{G}$ and S_V be the dual graph and the semigroup of values of a set $V = \{\nu_1, \dots, \nu_r\}$ of divisorial valuations, $r > 1$. A **general curve** of V is a reduced plane curve with r branches defined by r different equations given by general elements of each valuation ν_i. An element $\underline{m} \in S_V$ is said to be **indecomposable** if we cannot write $\underline{m} = \underline{n} + \underline{k}$ with $\underline{n}, \underline{k} \in S_V \setminus \{0\}$.

For $1 \leqslant i \leqslant r$ set $\alpha(i) = \alpha(\nu_i)$, and for each vertex $\rho \in \mathcal{E}$ denote by β_ρ the nearest vertex to ρ in $\Omega = \bigcup_{i=1}^r [\mathbf{1}, \alpha(i)]$ (i.e. $\beta_\rho = \max(\Omega \cap [\mathbf{1}, \rho])$). Consider the set

$$\mathcal{H} = \{\mathbf{1}\} \cup \mathcal{E} \cup (\Omega \setminus \{\Gamma \cup \{\beta_\rho \mid \rho \in \mathcal{E}\}\}),$$

where $\Gamma = \bigcap_{i=1}^r [\mathbf{1}, \alpha(i)]$. Then we can state the following

Theorem 1. *The set of indecomposable elements of the semigroup of values S_V is the set $\{\underline{\nu}(Q_\alpha) \mid \alpha \in \mathcal{H}\}$. In particular, S_V is finitely generated.*

This theorem is the divisorial version of the next one which holds for a reduced plane curve C with r branches [4]. In it, we consider the dual graph of $C = \bigcup_{i=1}^r C_i$ and define $\mathcal{H}$ as above, and $\alpha(1), \dots, \alpha(r)$ are the vertices with arrows, corresponding to the branches $C_1, \dots, C_r$ of C.

Theorem 2. *The set of indecomposable elements of the semigroup S_C is*

$$\{\underline{v}(Q_\alpha) \mid \alpha \in \mathcal{H}\} \cup \{\underline{v}(Q_{\alpha(i)}) + (0, \dots, 0, k, 0, \dots, 0) \mid i = 1, \dots, r,\ k \geqslant 1\},$$

where k is in the ith component.

Proof. Let us prove Theorem 1. If $C = \bigcup_{i=1}^{r} C_i$ is any general curve of V, that is, C_i is general for ν_i, then $S_V \subseteq S_C$, therefore, by Theorem 2, elements in the set $\{\underline{\nu}(Q_\alpha) \mid \alpha \in \mathcal{H}\}$ are indecomposable. Conversely, given $h \in R$ such that $\underline{\nu}(h)$ is indecomposable in S_V, choose a general curve C of V such that the strict transforms of C and C_h by the minimal resolution of V do not intersect. So, from equality (1), $\underline{\nu}(h) = \underline{v}(h)$ and $\underline{\nu}(Q_\alpha) = \underline{v}(Q_\alpha)$ for any vertex α, $\underline{v}$ given by the valuations associated to C. Moreover, h must be irreducible and by the proof of Theorem 2 [4], $\underline{v}(h)$ decomposes in S_C as a sum of elements $\underline{v}(Q_\gamma)$ with $\gamma \in \mathcal{H}$, which proves that $\underline{\nu}(h) = \underline{\nu}(Q_\alpha)$ for some $\alpha \in \mathcal{H}$. □

Remark. A consequence of Theorem 1 is that the semigroup S_V does not have conductor whenever $r > 1$, that is, there is no element $\delta \in S_V$ such that $\delta + \mathbb{Z}^r_{\geqslant 0} \subseteq S_V$. However, the semigroup of values of a curve with r branches does have a conductor δ [7, Th. 2.7], and thus, it cannot be finitely generated if $r > 1$. In particular, if C is any general curve of V, $S_V \neq S_C$ when $r > 1$ (recall that $S_V = S_C$ if $r = 1$).

Considering the ordering over $\mathbb{Z}^r$ given by $\underline{n} \leqslant \underline{m} \Leftrightarrow \underline{m} - \underline{n} \in \mathbb{Z}^r_{\geqslant 0}$, a finite set of divisorial valuations $V = \{\nu_1, \dots, \nu_r\}$ induces a multi-index filtration of the ring R by means of the valuation ideals $J(\underline{m})$, $\underline{m} = (m_1, \dots, m_r) \in \mathbb{Z}^r_{\geqslant 0}$:

$$J(\underline{m}) := \{ g \in R \mid \underline{\nu}(g) \geqslant \underline{m} \}.$$

For $J \subset \{1, \dots, r\}$ denote by $\underline{e}_J$ the element of $\mathbb{Z}^r_{\geqslant 0}$ whose ith component is equal to 1 (respectively, to 0) if $i \in J$ (respectively, $i \notin J$); denote $\underline{e} = \underline{e}_{\{1,\dots,r\}}$. We will use $\underline{e}_i$ instead of $\underline{e}_{\{i\}}$.

We will denote $D(\underline{m}) = J(\underline{m})/J(\underline{m} + \underline{e})$ and $D_i(\underline{m}) = J(\underline{m})/J(\underline{m} + \underline{e}_i)$ for $1 \leqslant i \leqslant r$. It is clear that the natural homomorphism $D(\underline{m}) \to D_1(\underline{m}) \times \cdots \times D_r(\underline{m})$ is injective. For $h \in J(\underline{m}) \setminus J(\underline{m} + \underline{e}_i)$ we will denote $\mathtt{in}_{\nu_i}(h) = h + J(\underline{m} + \underline{e}_i) \in D_i(\underline{m})$, and call it **the initial form of** h with respect to ν_i.

When $r = 1$, Nakayama's Lemma proves that for any $m \in \mathbb{Z}$, $D(m)$ is a finite-dimensional K-vector space and, therefore, so are $D(\underline{m})$ and $D_i(\underline{m})$ for $\underline{m} \in \mathbb{Z}^r_{\geqslant 0}$. Set $d(\underline{m}) = \dim D(\underline{m})$ and $d_i(\underline{m}) := \dim D_i(\underline{m})$.

In the sequel, we will set $B^i = \underline{\nu}(Q_{\alpha(i)})$, $i = 1, \dots, r$. Let $f \in R$ be such that $\nu_i(f) = \nu_i(Q_{\alpha(i)})$ (remember that $\alpha(i)$ denotes the vertex $\alpha(\nu_i)$ corresponding to the divisor that defines ν_i). Then by Proposition 1, $\nu_j(f) \geqslant \nu_j(Q_{\alpha(i)})$ for $j = 1, \dots, r$. Moreover, by Lemma 1, if $\widetilde{C}_f \cap \widetilde{C}_{Q(\alpha(i))} = \emptyset$, there exists a unique point $P(f)$ in $E_{\alpha(i)}$ mapped to ∞ by the lifting of the rational function $\varphi = Q_{\alpha(i)}/f$, namely, $P(f) = \widetilde{C}_f \cap E_{\alpha(i)}$ if $\widetilde{C}_f \cap E_{\alpha(i)} \neq \emptyset$ and $P(f) = E_\Delta \cap E_{\alpha(i)}$ if $\widetilde{C}_f \cap E_{\alpha(i)} = \emptyset$ and Δ is the connected component of $G(\pi) \setminus \{\alpha(i)\}$ such that $\widetilde{C}_f \cap E_\Delta \neq \emptyset$. Furthermore, we denote $P(f) = \widetilde{C}_{Q_{\alpha(i)}} \cap E_{\alpha(i)}$ whenever $\widetilde{C}_f \cap \widetilde{C}_{Q_{\alpha(i)}} \neq \emptyset$.

Proposition 5. *The map $\Phi : \mathbb{P}D_i(B^i) \to E_{\alpha(i)}$ from the projectivization of the vector space $D_i(B^i)$ to the exceptional component $E_{\alpha(i)}$, which sends the class $\mathtt{in}_{\nu_i}(f)$ to $P(f)$, is an isomorphism. In particular, $d_i(B^i) = 2$ and a basis of $D_i(B^i)$ is given by the initial forms of two elements f and g such that $P(f) \neq P(g)$ (e.g., two $Q_{\alpha(i)}$ elements at two different points in $E_{\alpha(i)}$).*

Proof. First of all, we assert that Φ is well defined. In fact, given $f, g \in J(B^i) \setminus J(B^i + \underline{e}_i)$ such that $\mathtt{in}_{\nu_i}(f) = \lambda\, \mathtt{in}_{\nu_i}(g)$, that is, $\nu_i(f - \lambda g) > \nu_i(f) = \nu_i(g) = \nu_i(Q_{\alpha(i)})$ for some $\lambda \in K \setminus \{0\}$, the liftings $\widetilde{\varphi}_1$ and $\widetilde{\varphi}_2$ of the rational functions $f/Q_{\alpha(i)}$ and $g/Q_{\alpha(i)}$ are defined in $E_{\alpha(i)}$, hence the lifting of $(f-\lambda g)/Q_{\alpha(i)}$ is also defined and it vanishes in $E_{\alpha(i)}$. This means that $\widetilde{\varphi}_1 = \lambda\widetilde{\varphi}_2$ in $E_{\alpha(i)}$ and so $P(f)=P(g)$.

It is evident that Φ is surjective, let us see that it is injective. Take $f, g \in J(B^i) \setminus J(B^i + \underline{e}_i)$ such that $P(f)=P(g)$. If $\widetilde{C}_f \cap \widetilde{C}_{Q_{\alpha(i)}} = \emptyset$, then $\widetilde{C}_g \cap \widetilde{C}_{Q_{\alpha(i)}} = \emptyset$, and, perhaps with some additional blowing-ups, we are in the situation of Proposition 2, so there exists $\lambda \in K \setminus \{0\}$ such that $\nu_i(f-\lambda g) > \nu_i(f)$, that is, $\mathtt{in}_{\nu_i}(f) = \lambda\, \mathtt{in}_{\nu_i}(g)$ as we want. Otherwise, $\widetilde{C}_g \cap \widetilde{C}_{Q_{\alpha(i)}} \neq \emptyset$ and since $\nu_i(f)=\nu_i(g)=\nu_i(Q_{\alpha(i)})$, f, g and $Q_{\alpha(i)}$ are irreducible, smooth and transversal to $E_{\alpha(i)}$. Making an additional blowing-up at the point $P = \widetilde{C}_g \cap E_{\alpha(i)} = \widetilde{C}_f \cap E_{\alpha(i)}$, we can conclude, applying again Proposition 2, that $\mathtt{in}_{\nu_i}(f) = \lambda\, \mathtt{in}_{\nu_i}(g)$ for some $\lambda \in K \setminus \{0\}$. □

Proposition 6. *The map $\widetilde{\Phi} : \mathbb{P}D(B^i) \to E_{\alpha(i)}$ which sends the class of f to $P(f)$, is an isomorphism. In particular $d(B^i)=2$.*

Proof. The result is a consequence of Proposition 5 and of the next lemma. □

Lemma 2. *The natural homomorphism $D(B^i) \to D_i(B^i)$ is an isomorphism.*

Proof. Let $f \in R$ be such that $\nu_j(f) \geqslant B^i_j = \nu_j(Q_{\alpha(i)})$ for every $j \in \{1,\ldots,r\}$ and $\nu_i(f) > B^i_i = \nu_i(Q_{\alpha(i)})$. We need to prove that $\nu_j(f) > B^i_j$ for any j.

Denote by Δ the maximal connected subset of $\mathcal{G}(\pi)$ such that $\alpha(i) \in \Delta$ and $\nu_\beta(f) > \nu_\beta(Q_{\alpha(i)})$ for every $\beta \in \Delta$. Notice that the lifting $\widetilde{\varphi}$ of the function $\varphi = f/Q_{\alpha(i)}$ is defined and it is identically 0 in E_Δ, in particular E_β is not dicritical for the pencil $L = \{\lambda f + \mu Q_{\alpha(i)} \mid \lambda, \mu \in K\}$ for any $\beta \in \Delta$. Let us see that $\Delta = \mathcal{G}(\pi)$, which proves the lemma.

Otherwise, we could choose a divisor E_β such that $E_\beta \cap E_\Delta \neq \emptyset$ and $\beta \notin \Delta$, that is, $\nu_\beta(f) \leqslant \nu_\beta(Q_{\alpha(i)})$. By making some additional blowing-ups, we can suppose that in fact $\nu_\beta(f) = \nu_\beta(Q_{\alpha(i)})$, then $\widetilde{\varphi}$ is defined and it is not constant in E_β, so it is dicritical for L. Hence, there exists a point $P \in E_\beta$, $P \neq E_\beta \cap E_\Delta$, such that $\widetilde{\varphi}(P) = \infty$, and this means that $Q_{\alpha(i)}$ meets either E_β at P or $E_{\Delta'}$, Δ' being the connected component of P in $\mathcal{G}(\pi) \setminus \{\beta\}$. But both things are impossible, as $Q_{\alpha(i)}$ only meets E at $E_{\alpha(i)}$, and $\alpha(i) \in \Delta \subset \mathcal{G}(\pi) \setminus \Delta'$. □

The following two lemmas are devoted to prove Theorem 3 which gives an explicit description of the semigroup S_V and clarifies the special role of the elements $B^1, \ldots, B^r$. Fix $\underline{m} \in \mathbb{Z}^r$ and i such that $1 \leqslant i \leqslant r$.

Lemma 3. *$d_i(\underline{m}) \geqslant 2$ if and only if $d_i(\underline{m} - B^i) \geqslant 1$. Moreover, if $\underline{m} \in S_V$, then $d_i(\underline{m}) \geqslant 2$ if and only if $\underline{m} - B^i \in S_V$.*

Proof. If $d_i(\underline{m} - B^i) \geqslant 1$, take $h \in J(\underline{m} - B^i) \setminus J(\underline{m} - B^i + \underline{e}_i)$ and choose a basis $\{\mathtt{in}_{\nu_i}(h_1), \mathtt{in}_{\nu_i}(h_2)\}$ of $D_i(B^i)$. Then $\mathtt{in}_{\nu_i}(hh_1)$, $\mathtt{in}_{\nu_i}(hh_2)$ are linearly independent vectors in $D_i(\underline{m})$.

Conversely, pick $h_1, h_2 \in J(\underline{m}) \setminus J(\underline{m} + \underline{e}_i)$ whose classes in $D_i(\underline{m})$ are linearly independent. Every nonzero function of the pencil L generated by h_1 and h_2, $L = \{\lambda h_1 + \mu h_2 \mid \lambda, \mu \in K\}$, satisfies $\nu_i(\lambda h_1 + \mu h_2) = m_i$, so $E_{\alpha(i)}$ is dicritical for L. Therefore, the restriction to $E_{\alpha(i)}$ of the lifting to X, $\widetilde{\varphi}$, of the rational function $\varphi = h_1/h_2$ defines an s to 1 surjective morphism from $E_{\alpha(i)}$ onto $\mathbb{P}^1_K$. Then, a generic fiber $h = \lambda h_1 + \mu h_2$ of L can be factorized in R as $h = h' \prod_{l=1}^{s} g_l$, where the g_l are irreducible, $g_l \neq g_j$ when $l \neq j$ and the strict transform of each curve C_{g_l} is smooth and transversal to $E_{\alpha(i)}$ in a smooth point. Therefore $\underline{\nu}(g_l) = B^i$ and $h/g_i \in J(\underline{m} - B^i)$ but $h/g_i \notin J(\underline{m} - B^i + \underline{e}_i)$.

Moreover, if $\underline{m} \in S_V$ then h_2 can be chosen in such a way that $\underline{\nu}(h_2) = \underline{m}$ and so for λ and μ generic we have $\underline{\nu}(h) = \underline{m}$ and $\underline{\nu}(h/g_i) = \underline{m} - B^i \in S_V$. □

Lemma 4.

1. *If* $\underline{m} \in S_V$ *and* $j \neq i$ *then* $d_i(\underline{m} + B^j) = d_i(\underline{m})$.
2. *If* $d_i(\underline{m}) \neq 0$ *then* $d_i(\underline{m} + B^i) = 1 + d_i(\underline{m})$.

Proof. First, we will prove that if $j \neq i$ and $\underline{m} \in S_V$ then the multiplication by $Q_{\alpha(j)}$ provides a linear bijective map $\psi : D_i(\underline{m}) \to D_i(\underline{m} + B^j)$. Clearly it is injective, let us see that it is also surjective. Pick an element $f \in R$ such that $\underline{\nu}(f) = \underline{m}$ and take $h \in J(\underline{m} + B^j) \setminus J(\underline{m} + B^j + \underline{e}_i)$. Notice that $\nu_j(h) \geqslant \nu_j(f Q_{\alpha(j)})$ and so $\nu_j(h - \lambda f Q_{\alpha(j)}) \geqslant \nu_j(f Q_{\alpha(j)})$ for $\lambda \in K$.

If $\nu_j(h - \lambda f Q_{\alpha(j)}) > \nu_j(f Q_{\alpha(j)})$, for some $\lambda \in K$, then there exists an irreducible component g of h such that the strict transforms of C_g and $C_{Q_{\alpha(j)}}$ by the minimal resolution of V intersect $E_{\alpha(j)}$ at the same point, and then $\mathtt{in}_{\nu_j}(g) = b \cdot \mathtt{in}_{\nu_j}(Q_{\alpha(j)})^c$ for some $c \geqslant 1$ and $b \in K \setminus \{0\}$ (see the remark after Proposition 2). Thus $h' = bhQ^c_{\alpha(j)}/g$ and h have the same value and initial form with respect to ν_k for $1 \leqslant k \leqslant r$. In particular, $\mathtt{in}_{\nu_i}(h) = \mathtt{in}_{\nu_i}(h') \in \mathtt{Im}\,\psi$.

Otherwise, $\nu_j(h - \lambda f Q_{\alpha(j)}) = \nu_j(f Q_{\alpha(j)})$ for all $\lambda \in K$ and then $E_{\alpha(j)}$ is a dicritical divisor of the pencil generated by h and $f Q_{\alpha(j)}$. Thus, for a generic λ, $h - \lambda f Q_{\alpha(j)}$ has an irreducible component g such that $\widetilde{C}_g$ is smooth and transversal to $E_{\alpha(j)}$ at a smooth point. As $i \neq j$, by Proposition 2, there exists $b \in K \setminus \{0\}$ such that $\mathtt{in}_{\nu_i}(g) = b\,\mathtt{in}_{\nu_i}(Q_{\alpha(j)})$. Then $h' = (h - \lambda f Q_{\alpha(j)})/g \in J(\underline{m}) \setminus J(\underline{m} + \underline{e}_i)$ and $\mathtt{in}_{\nu_i}(gh') = \mathtt{in}_{\nu_i}(bQ_{\alpha(j)}h') \in \mathtt{Im}\,\psi$. Hence $\mathtt{in}_{\nu_i}(h) = \lambda\,\mathtt{in}_{\nu_i}(f Q_{\alpha(j)}) + \mathtt{in}_{\nu_i}(gh') \in \mathtt{Im}\,\psi$.

Now, we will prove 2. Assume $j = i$ and pick elements $h_1, \ldots, h_s \in J(\underline{m}) \setminus J(\underline{m} + \underline{e}_i)$ such that the set $\{\mathtt{in}_{\nu_i}(h_l) \mid 1 \leqslant l \leqslant s\}$ is a basis of $D_i(\underline{m})$. Take an irreducible element $g \in R$ such that $\widetilde{C}_g$ is smooth and transversal to $E_{\alpha(i)}$ at a smooth point P, $\widetilde{C}_g \cap \widetilde{C}_{Q_{\alpha(i)}} = \emptyset$ and $\widetilde{C}_g \cap \widetilde{C}_{h_s} = \emptyset$. Then $\mathtt{in}_{\nu_i} h_1 g, \ldots, \mathtt{in}_{\nu_i} h_s g, \mathtt{in}_{\nu_i} h_s Q_{\alpha(i)}$ are linearly independent in the vector space $D_i(\underline{m} + B^i)$, because in other case we could find $h = \sum \lambda_i h_i \in J(\underline{m}) \setminus J(\underline{m} + \underline{e}_i)$ and $\lambda \neq 0$ with $\nu_i(hg - \lambda h_s Q_{\alpha(i)}) > \nu_i(hg)$ and then $\widetilde{C}_{h_s Q_{\alpha(i)}}$ must intersect $E_{\alpha(i)}$ at the point P (see again the remark after Proposition 2), in contradiction with the election of g. Hence, $d_i(\underline{m} + B^i) \geqslant d_i(\underline{m}) + 1$.

To finish the proof, it suffices to show that if $d_i(\underline{m} + B^i) = t \geqslant 2$ then $d_i(\underline{m}) \geqslant t - 1$. In fact, let $\{\mathtt{in}_{\nu_i} g_1, \ldots, \mathtt{in}_{\nu_i} g_t\}$ be a basis of $D_i(\underline{m} + B^i)$ and consider the family of pencils $L_k = \{\lambda g_1 + \mu g_k\}$, $2 \leqslant k \leqslant t$. Fix a smooth point $P \in E_{\alpha(i)}$ in such a way that P is noncritical for all the pencils L_k. For each $k = 2, \ldots, t$, let $\lambda g_1 + \mu g_k = \varphi_k g'_k$ be the fiber of L_k corresponding to P and φ_k the unique irreducible component of such fiber by P. In this way, all the initial forms of φ_k are equal (up to product by constants). Set $\mathcal{B} = \{g_1, \varphi_2 g'_2, \ldots, \varphi_t g'_t\}$. Then, for generic P, $\mathtt{in}_{\nu_i}(\mathcal{B})$ is a basis of $D_i(\underline{m} + B^i)$, and $\mathtt{in}_{\nu_i}(g'_2), \ldots, \mathtt{in}_{\nu_i}(g'_t) \in D_i(\underline{m})$ are linearly independent elements. Thus $d_i(\underline{m}) \geqslant t - 1$ and the proof is finished. □

Theorem 3. *For any* $\underline{m} \in S_V$ *there exist unique* $a_1, \ldots, a_r \in \mathbb{Z}_{\geqslant 0}$ *and* $\underline{n} \in S_V$ *such that*

1. $\underline{m} = \underline{n} + a_1 B^1 + \cdots + a_r B^r$.
2. $d_i(\underline{n}) = 1$ *for every* $i = 1, \ldots, r$.

In fact $a_i = \max\{k \in \mathbb{Z}_{\geqslant 0} \mid \underline{m} - kB^i \in S_V\} = d_i(\underline{m}) - 1$ *for* $i = 1, \ldots, r$.

Proof. Assume the existence of the values $a_1, \dots, a_r, \underline{n}$, then, by Lemma 4, $1 = d_i(\underline{n}) = d_i(\underline{m} - a_i B^i) = d_i(\underline{m}) - a_i$, and by Lemma 3, $\underline{n} - B^i \notin S_V$, so $a_i = \max\{k \in \mathbb{Z}_{\geqslant 0} \mid \underline{m} - kB^i \in S_V\}$, and we have the uniqueness. We also have $a_i = d_i(\underline{m}) - 1$.

For the existence, define $a_i = \max\{k \in \mathbb{Z}_{\geqslant 0} \mid \underline{m} - kB^i \in S_V\}$ and $\underline{n} = \underline{m} - \sum_k a_k B^k$. To prove $\underline{n} \in S_V$ it suffices to prove that if $\underline{m} - B^i \in S_V$ and $\underline{m} - B^j \in S_V$ then $\underline{m} - B^i - B^j \in S_V$. The conditions $\underline{m} - B^i \in S_V$ and $\underline{m} - B^j \in S_V$ imply, by Lemmas 3 and 4 that $d_j(\underline{m} - B^i) = d_j(\underline{m}) \geqslant 2$ and hence that $\underline{m} - B^i - B^j \in S_V$. $\square$

Corollary 1. *Given a modification π and the family of all the valuations associated to the components $\{E_1, \dots, E_s\}$ of the exceptional divisor of π, $W = \{\nu_1, \dots, \nu_s\}$, it holds that $S_W = \langle B^1, \dots, B^s \rangle \cong \mathbb{Z}^s_{\geqslant 0}$.*

Remark. The above corollary, established here as a consequence of Theorem 3, was already known, since the determinant of the intersection matrix of the components $\{E_1, \dots, E_s\}$ of the exceptional divisor of π, $M = (E_i \cdot E_j)$, is -1, the s rows of $A = -M^{-1}$ are exactly the values $\{B^1, \dots, B^s\}$ and $S_W = \{\underline{m} \in \mathbb{Z}^s_{\geqslant 0} \mid -\underline{m}M \geqslant 0\}$. Thus the semigroup is the free semigroup generated by the vectors $B^1, \dots, B^s$.

In the general case, valuations in V are those corresponding to a subset L of $\{1, \dots, s\}$, $V = V_L = \{\nu_l \mid l \in L\}$, and then S_{V_L} is the projection over $\mathbb{Z}^{|L|}_{\geqslant 0}$ (that is, over the coordinates in L) of the semigroup S_W, so it is contained in the convex polyhedral cone in $\mathbb{R}^{|L|}_{\geqslant 0}$ generated by the elements $\{B_l \mid l \in L\}$.

3. Graded algebra and generating sequences

Throughout this section, we will consider a nonempty finite set of divisorial valuations $V = \{\nu_1, \dots, \nu_r\}$ and we will use the notations of the above sections. The **graded K-algebra** associated to V is defined to be

$$\mathrm{gr}_V R := \bigoplus_{\underline{m} \in \mathbb{Z}^r_{\geqslant 0}} \frac{J(\underline{m})}{J(\underline{m} + \underline{e})}.$$

Set $\Lambda = \{u_j\}_{j \in J}$ a subset of the maximal ideal $\mathfrak{m}$ of R. A monomial in Λ is a product $\prod_{j \in J} u_j^{\gamma_j}$ with $\gamma_j \in \mathbb{Z}_{\geqslant 0}$ and $\gamma_j = 0$ except for a finite subset of J. Let $\mathcal{M}(\Lambda)$ denote the set of monomials in Λ, we will say that Λ is a **generating sequence** of V if for each $\underline{m} \in \mathbb{Z}^r_{\geqslant 0}$ the ideal $J(\underline{m})$ is generated by $\mathcal{M}_{\underline{m}}(\Lambda) := \mathcal{M}(\Lambda) \cap J(\underline{m})$. In particular, Λ is a system of generators of $\mathfrak{m}$.

A generating sequence Λ of V is said to be minimal whenever each proper subset of Λ fails to be a generating sequence. In this case V is said to be **monomial with respect to Λ**. Generating sequences of a family V and its graded algebra $\mathrm{gr}_V R$ are closely related, as the following result (proved in [2] in a more general context) shows:

Theorem 4. *Assume that there exists a finite generating sequence for some valuation of V. Then, a system of generators $\Lambda = \{u_j\}_{j \in J}$ of the maximal ideal $\mathfrak{m}$ is a generating sequence of V if and only if the K-algebra $\mathrm{gr}_V R$ is generated by the set $\bigcup_{j \in J} [u_j]$, where $[u_j]$ denotes the cosets that u_j defines in $\mathrm{gr}_V R$.*

It is convenient to clarify the sense of the notation $[u]$ in the above theorem: if $u \in \mathtt{m}$ and $\underline{m} = \underline{v}(u)$, then $u \in J(\underline{n})$ for any $\underline{n} \leqslant \underline{m}$. Denote $[u]_{\underline{n}} := u + J(\underline{n} + \underline{e})$. So, $[u]_{\underline{n}} \neq 0$ if, and only if, $\underline{n} + \underline{e} \nleqslant \underline{m}$ (that is, $n_i = m_i$ for some index $i \in \{1, \dots, r\}$). Then, $[u]$ in Theorem 4 means $[u] := \{[u]_{\underline{n}} \mid \underline{n} \leqslant \underline{m} \text{ and } \underline{n} + \underline{e} \nleqslant \underline{m}\}$.

Denote by $\mathcal{E}$ the set of dead ends of the dual graph of V and fix an element $Q_\rho \in R$ for each $\rho \in \mathcal{E}$. Set

$$\Lambda_{\mathcal{E}} = \{Q_\rho \mid \rho \in \mathcal{E}\}.$$

Next result is the analogous of Proposition 4 for initial forms of elements in R.

Proposition 7. *Given $h \in R$ and $i \in \{1, \dots, r\}$, there exists a linear combination of monomials $q = \sum a_\lambda q^\lambda$, $q^\lambda = \prod_{\rho \in \mathcal{E}} Q_\rho^{\lambda_\rho}$, such that $\nu_i(q) = \nu_i(h)$, $\nu_i(h - q) > \nu_i(h)$ and $\nu_j(q) \geqslant \nu_j(h)$ for every index $j \neq i$.*

Proof. Note that the condition $\nu_i(h - q) > \nu_i(h)$ is equivalent to $\mathtt{in}_{\nu_i}(q) = \mathtt{in}_{\nu_i}(h)$. Thus, it suffices to prove the result for h irreducible. Let π be the minimal modification such that π is a resolution of V and the strict transform $\widetilde{C}_h$ of C_h by π only meets one irreducible component of the exceptional divisor of π, $E_{\alpha(h)}$.

If $\alpha(h) \neq \alpha(i)$ then, by Proposition 2, there exists λ such that λq, q being the monomial constructed in Proposition 4, satisfies the result. Assume that $\alpha(h) = \alpha(i)$ and choose $Q_{\alpha(h)}$ such that $\widetilde{C}_{Q_{\alpha(h)}}$ goes through the intersection point $E_\alpha \cap \widetilde{C}_h$. Denoting $m = (E_{\alpha(h)}, \widetilde{C}_h)$, we have $\nu_j(h) = \nu_j(Q_\alpha^m)$ for $1 \leqslant j \leqslant r$, and $\mathtt{in}_{\nu_i}(h) = \lambda\, \mathtt{in}_{\nu_i}(Q_\alpha^m)$ for some $\lambda \in K \setminus \{0\}$ (see the remark after Proposition 2), so we only need to prove the statement for $h = Q_{\alpha(i)}$.

Let $\Delta_0, \dots, \Delta_s$ be the connected components of $\mathcal{G}(\pi) \setminus \{\alpha(i)\}$ and q_{Δ_i}, $0 \leqslant i \leqslant s$ the monomial constructed in Proposition 3 for Δ_i. If $s \geqslant 1$, by Proposition 5, the classes of any pair q' and q'' of such monomials are a basis of $D_i(\underline{v}(Q_{\alpha(i)}))$, thus $\mathtt{in}_{\nu_i}(h) = \lambda\, \mathtt{in}_{\nu_i}(q') + \mu\, \mathtt{in}_{\nu_i}(q'')$ for some $\lambda, \mu \in K$ and the linear combination $q = \lambda q' + \mu q''$ satisfies the requirements of the statement. Finally, if $s = 0$, the vertex $\alpha(i)$ is an end vertex and we can use $Q_{\alpha(i)}$ together with q_{Δ_0} to have a basis of $D_i(\underline{v}(Q_{\alpha(i)}))$. $\square$

Now, let C be a reduced plane curve with r branches, $C_1, \dots, C_r$, and local ring $\mathcal{O} = R/(f)$, and denote $\underline{v} := (v_1, \dots, v_r)$, where v_i is the valuation associated to C_i. We will say that $\Lambda \subset \mathtt{m}$ is a generating sequence of C if the valuation ideals $J^C(\underline{m}) = \{g \in \mathcal{O} \mid \underline{v}(g) \geqslant \underline{m}\}$ are generated by the images in $\mathcal{O}$ of the monomials in Λ. We will set $c(\underline{m}) := \dim C(\underline{m})$, where $C(\underline{m}) = \frac{J^C(\underline{m})}{J^C(\underline{m}+\underline{e})}$ is the corresponding vector space of initial forms. Finally, we define the graded K-algebra of $\mathcal{O}$ as

$$\operatorname{gr}\mathcal{O} := \bigoplus_{\underline{m} \in \mathbb{Z}^r_{\geqslant 0}} \frac{J^C(\underline{m})}{J^C(\underline{m} + \underline{e})}.$$

Denote by $\mathcal{E}$ the set of dead ends of the dual graph of C and let f_i be an element in R that gives an equation for C_i, $(1 \leqslant i \leqslant r)$. We define

$$\Lambda_{\overline{\mathcal{E}}} = \{Q_\alpha \mid \alpha \in \mathcal{E}\} \cup \{f_1, \dots, f_r\},$$

where we do not include $f = f_1$ if $r = 1$.

Reduced curves can be approached by finite sets of divisorial valuations. Indeed, denote by $\pi^{(0)}: X^{(0)} \to \operatorname{Spec} R$ the minimal embedded resolution of the curve C, and by $\pi^{(k)}: (X^{(k)}, E^{(k)}) \to \operatorname{Spec} R$ the composition of $\pi^{(k-1)}$ with r additional blowing-ups, one at each point where the strict transform of C intersects $E^{(k-1)}$. Set $\nu_i^{(k)}$ the $E_{\alpha^{(k)}(i)}$-valuation, $E_{\alpha^{(k)}(i)}$ being the irreducible component of the exceptional divisor $E^{(k)}$ intersected by the strict transform of the branch C_i. Then, the sequence $V^{(k)} = \{\nu_1^{(k)}, \dots, \nu_r^{(k)}\}$ approaches C, in the sense that for any element $h \in R$ which is not divisible by any f_i, $\nu_i^{(k)}(h) = (f_i, h) = v_i(h)$ for $k \gg 0$ and $1 \leqslant i \leqslant r$.

For $1 \leqslant i \leqslant r$, denote by $\alpha^{(0)}(i)$ the vertex of $\mathcal{G}(\pi^{(0)})$ such that the strict transform of the branch C_i meets $E_{\alpha^{(0)}(i)}$. Then, for any $k > 0$, the graph of $V^{(k)}$, $\mathcal{G}(\pi^{(k)})$, is obtained by adding r vertices $\alpha^{(k)}(1), \dots, \alpha^{(k)}(r)$ (corresponding to the components $E_{\alpha^{(k)}(i)}$) to $\mathcal{G}(\pi^{(k-1)})$, each $\alpha^{(k)}(i)$ adjacent to $\alpha^{(k-1)}(i)$. Denoting by $\mathcal{E}^{(k)}$ the set of dead ends of $\mathcal{G}(\pi^{(k)})$, it is clear that $\mathcal{E}^{(k)} = (\mathcal{E}^{(k-1)} \setminus \{\alpha^{(k-1)}(1), \dots, \alpha^{(k-1)}(r)\}) \cup \{\alpha^{(k)}(1), \dots, \alpha^{(k)}(r)\}$ for $k \geqslant 1$.

Moreover, for each $k \geqslant 0$, the strict transform by $\pi^{(k)}$ of the branch C_i is smooth and meets $E_{\alpha^{(k)}(i)}$ transversally at a nonsingular point, so we can choose $Q_{\alpha^{(k)}(i)} = f_i$. In this way, for every $k \geqslant 1$, when $r > 1$ we have $\Lambda_{\mathcal{E}^{(k)}} = \Lambda_{\overline{\mathcal{E}}}$, and $\Lambda_{\mathcal{E}^{(k)}} = \Lambda_{\overline{\mathcal{E}}} \cup \{f_1\}$ in the case $r = 1$.

Note that $\mathcal{G}(\pi^{(k)}) \subset \mathcal{G}(\pi^{(k+1)})$, and the (infinite) graph obtained by blowing-up every infinitely near point of C, is exactly the union $\bigcup_{k\geqslant 0} \mathcal{G}(\pi^{(k)})$. Analogously, if $S_{V^{(k)}}$ denotes the value semigroup of the set $V^{(k)}$, one gets the inclusion chain $S_{V^{(0)}} \subseteq S_{V^{(1)}} \subseteq \cdots$ and the equality $S_C = \bigcup_{k\geqslant 0} S_{V^{(k)}}$.

Theorem 5. *Let V and C be as above. Then, $\Lambda_{\mathcal{E}}$ ($\Lambda_{\overline{\mathcal{E}}}$, respectively) is a minimal generating sequence of V (C, respectively).*

Proof. Let us prove first the result for the curve C. Consider the sequence $V^{(k)}$ as explained above, in such a way that $\Lambda_{\mathcal{E}^{(k)}} = \Lambda_{\overline{\mathcal{E}}}$ for every k if $r > 1$ and $\Lambda_{\overline{\mathcal{E}}} = \Lambda_{\mathcal{E}^{(k)}} \setminus \{f_1\}$ if $r = 1$. Let $h \in R$ be such that $h \notin (f)$. We claim that there exists a monomial q_1 in $\Lambda_{\overline{\mathcal{E}}}$ such that $v_1(h) = v_1(q_1)$ and $v_i(h) \leqslant v_i(q_1)$ for $i = 1, \dots, r$. To prove the claim it is enough to find such a monomial for each irreducible component of h, so assume h irreducible. Moreover, if $h = f_i$ for some i, then we can take $q_1 = f_i$. Otherwise, take $k \gg 0$ such that the strict transform of C_h by $\pi^{(k)}$ does not intersect any of the components $E_{\alpha^{(k)}(i)}$, $1 \leqslant i \leqslant r$. Then, $\nu_i^{(k)}(h) = v_i(h)$ for $i = 1, \dots, r$, and applying Proposition 4 to the set $V^{(k)}$ we find a monomial $q_1 = \prod_{\rho \in \mathcal{E}^{(k)}} Q_\rho^{\lambda_\rho}$ such that $\nu_1^{(k)}(h) = \nu_1^{(k)}(q_1)$, $\nu_i^{(k)}(h) \leqslant \nu_i^{(k)}(q_1)$ for $i = 1, \dots, r$ and $\lambda_{\alpha^{(k)}(1)} = 0$. In particular, $f_1 = Q_{\alpha^{(k)}(1)}$ does not appear in the expression of q_1, so q_1 is a monomial in $\Lambda_{\overline{\mathcal{E}}}$ even in case $r = 1$. Moreover, $\nu_1^{(k)}(q_1) = v_1(q_1)$ and $\nu_i^{(k)}(q_1) \leqslant v_i(q_1)$ for any i, therefore $v_1(h) = v_1(q_1)$ and $v_i(h) \leqslant v_i(q_1)$ for $i = 1, \dots, r$.

So, for $h \in R \setminus (f)$ we have the monomial q_1 of the claim, and there exists a nonzero constant a_1 with $v_1(h - a_1 q_1) > v_1(h)$ and $v_i(h - a_1 q_1) \geqslant v_i(h)$ for $i \geqslant 2$. The same claim can be applied to an index (if it exists) $i \geqslant 2$ such that $v_i(h - a_1 q_1) = v_i(h)$ and the element $h - a_1 q_1$, and iteratively we find a linear combination of monomials $p = \sum a_i q_i$ satisfying $\underline{v}(h - p) \geqslant \underline{v}(h) + \underline{e}$. Now, if $\mathcal{E} \neq \emptyset$ choose any $\rho \in \mathcal{E}$ and set $Q = Q_\rho$, and if $\mathcal{E} = \emptyset$ (in particular $r \geqslant 2$) choose generic $\lambda_1, \dots, \lambda_r$ in K^* and set $Q = \lambda_1 f_1 + \cdots + \lambda_r f_r$. Repeating the above procedure the times we need, we can finally obtain a finite linear combination q of monomials in $\mathcal{M}(\Lambda_{\overline{\mathcal{E}}})$ such that $\underline{v}(h - q) \geqslant \delta + \underline{v}(Q)$ where δ is the conductor of the semigroup S_C. The element $g = (h - q)/Q$ of the total ring of fractions of $\mathcal{O}$ has value $\underline{v}(g) \geqslant \delta$, in particular it belongs to the integral closure $\overline{\mathcal{O}}$ of the ring $\mathcal{O}$ in its total ring of fractions (since $\overline{\mathcal{O}}$ is in fact the set of elements ϕ of the total ring of fractions such that $v_i(\phi) \geqslant 0$ for all $i = 1, \ldots, r$). Moreover, the conductor ideal of $\overline{\mathcal{O}}$ in $\mathcal{O}$ coincides with the valuation ideal $J^C(\delta)$, so $g \in \mathcal{O}$ and then $h = q + gQ$ belongs to the ideal generated by $\mathcal{M}_{\underline{v}(h)}(\Lambda_{\bar{\mathcal{E}}})$. Thus, the set $\Lambda_{\bar{\mathcal{E}}}$ is a finite generating sequence for the plane curve C.

Now, we prove the theorem for the set $V = \{\nu_1, \ldots, \nu_r\}$. The case $r = 1$ is proved in [16], hence, by Theorem 4, it suffices to show that for any $h \in R$, one can find a linear combination of monomials q in $\Lambda_{\mathcal{E}}$ such that $\nu_i(h - q) > \nu_i(h)$ for all $i = 1, \ldots, r$. Proposition 7, applied recursively for $i = 1, \ldots, r$, gives a finite sequence of polynomials $q_1, \ldots, q_r$ in $\Lambda_{\mathcal{E}}$ such that $\nu_j(h - \sum_{k=1}^{i} q_k) > \nu_j(h)$ for $j \leqslant i$ and $\nu_j(h - \sum_{k=1}^{i} q_k) \geqslant \nu_j(h)$ for $j = 1, 2, \ldots, r$. Hence, $q = \sum_{k=1}^{r} q_k$ satisfies our requirements.

To prove the minimality, it is enough to check that any generating sequence must have an element of type Q_ρ (that is, irreducible and with strict transform smooth and transversal to E_ρ at a nonsingular point) for each $\rho \in \mathcal{E}$.

Suppose $r = 1$, and consider the minimal set of generators of the semigroup S_C or S_V, $\bar{\beta}_0, \ldots, \bar{\beta}_g$ (corresponding to $\mathcal{E} = \{\rho_0, \ldots, \rho_g\}$). In order to generate $J(\bar{\beta}_i)$ $(0 \leqslant i \leqslant g)$, we need at least an element $h \in R$ such that $v_1(h) = \nu_1(h) = \bar{\beta}_i$. But it is known (see e.g. [7]) that in this case h must be of type Q_{ρ_i}. In the divisorial case, if $\alpha(1)$ is a dead end, moreover we have to consider $\bar{\beta}_{g+1} = \nu_1(Q_{\alpha(1)})$ and from Proposition 5 we deduce that to generate $J(\bar{\beta}_{g+1})$ we need some element of type $Q_{\alpha(1)}$, since all the elements $h \in R$ such that $\nu_1(h) = \bar{\beta}_{g+1}$ and $\widetilde{C}_h \cap E_{\alpha(1)} = \emptyset$ have the same initial form.

Now, assume $r > 1$. Notice that for any $W \subset V$ and $\underline{m}' \in S_W$, $J(\underline{m}') = J(\min pr_W^{-1}(\underline{m}'))$, where $pr_W : S_V \to S_W$ is the projection map. Thus, any generating sequence for V is also a generating sequence for W and in particular for ν_i, $1 \leqslant i \leqslant r$. On the other hand, if $\rho \in \mathcal{G}(\pi)$ is a dead end of the minimal resolution π of V, then there exists $i \in \{1, \ldots, r\}$ such that ρ is a dead end of the minimal resolution of ν_i. Therefore we cannot delete any Q_ρ in our generating sequence and a similar argument holds for curves. Finally, in this last case, if $v_j(h) = v_j(f_i)$, $j \neq i$, and $h \neq f_i$, then $v_i(h) < k$ for some positive integer k and $\underline{m} = (v_1(f_i), \ldots, k, \ldots, v_r(f_i))$, where k is in the ith coordinate, belongs to S_C; hence no f_i can be omitted to generate $J^C(\underline{m})$. □

Remark. We have also proved that minimal generating sequences for V and C must be of the form given in Theorem 5. On the other hand, Theorem 4 is also true for the case of curves, so $[\Lambda_{\bar{\mathcal{E}}}] = \{[Q_\rho]\} \cup \{[f_1], \ldots, [f_r]\}$ is a set of generators of $\operatorname{gr}\mathcal{O}$. However, here the set $[\Lambda_{\bar{\mathcal{E}}}]$ has **infinitely many elements**, because $[f_i]_{\underline{n}} \neq 0$ for infinitely many elements $\underline{n} \in S_C$, since $v_i(f_i) = \infty$.

4. Poincaré series

Along this section we will suppose $r > 1$. Let $\mathcal{L} := \mathbb{Z}[\![t_1, t_1^{-1}, \ldots, t_r, t_r^{-1}]\!]$ be the set of formal Laurent series in $t_1, \ldots, t_r$ and $\underline{t}^{\underline{m}} := t_1^{m_1} \cdot \cdots \cdot t_r^{m_r}$ for $\underline{m} = (m_1, \ldots, m_r) \in \mathbb{Z}^r$. $\mathcal{L}$ is not a ring, but it is a $\mathbb{Z}[t_1, \ldots, t_r]$-module and a $\mathbb{Z}[t_1, t_1^{-1}, \ldots, t_r, t_r^{-1}]$-module. For a reduced plane curve C with r branches, the formal Laurent series

$$L_C(t_1, \ldots, t_r) = \sum_{\underline{m} \in \mathbb{Z}^r} c(\underline{m}) \cdot \underline{t}^{\underline{m}} \in \mathcal{L}$$

was introduced in [6], where it was shown that

$$P'_C(t_1,\dots,t_r) = L_C(t_1,\dots,t_r)\cdot\prod_{i=1}^{r}(t_i-1)$$

is in fact a polynomial that is divisible by $t_1\cdots t_r - 1$. The Poincaré series for the curve C was defined as the polynomial with integer coefficients

$$P_C(t_1,\dots,t_r) = \frac{P'_C(t_1,\dots,t_r)}{t_1\cdots t_r - 1}.$$

Analogously, for a set of divisorial valuations $V = \{\nu_1,\dots,\nu_r\}$, we define

$$L_V(t_1,\dots,t_r) = \sum_{\underline{m}\in\mathbb{Z}^r} d(\underline{m})\cdot \underline{t}^{\underline{m}} \in \mathcal{L}.$$

L_V is a Laurent series, but, since $d(\underline{m})$ can be positive even if $\underline{m}$ have some negative component m_i, it is not a power series (in fact, as in the case of a curve, it contains infinitely many terms with negative powers). In Proposition 8, we will show that

$$P'_V(t_1,\dots,t_r) = L_V(t_1,\dots,t_r)\cdot\prod_{i=1}^{r}(t_i-1) \in \mathbb{Z}[\![t_1,\dots,t_r]\!].$$

Thus, we define the **Poincaré series** of V as the formal power series with integer coefficients

$$P_V(t_1,\dots,t_r) = \frac{P'_V(t_1,\dots,t_r)}{t_1\cdots t_r - 1}.$$

Remark. It is not clear a priori whether the Laurent series L_V can be computed from the Poincaré series P_V, since in $\mathcal{L}$ there are elements which vanish after multiplication by $\prod_{i=1}^{r}(t_i-1)$. So, it is not obvious how to recover neither the Hilbert function of the graded ring $\mathrm{gr}_V R$, $d(\underline{m})$, $\underline{m}\in\mathbb{Z}^r_{\geqslant 0}$, nor the Hilbert function of the multi-index filtration of the ring R, $h(\underline{m}) := \dim R/J(\underline{m})$, $\underline{m}\in\mathbb{Z}^r_{\geqslant 0}$. This can be done, following [8], as follows: denote $I = \{1,\dots,r\}$, and define

$$\widetilde{L}_V(t_1,\dots,t_r) = \sum_{\underline{m}\in\mathbb{Z}^r_{\geqslant 0}} d(\underline{m})\cdot \underline{t}^{\underline{m}} \in \mathbb{Z}[\![t_1,\dots,t_r]\!],$$

and $\widetilde{P}'_V(t_1,\dots,t_r) = \widetilde{L}_V(t_1,\dots,t_r)\cdot\prod_{i=1}^{r}(t_i-1)$. The formula

$$\widetilde{P}'_V(t_1,\dots,t_r) = \sum_{J\subset I}(-1)^{\#J} P'_V(t_1,\dots,t_r)|_{\{t_i=1 \text{ for } i\in J\}}$$

($\#J$ denoting the cardinality of J) allows to determine the series $\widetilde{P}'_V$ from the series P'_V, and as a consequence the power series $\widetilde{L}_V$. Finally, $\widetilde{L}_V$ determines the Laurent series L_V, since $d(\underline{m}) = d(\max(m_1,0),\dots,\max(m_r,0))$ for $\underline{m}\nleqslant -\underline{1}$ and $d(\underline{m}) = 0$ for $\underline{m}\leqslant -\underline{1}$.

To get h, set $H(t_1,\dots,t_r) := \sum_{\underline{m}\in\mathbb{Z}^r} h(\underline{m})\cdot \underline{t}^{\underline{m}} \in \mathcal{L}$, where $h(\underline{m}) = \dim R/J(\underline{m})$ for $\underline{m}\in\mathbb{Z}^r$ (notice that $h(\underline{m}) = 0$ if $\underline{m} \leqslant 0$). The equality $H(t_1,\dots,t_r) = L_V(t_1,\dots,t_r)(1+\underline{t}^{(1,\dots,1)} + \underline{t}^{(2,\dots,2)} + \cdots)$ solves our problem. Note that the right-hand side of the last equality makes sense since $d(\underline{m}) = 0$ for $\underline{m} \leqslant -\underline{1}$.

The following results involve dimensionality of the homogeneous components of the graded algebra relative to finite sets $V = \{\nu_1,\dots,\nu_r\}$ of divisorial valuations. They will be useful to relate the Poincaré series of V and the one of any general curve for V. For $i\in I = \{1,2,\dots,r\}$, define

$$p_i(\underline{m}) := \sum_{J\subset I\setminus\{i\}} (-1)^{\#J} d_i(\underline{m}+\underline{e}_J) \quad \text{and}$$

$$P_i(t_1,\dots,t_r) := \sum_{\underline{m}\in\mathbb{Z}^r} p_i(\underline{m})\underline{t}^{\underline{m}} \in \mathcal{L}. \tag{3}$$

Proposition 8. *Let V be a finite set of r divisorial valuations, then*

$$P'_V(t_1,\dots,t_r) = (t_1t_2\cdots t_r - 1)P_i(t_1,\dots,t_r) \in \mathbb{Z}[[t_1,\dots,t_r]].$$

As a consequence $P_V(t_1,\dots,t_r) = P_i(t_1,\dots,t_r)$ does not depend on the index i chosen. Moreover, if we write $P_V(t_1,\dots,t_r) = \sum_{\underline{m}\in\mathbb{Z}^r} p(\underline{m})\underline{t}^{\underline{m}}$, then $\underline{m}\in S_V$ whenever $p(\underline{m}) \neq 0$ and so $P_V(t_1,\dots,t_r) \in \mathbb{Z}[[t_1,\dots,t_r]]$.

Proof. We shall show the first statement for $i = 1$ for the sake of simplicity. Write $P'_V(t_1,\dots, t_r) = \sum_{\underline{m}\in\mathbb{Z}^r} \ell(\underline{m})\underline{t}^{\underline{m}} \in \mathcal{L}$. Then

$$\begin{aligned}
\ell(\underline{m}) &= \sum_{J\subset I} (-1)^{\#J} d(\underline{m} - \underline{e} + \underline{e}_J) \\
&= \sum_{J\subset I\setminus\{1\}} (-1)^{\#J} \big(d(\underline{m} - \underline{e} + \underline{e}_J) - d(\underline{m} - \underline{e} + \underline{e}_J + \underline{e}_1)\big).
\end{aligned}$$

On the other hand, if $\underline{n}\in\mathbb{Z}^r$, then, for any arrangement $(i_1,\dots,i_r)$ of the elements in the set I, $D(\underline{n}) \simeq \bigoplus_{j=1}^r D_{i_j}(\underline{n} + \underline{e}_{i_1} + \cdots + \underline{e}_{i_{j-1}})$. So, $d(\underline{n}) = \sum_{j=1}^r d_{i_j}(\underline{n} + \underline{e}_{i_1} + \cdots + \underline{e}_{i_{j-1}})$.

Applying the above decomposition for $\underline{n} = \underline{m} - \underline{e} + \underline{e}_J$ with the natural arrangement $(1,2,\dots,r)$ and for $\underline{n} = \underline{m} - \underline{e} + \underline{e}_J + \underline{e}_1$ with the arrangement $(2,3,\dots,r,1)$ we get:

$$\begin{aligned}
\ell(\underline{m}) &= \sum_{J\subset I\setminus\{1\}} (-1)^{\#J} \big(d(\underline{m} - \underline{e} + \underline{e}_J) - d(\underline{m} - \underline{e} + \underline{e}_J + \underline{e}_1)\big) \\
&= \sum_{J\subset I\setminus\{1\}} (-1)^{\#J} \big(d_1(\underline{m} - \underline{e} + \underline{e}_J) - d_1(\underline{m} + \underline{e}_J)\big) \\
&= p_1(\underline{m} - \underline{e}) - p_1(\underline{m}),
\end{aligned}$$

and thus, we obtain the formula for P'_V given in the statement.

Now, let $\underline{m} \in \mathbb{Z}^r$ be such that $\underline{m} \notin S_V$. Then there exists an index $i \in I$ such that $d_i(\underline{m}) = 0$, since otherwise

$$\underline{m} = \min\{\underline{\nu}(g_i) \mid \mathtt{in}_{\nu_i}(g_i) \in D_i(\underline{m}) \setminus \{0\},\ 1 \leqslant i \leqslant r\}$$

is in S_V. $D_i(\underline{m} + \underline{e}_J) \subset D_i(\underline{m})$ for any $J \subset I$ with $i \notin J$, thus $d_i(\underline{m} + \underline{e}_J) = 0$ and so $p_i(\underline{m}) = 0$, which ends the proof. □

We will say that the divisorial valuation $\nu_j \in V$ is **extremal** if $\alpha(j)$ is a dead end of $\mathcal{G}(\pi)$, π being the minimal resolution of V.

Lemma 5. *Take $\underline{m} \in S_V$ and let $\nu_j \in V$ be an extremal valuation. Then, for every $J \subset I$ with $j \notin J$ the equality $d_j(\underline{m} + B^j + \underline{e}_J) = d_j(\underline{m} + \underline{e}_J) + 1$ holds. As a consequence, $p(\underline{m}) = p(\underline{m} + B^j)$.*

Proof. Set $I' = I \setminus \{j\}$ and $B^j = (B_1^j, \dots, B_r^j)$. Since ν_j is extremal, by Proposition 3 there exists a monomial q such that $\nu_j(q) = B_j^j$ and $\nu_i(q) > B_i^j$ for $i \in I'$. Let $h \in R$ be such that $\underline{\nu}(h) = \underline{m}$; then $\underline{\nu}(hq) \geqslant \underline{m} + \underline{e}_{I'} + B^j$ but $\underline{\nu}(hq) \ngeqslant \underline{m} + \underline{e} + B^j$. In particular, for any $J \subset I'$, $\underline{\nu}(hq) \geqslant \underline{m} + \underline{e}_J + B^j$ but $\underline{\nu}(hq) \ngeqslant \underline{m} + \underline{e}_J + \underline{e}_j + B^j$ and so, $d_j(\underline{m} + B^j + \underline{e}_J) \neq 0$.

Consider again $J \subset I$ such that $j \notin J$. If $d_j(\underline{m} + \underline{e}_J) \neq 0$ then, by Lemma 4, $d_j(\underline{m} + B^j + \underline{e}_J) = d_j(\underline{m} + \underline{e}_J) + 1$. If, otherwise, $d_j(\underline{m} + \underline{e}_J) = 0$, then $d_j(\underline{m} + \underline{e}_J + B^j) = 1$, since $d_j(\underline{m} + \underline{e}_J + B^j) \geqslant 2$ implies $d_j(\underline{m} + \underline{e}_J) \geqslant 1$ (Lemma 3).

Finally, the fact that $p(\underline{n}) = p_j(\underline{n})$ for any $\underline{n} \in \mathbb{Z}^r$ and the following equalities chain conclude the proof (recall that $r > 1$)

$$\begin{aligned} p_j(\underline{m} + B^j) &= \sum_{j \notin J \subset I} (-1)^{\#J} d_j(\underline{m} + B^j + \underline{e}_J) \\ &= \sum_{j \notin J \subset I} (-1)^{\#J} \big(d_j(\underline{m} + \underline{e}_J) + 1\big) \\ &= p_j(\underline{m}) + \sum_{j \notin J \subset I} (-1)^{\#J} = p(\underline{m}). \qquad \square \end{aligned}$$

Let $C = C_f$ be a general curve of V and consider the sequence of families of valuations $\{V^{(k)}\}$ constructed for C in Section 3, where $\pi^{(0)}$ is the minimal resolution of V. Next proposition allows to write the Poincaré series, $P_{V^{(k)}}(t_1, \dots, t_r)$, as a quotient of two series in such a way that the numerator does not depend on k. Stand $B^i_{(k)}$ for the value B^i associated to the family $V^{(k)}$.

Proposition 9. *For every $k \geqslant 0$,*

$$P_{V^{(k)}}(t_1, \dots, t_r) \cdot \prod_{i=1}^r \big(1 - \underline{t}^{B^i_{(k)}}\big) = P_V(t_1, \dots, t_r) \cdot \prod_{i=1}^r \big(1 - \underline{t}^{B^i}\big).$$

Proof. It suffices to show the formula for $k = 1$. Let $E_{\widetilde{\alpha}(1)}$ be the exceptional divisor created by blowing-up at a smooth point $P \in E_{\alpha(1)}$. Set $\widetilde{\nu}_1$ the $E_{\widetilde{\alpha}(1)}$-valuation and consider the set of divisorial valuations $\widetilde{V}=\{\widetilde{\nu}_1,\nu_2,\ldots,\nu_r\}$. Stand $P_{\widetilde{V}}(t_1,\ldots,t_r)$ for the Poincaré series of $\widetilde{V}$ and set $\widetilde{B}^1=(\widetilde{\nu}_1(Q_{\widetilde{\alpha}(1)}),\nu_2(Q_{\widetilde{\alpha}(1)}),\ldots,\nu_r(Q_{\widetilde{\alpha}(1)}))\in S_{\widetilde{V}}$. If we prove that

$$\left(1-\underline{t}^{\widetilde{B}^1}\right)P_{\widetilde{V}}(t_1,\ldots,t_r)=\left(1-\underline{t}^{B^1}\right)P_V(t_1,\ldots,t_r),$$

then the result, for $k=1$, follows after iterating the same procedure for the remaining valuations ν_i.

Let us write $P_{\widetilde{V}}(t_1,\ldots,t_r)=\sum_{\underline{m}\in\mathbb{Z}^r}\widetilde{p}(\underline{m})\underline{t}^{\underline{m}}$. We only need to prove, for any $\underline{m}\in\mathbb{Z}^r$, the following equality:

$$\widetilde{p}(\underline{m})-\widetilde{p}\left(\underline{m}-\widetilde{B}^1\right)=p(\underline{m})-p\left(\underline{m}-B^1\right). \tag{4}$$

Indeed, if $\underline{m}\notin S_V$ (respectively, $\underline{m}\notin S_{\widetilde{V}}$) then the right-(respectively, the left-)hand side of equality (4) vanishes, since both involved terms are equal to zero. Moreover, if $\underline{m}\in S_{\widetilde{V}}\setminus S_V$ then by Theorem 3, $\underline{m}=\underline{n}+s\widetilde{B}^1$ for some $\underline{n}\in S_V$ and $s\geqslant 1$ (since otherwise $\underline{m}\in S_V$). In particular, $\underline{m}-\widetilde{B}^1\in S_{\widetilde{V}}$ and, since $\widetilde{\nu}_1$ is extremal, Lemma 5 implies $\widetilde{p}(\underline{m})=\widetilde{p}(\underline{m}-\widetilde{B}^1)$. Therefore the left-hand side of equality (4) is also equal to zero. So, from now on we assume that $\underline{m}\in S_V$.

Denote by $\widetilde{J}(\underline{m})$ the valuation ideal of $\underline{m}$ for $\widetilde{V}$. Set $\widetilde{d}_1(\underline{m})=\dim\widetilde{J}(\underline{m})/\widetilde{J}(\underline{m}+\underline{e}_1)$. Taking into account the formulae in (3), to prove (4) we only need to show the following equality for any $J\subset I\setminus\{1\}$:

$$\widetilde{d}_1(\underline{m}+\underline{e}_J)-\widetilde{d}_1\left(\underline{m}-\widetilde{B}^1+\underline{e}_J\right)=d_1(\underline{m}+\underline{e}_J)-d_1\left(\underline{m}-B^1+\underline{e}_J\right). \tag{5}$$

Let us assume that either $d_1(\underline{m}+\underline{e}_J)\neq 0$ or $\widetilde{d}_1(\underline{m}+\underline{e}_J)=0$. Since $S_V\subset S_{\widetilde{V}}$, we have $d_1(\underline{n})=0$ if $\widetilde{d}_1(\underline{n})=0$, for any $\underline{n}\in\mathbb{Z}^r$. Therefore, if $\widetilde{d}_1(\underline{m}-B^1+e_J)=0$, then $d_1(\underline{m}-B^1+e_J)=0$, and by Lemma 3, $\widetilde{d}_1(\underline{m}+e_J)\in\{0,1\}$ and $d_1(\underline{m}+e_J)\in\{0,1\}$. Since our assumption excludes the case $\widetilde{d}_1(\underline{m}+e_J)=1$, $d_1(\underline{m}+e_J)=0$, the equality (5) holds. Otherwise, $\widetilde{d}_1(\underline{m}-B^1+e_J)\neq 0$, by Lemma 4 the left-hand side of the equality (5) is equal to 1. Again by our assumption, we cannot have $d_1(\underline{m}+e_J)=0$ and applying Lemma 3 when $d_1(\underline{m}-B^1+e_J)=0$ and Lemma 4 otherwise we prove that the right-hand side also equals 1.

To finish the proof, we will prove that there is no $J\subset I\setminus\{1\}$ such that $d_1(\underline{m}+\underline{e}_J)=0$ and $\widetilde{d}_1(\underline{m}+\underline{e}_J)\neq 0$. If $d_1(\underline{m}+\underline{e}_J)=0$ and $\widetilde{d}_1(\underline{m}+\underline{e}_J)\neq 0$, pick $h\in R$ such that its image in $\widetilde{D}_1(\underline{m}+\underline{e}_J)$ does not vanish. Let $\widetilde{\pi}$ be the minimal resolution of $\widetilde{V}$. Clearly, $\mathcal{G}(\widetilde{\pi})\setminus\{\widetilde{\alpha}(1)\}$ is connected. Since $d_1(\underline{m}+\underline{e}_J)=0$, we have $\widetilde{\nu}_1(h)\neq\nu_1(h)$, and as a consequence the strict transform of C_h by $\widetilde{\pi}$ intersects $E_{\widetilde{\alpha}(1)}$. Let $h=\varphi h'$ be such that the strict transform of $C_{h'}$ by $\widetilde{\pi}$ does not intersect $E_{\widetilde{\alpha}(1)}$ and the strict transforms by $\widetilde{\pi}$ of all the irreducible components of C_φ intersect $E_{\widetilde{\alpha}(1)}$.

Applying Proposition 3 to the connected component $\Delta=\mathcal{G}(\widetilde{\pi})\setminus\{\widetilde{\alpha}(1)\}$ one can show that there exists a monomial q in the elements $\{Q_\rho\mid\rho\in\mathcal{E}\cap\Delta\}$, such that $\widetilde{\nu}_1(q)=\widetilde{\nu}_1(\varphi)$ and $\nu_i(q)>\nu_i(\varphi)$ for $i=2,\ldots,r$. As the irreducible components of the strict transforms of C_q do not meet the divisor $E_{\widetilde{\alpha}(1)}$, one has $\nu_1(q)=\widetilde{\nu}_1(q)$ and so $\nu_1(h'q)=\widetilde{\nu}_1(h'q)=\widetilde{\nu}_1(h)=m_1$ and $\nu_i(h'q)>\nu_i(h)\geqslant m_i$. As a consequence, $h'q\in D_1(\underline{m}+\underline{e}_{I\setminus\{1\}})\setminus\{0\}$, and then $d_1(\underline{m}+\underline{e}_J)\neq 0$, which is a contradiction. □

Now, we state the relationship between the Poincaré series of a finite set of divisorial valuations V and the Poincaré polynomial of a general curve, C, of V.

Theorem 6. *Let V and C be as above. Then,*

$$P_V(t_1,\dots,t_r)=\frac{P_C(t_1,\dots,t_r)}{\prod_{i=1}^r(1-\underline{t}^{B^i})}.$$

Proof. By Proposition 9, it suffices to prove the result for the set of valuations $V^{(k)}$, for some k. In particular, we can assume that all the divisorial valuations $\nu_1,\dots,\nu_r$ are extremal. Fix some k and, for simplicity, write $\widetilde{V}=V^{(k)}$, $P_{\widetilde{V}}(t_1,\dots,t_r)=\sum_{\underline{m}\in\mathbb{Z}^r_{\geqslant 0}}\widetilde{p}(\underline{m})\underline{t}^{\underline{m}}$, and set $\widetilde{d}_i$ for the corresponding dimensions, and recall that $\widetilde{p}(\underline{m})=0$ when $\underline{m}\notin S_{\widetilde{V}}$.

The coefficient of $\underline{t}^{\underline{m}}$ in the series $P_{\widetilde{V}}(t_1,\dots,t_r)\cdot\prod_{i=1}^r(1-\underline{t}^{\widetilde{B}^i})$ is

$$\lambda_{\underline{m}}=\sum_{J\subset I}(-1)^{\#J}\widetilde{p}\left(\underline{m}-\sum_{i\in J}\widetilde{B}^i\right).$$

Now, if $J_{\underline{m}}=\{i\in I\mid \underline{m}-\widetilde{B}^i\in S_{\widetilde{V}}\}$, we have $\underline{m}-\sum_{i\in J}\widetilde{B}^i\in S_{\widetilde{V}}$ if and only if $J\subset J_{\underline{m}}$ (see Theorem 3), and in this case, by Lemma 5, $\widetilde{p}(\underline{m}-\sum_{i\in J}\widetilde{B}^i)=\widetilde{p}(\underline{m})$. Therefore,

$$\lambda_{\underline{m}}=\sum_{J\subset J_{\underline{m}}}(-1)^{\#J}\widetilde{p}\left(\underline{m}-\sum_{i\in J}\widetilde{B}^i\right)=\sum_{J\subset J_{\underline{m}}}(-1)^{\#J}\widetilde{p}(\underline{m}),$$

which is 0 if $J_{\underline{m}}\neq\emptyset$ and $\widetilde{p}(\underline{m})$ if $J_{\underline{m}}=\emptyset$, that is, if $\widetilde{d}_i(\underline{m})=1$ for $1\leqslant i\leqslant r$ (Theorem 3). Hence,

$$P_{\widetilde{V}}(t_1,\dots,t_r)\cdot\prod_{i=1}^r\left(1-\underline{t}^{\widetilde{B}^i}\right)=\sum_{m\in A}\widetilde{p}(\underline{m})\underline{t}^{\underline{m}},$$

where

$$A:=\left\{\underline{m}\in S_{\widetilde{V}}\;\middle|\;\widetilde{d}_i(\underline{m})=1\text{ for }1\leqslant i\leqslant r\right\}.$$

For $J\subset I\setminus\{1\}$ we have $D_1(\underline{m}+e_J)\subset D_1(\underline{m})$, hence $\widetilde{d}_1(\underline{m}+e_J)\leqslant 1$ for any $\underline{m}\in A$. Thus, all the summands in the formula $\widetilde{p}(\underline{m})=\widetilde{p}_1(\underline{m})=\sum_{J\subset I\setminus\{1\}}(-1)^{\#J}\widetilde{d}_1(\underline{m}+\underline{e}_J)$ are 1 or 0.

On the other hand, if $P_C(t_1,\dots,t_r)=\sum_{\underline{m}}\bar{p}(\underline{m})\underline{t}^{\underline{m}}$ is the Poincaré polynomial of the curve C, it is straightforward to deduce that the coefficients $\bar{p}(\underline{m})$ satisfy a formula similar to (3), in particular the following equality holds

$$\bar{p}(\underline{m})=\bar{p}_1(\underline{m})=\sum_{J\subset I\setminus\{1\}}(-1)^{\#J}c_1(\underline{m}+\underline{e}_J)$$

where $c_1(\underline{n})=\dim J^C(\underline{n})/J^C(\underline{n}+e_1)$ for any $\underline{n}\in\mathbb{Z}^r$. And in this case, the dimensions $c_1(\underline{n})$ only could be 1 or 0, because $J^C(\underline{n})/J^C(\underline{n}+e_1)$ can be regarded as a vector subspace of $J^{C_1}(\underline{n})/J^{C_1}(\underline{n}+e_1)$ (C_1 being one of the branches of C), whose dimension is 1 or 0.

We claim that there exists some k such that for any $\underline{m}\in A$ and $J\subset I\setminus\{1\}$ it happens that $\widetilde{d}_1(\underline{m}+\underline{e}_J)=0$ if and only if $c_1(\underline{m}+\underline{e}_J)=0$, and such that $\bar{p}(\underline{m})=0$ for any $\underline{m}\notin A$. Then, we deduce $P_{\widetilde{V}}(t_1,\dots,t_r)=\frac{P_C(t_1,\dots,t_r)}{\prod_{i=1}^r(1-\underline{t}^{B^i})}$, as we wanted to prove (recall that $\widetilde{V}$ depends on k).

Since $S_{\widetilde{V}} \subset S_C$, then $\widetilde{d}_1(\underline{n}) \neq 0$ implies $c_1(\underline{n}) \neq 0$ for any $\underline{n} \in \mathbb{Z}^r$. Moreover, for $\underline{m}$ fixed, there exists k_0 such that, for any $J \subset I \setminus \{1\}$ and $k \geqslant k_0$, we have $d_1^{(k)}(\underline{m} + \underline{e}_J) \neq 0$ if $c_1(\underline{m} + \underline{e}_J) \neq 0$. But P_C is a polynomial, so $B := \{\underline{m} \in S_C \mid \bar{p}(\underline{m}) \neq 0\}$ is a finite set and for k large enough we find $\widetilde{d}_1(\underline{m} + \underline{e}_J) = 0$ if and only if $c_1(\underline{m} + \underline{e}_J) = 0$, for any $J \subset I \setminus \{1\}$ and $\underline{m} \in B$; if moreover we pick k such that $\underline{m} \ngeqslant B^i_{(k)}$ for every $\underline{m} \in B$, we have $B \subset A$ (see Theorem 3), that is, $\bar{p}(\underline{m}) = 0$ for any $\underline{m} \notin A$, which proves our claim. □

Next corollary gives a precise meaning to the fact that the valuations defined by a curve singularity can be approached by families of divisorial valuations:

Corollary 2. *Let V and C as above and $V^{(k)}$ ($k \geqslant 0$) the finite sets of divisorial valuations defined in Section* 3. *Then,*

$$\lim_{k \to \infty} P_{V^{(k)}}(t_1, \ldots, t_r) = P_C(t_1, \ldots, t_r).$$

Finally, assume that $R = \mathcal{O}_{\mathbb{C}^2,0}$ is the local ring of germs of holomorphic functions at the origin of the complex plane $\mathbb{C}^2$. For a vertex α of the dual graph $\mathcal{G}$ of a set of valuations V as above, denote by $\overset{\bullet}{E}_\alpha = E_\alpha \setminus (\overline{E - E_\alpha})$ the smooth part of an irreducible component E_α in the exceptional divisor, E, of the minimal resolution of V and by $\chi(\overset{\bullet}{E}_\alpha)$ its Euler characteristic. In addition, set $\underline{v}^\alpha = \underline{v}(Q_\alpha)$. Then the following formula of A'Campo's type [1], firstly proved in [8], holds.

Corollary 3.

$$P_V(t_1, \ldots, t_r) = \prod_{E_\alpha \subset E} \left(1 - \underline{t}^{\underline{v}^\alpha}\right)^{-\chi(\overset{\bullet}{E}_\alpha)}.$$

Proof. E_α is isomorphic to the complex line $\mathbb{P}^1_{\mathbb{C}}$, so $\chi(\overset{\bullet}{E}_\alpha) = 2 - b(\alpha)$, where $b(\alpha)$ denotes the number of singular points of E_α in E (i.e., the number of connected components of $\mathcal{G} \setminus \{\alpha\}$).

Since the Poincaré polynomial $P_C(t_1, \ldots, t_r)$ coincides with the Alexander polynomial $\Delta^C(t_1, \ldots, t_r)$ (see [6]) and by using the Eisenbud–Neumann formula for $\Delta^C(t_1, \ldots, t_r)$ [11], we obtain:

$$P_C(t_1, \ldots, t_r) = \Delta^C(t_1, \ldots, t_r) = \prod_{E_\alpha \subset E} \left(1 - \underline{t}^{\underline{\upsilon}^\alpha}\right)^{-\chi(\overset{\circ}{E}_\alpha)},$$

where $\underline{\upsilon}^\alpha = \underline{\upsilon}(Q_\alpha)$, $\underline{\upsilon}$ being the above described valuation sequence given by C, and $\overset{\circ}{E}_\alpha$ is the smooth part of E_α in the total transform of C by the minimal resolution of V. $\chi(\overset{\circ}{E}_\alpha) = 2 - b(\alpha) = \chi(\overset{\bullet}{E}_\alpha)$ for those $\alpha \notin \{\alpha(1), \ldots, \alpha(r)\}$ and $\chi(\overset{\circ}{E}_{\alpha(i)}) = 2 - (b(\alpha(i)) + 1) = \chi(\overset{\bullet}{E}_{\alpha(i)}) - 1$ for $i = 1, \ldots, r$.

Finally, $\underline{\upsilon}^\alpha = \underline{v}^\alpha$ for $\alpha \in \mathcal{G}$ and, so,

$$P_V(t_1, \ldots, t_r) = \frac{\prod_{E_\alpha \subset E}(1 - \underline{t}^{\underline{\upsilon}^\alpha})^{-\chi(\overset{\circ}{E}_\alpha)}}{\prod_{i=1}^r (1 - \underline{t}^{B^i})} = \prod_{E_\alpha \subset E} \left(1 - \underline{t}^{\underline{v}^\alpha}\right)^{-\chi(\overset{\bullet}{E}_\alpha)}. \qquad \square$$

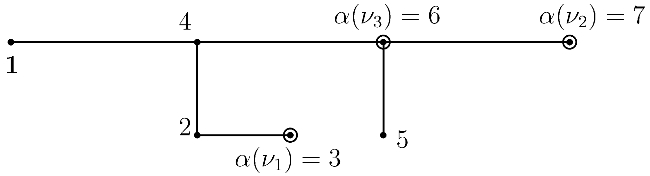

Fig. 2. Dual graph.

Remark. The proof of the equality between the Poincaré and the Alexander polynomials in [6] uses the topology of the complex field. However, the authors have informed us about the existence of a nonpublished alternative proof which avoids this, by using deeper properties of the semigroup S_C. Thus, writing $2 - b(\alpha)$ instead $\chi(\overset{\bullet}{E}_\alpha)$, the above formula holds also in the non-complex case.

We conclude this paper giving an illustrative example.

Example. Let x, y be independent variables and set $T = \mathbb{C}[x, y]_{(x,y)}$. Consider the set $V = \{\nu_1, \nu_2, \nu_3\}$ of divisorial valuations of $\mathbb{C}(x, y)$ centered at T, whose minimal resolution is given by the sequence of ideals:

- ν_1: $\mathfrak{m}_0 = \langle x, y\rangle, \mathfrak{m}_1 = \langle x, \frac{y}{x}\rangle, \mathfrak{m}_2 = \langle x, \frac{y}{x^2} - 1\rangle$;
- ν_2: $\mathfrak{m}_0, \mathfrak{m}_1, \mathfrak{m}_3 = \langle \frac{y}{x}, \frac{x^2}{y}\rangle, \mathfrak{m}_4 = \langle \frac{y}{x}, \frac{x^3-y^2}{y^2}\rangle, \mathfrak{m}_5 = \langle \frac{x^3-y^2}{y^2}, \frac{y^3}{x(x^3-y^2)}\rangle, \mathfrak{m}_6 = \langle \frac{x^3-y^2}{y^2}, \frac{y^5}{x(x^3-y^2)^2} - 1\rangle$;
- ν_3: $\mathfrak{m}_0, \mathfrak{m}_1, \mathfrak{m}_3, \mathfrak{m}_4, \mathfrak{m}_5$.

The dual graph of V has the shape of Fig. 2. From it, we can deduce the values $\underline{\nu}^{\mathbf{1}} = (1, 4, 4)$, $\underline{\nu}^2 = (2, 6, 6)$, $\underline{\nu}^3 = (3, 6, 6)$, $\underline{\nu}^4 = (3, 12, 12)$, $\underline{\nu}^5 = (3, 13, 13)$, $\underline{\nu}^6 = (6, 26, 26)$, $\underline{\nu}^7 = (6, 27, 26)$, as well as the values $\chi(\overset{\bullet}{E}_\alpha) = 2 - b(\alpha)$ (Corollary 3), giving the following expression for the Poincaré series

$$P_V = \frac{(1 - t_1^3 t_2^{12} t_3^{12})(1 - t_1^6 t_2^{26} t_3^{26})}{(1 - t_1 t_2^4 t_3^4)(1 - t_1^3 t_2^6 t_3^6)(1 - t_1^3 t_2^{13} t_3^{13})(1 - t_1^6 t_2^{27} t_3^{26})}.$$

Moreover, by Theorem 5, the set $\Lambda = \{Q_{\mathbf{1}} = x,\ Q_3 = y - x^2,\ Q_5 = y^2 - x^3,\ Q_7 = (y^2 - x^3)^2 - x^5 y\}$ is a minimal generating sequence of V since the set $\{\mathbf{1}, 3, 5, 7\}$ is the set of dead ends of the displayed dual graph.

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References

[1] N. A'Campo, La fonction zêta d'une monodromie, Comment. Math. Helv. 50 (1975) 233–248.
[2] A. Campillo, C. Galindo, The Poincaré series associated with finitely many monomial valuations, Math. Proc. Cambridge Philos. Soc. 134 (3) (2003) 433–443.

[3] A. Campillo, F. Delgado, S.M. Gusein-Zade, The extended semigroup of a plane curve singularity, Proc. Steklov Inst. Math. 221 (1998) 139–156.

[4] A. Campillo, F. Delgado, S.M. Gusein-Zade, On generators of the semigroup of a plane curve singularity, J. London Math. Soc. 60 (2) (1999) 420–430.

[5] A. Campillo, F. Delgado, S.M. Gusein-Zade, On the monodromy of a plane curve singularity and the Poincaré series of its ring of functions, Funct. Anal. Appl. 33 (1) (1999) 56–57.

[6] A. Campillo, F. Delgado, S.M. Gusein-Zade, The Alexander polynomial of a plane curve singularity via the ring of functions on it, Duke Math. J. 117 (2003) 125–156.

[7] F. Delgado, The semigroup of values of a curve singularity with several branches, Manuscripta Math. 59 (1987) 347–374.

[8] F. Delgado, S.M. Gusein-Zade, Poincaré series for several divisorial valuations, Proc. Edinb. Math. Soc. 46 (2) (2003) 501–509.

[9] F. Delgado, H. Maugendre, Special fibres and critical locus for a pencil of plane curve singularities, Compos. Math. 136 (1) (2003) 69–87.

[10] F. Delgado, C. Galindo, A. Nuñez, Saturation for valuations on two-dimensional regular local rings, Math. Z. 234 (2000) 519–550.

[11] D. Eisenbud, W. Neumann, Three-Dimensional Link Theory and Invariants of Plane Curve Singularities, Ann. of Math. Stud., vol. 110, Princeton Univ. Press, Princeton, NJ, 1985.

[12] C. Galindo, On the Poincaré series for a plane divisorial valuation, Bull. Belg. Math. Soc. 2 (1995) 65–74.

[13] S. Greco, K. Kiyek, General elements of complete ideals and valuations centered at two dimensional regular local rings, in: Algebra, Arithmethics and Geometry with Applications, Springer-Verlag, 2003, pp. 381–455.

[14] R. Lazarsfeld, Positivity in Algebraic Geometry, vol. II, Springer-Verlag, 2004.

[15] J. Lipman, K. Watanabe, Integrally closed ideals in two-dimensional regular local rings are multiplier ideals, Math. Res. Lett. 10 (2003) 1–12.

[16] M. Spivakovsky, Valuations in function fields of surfaces, Amer. J. Math. 112 (1990) 107–156.

[17] B. Teissier, Valuations, deformations, and toric geometry, in: Valuation Theory and Its Applications, vol. II, Saskatoon, SK, 1999, in: Fields Inst. Commun., vol. 33, Amer. Math. Soc., Providence, RI, 2003, pp. 361–459.

[18] O. Zariski, Le problème des modules pour les branches planes, with an appendix by B. Teissier, Hermann, Paris, 1986.