Pólya frequency sequences and real zeros of some $3\,F_2$ polynomials

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Abstract

We prove that the zeros of some families of $3\,F_2$ hypergeometric polynomials are all real and negative. This result has a connection with the theory of Pólya frequency sequences and functions. As a consequence, we establish the asymptotic distribution of these zeros when the degree of the polynomials tends to infinity.

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1. Introduction

Polynomials with only real zeros (also known as hyperbolic polynomials) are of interest in different branches of mathematics, such as combinatorics, probability, approximation theory and

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numerical analysis. The hyperbolicity of a given polynomial is a classical problem, addressed already by Gauss and Newton, that can be tackled by an ever increasing number of mathematical tools. It is a standard way of establishing the log-concavity and unimodality of the sequence of coefficients of a polynomial, otherwise demanding cumbersome combinatorial manipulations (see, e.g., [22] and [3]).

Our main concern here is sequences of hyperbolic polynomials of specific type with positive coefficients, whose zeros in consequence are all negative. The hypergeometric polynomial of degree $n$ is an expression of the form

$$pF_q\left(\begin{array}{c} -n, a_2, \ldots, a_p \\ b_1, \ldots, b_q \end{array}; x \right) = \sum_{k=0}^{n} \frac{(-n)_k(a_2)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \frac{x^k}{k!},$$

where $(\alpha)_k = \alpha(\alpha+1) \cdots (\alpha+k-1)$ is the Pochhammer symbol. These polynomials occur in many different contexts, including the study of combinatorial problems (cf. [2]) and in Padé approximation (cf. [8,19]). The location of their zeros plays a role in analysing the convergence of Padé approximants (cf. [19]) and in problems arising in Bergman spaces (cf. [24]). The hypergeometric polynomials whose properties are best known and understood, including the location of their zeros and the asymptotic zero distribution, are the $1F_1$ (or Whittaker) and the $2F_1$ (Gauss) classes, mostly because of the connection with orthogonal polynomials (cf. [12]).

A standard way of proving the hyperbolicity of a polynomial is showing that it is orthogonal with respect to a measure on $\mathbb{R}$. In the case of $2F_1$ polynomials, their connection with Jacobi polynomials $P_n^{(\alpha,\beta)}$ given by

$$P_n^{(\alpha,\beta)}(x) = \frac{(1+\alpha)_n}{n!} 2F_1\left(\begin{array}{c} -n, 1+\alpha + \beta + n \\ 1+\alpha \end{array}; \frac{1-x}{2} \right),$$

(1)

has led to significant information about their zeros, notably the values of the parameters $a_2 = 1+\alpha + \beta + n$ and $b_1 = 1+\alpha$ that guarantee a specific number of real zeros (cf. [21, p. 145, Theorem 6.72]).

For $3F_2$ polynomials however, there is no connection in general with classical orthogonal polynomials, and knowledge of the location of their zeros is restricted to some rather special classes. These special classes include $3F_2$ polynomials that permit decomposition into products or linear combinations of $2F_1$ polynomials. Information about the location of their zeros (or asymptotic properties of their zeros) then flows from the orthogonality or quasi-orthogonality of the factors (cf. [5–7,9]).

However, the connection between $3F_2$ polynomials and orthogonal or quasi-orthogonal polynomials is restricted to very special classes and it is natural to expect that one should move away from considerations of orthogonality or quasi-orthogonality in attempting to analyse the zeros of more general classes of $3F_2$ polynomials.

Our interest in the zeros of the three classes of $3F_2$ hypergeometric polynomials that we consider in this paper arose from the conjecture of C. Greene and H. Wilf (cf. [4]) that all zeros of the polynomial

$$3F_2\left(\begin{array}{c} -n, -n, 1/2 \\ 1, 1 \end{array}; 4x \right) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k} x^k$$

(2)

are real and negative. Our proof of the Wilf–Greene conjecture uses the relationship between Pólya frequency functions and the real zeros of polynomials linked with Pólya frequency functions.
In this paper, we prove that the zeros of three distinct classes of hypergeometric polynomials are all real. In addition to establishing the Wilf–Greene conjecture, we shall prove a slightly stronger assertion, namely that the zeros of the polynomial

\[ 3F_2 \left( -n, -m, a - \frac{1}{2}; \frac{a}{2a - 1}; x \right), \quad a > 0, \ m, n \in \mathbb{N}, \tag{3} \]

are all real and negative. Clearly, the case \( n = m \) and \( a = 1 \) in (3) yields (2). The methods we use further establish that the two families of polynomials

\[ 3F_2 \left( -n, -m, a + l; \frac{a}{b}; x \right), \quad a, b > 0 \text{ and } l, m, n \in \mathbb{N}, \tag{4} \]

and

\[ 3F_2 \left( -n, -m, -l; \frac{a}{b}; -x \right), \quad a, b > 0 \text{ and } l, m, n \in \mathbb{N}, \tag{5} \]

have only negative real zeros. We note that none of the classes of \( 3F_2 \) polynomials considered here can be dealt with using the methods in [5–7,9] and our approach is to use results on Pólya frequency sequences. As far as we know, Pólya frequency sequence properties of the hypergeometric series \( pFq \) have been studied only in the works of Richards and collaborators (see [18]).

The paper is organised as follows. In Section 2 we summarise known facts about Pólya frequency sequences that are relevant to our problem. In Section 3 we state and prove our results and in Section 4 we derive, subject to certain assumptions, the asymptotic distribution (as \( n \to \infty \)) of the zeros of two of the classes of polynomials considered in Section 3.

2. Pólya frequency sequences

Recall that a sequence of real numbers \( (a_k)_{k=0}^{\infty} \) is called a Pólya frequency (or PF) sequence if the infinite matrix \( (a_{j-i})_{i,j=0}^{\infty} \) is totally positive, (i.e., all its minors have a nonnegative determinant) where we adopt the convention that \( a_k = 0 \) for \( k < 0 \). This concept is extended to finite sequences in the obvious way by completing the sequence with zero terms.

The connection between finite PF sequences and the zeros of the corresponding polynomials is given by the following characterization (see, e.g., [1,14]):

**Characterization of Aissen–Schoenberg–Whitney.** Let \( c_0, c_1, \ldots, c_n \geq 0 \). Then

\( (c_0, c_1, \ldots, c_n) \) is a PF sequence \iff \( \sum_{k=0}^{n} c_k x^k \) has only real zeros.

The collection of all finite Pólya frequency sequences with positive terms is closed under various operations (cf. [11,14]) such as factorial tilting and this, together with the A–S–W characterization, leads to the following observations:

**Proposition 1.**

(i) If \( \sum_{k=0}^{n} a_k x^k \) has only real zeros then \( \sum_{k=0}^{n} \frac{a_k}{k!} x^k \) also has only real zeros.

(ii) Let \( a_i, b_i \geq 0 \) and suppose \( \sum_{i=0}^{d} a_i x^i \) and \( \sum_{i=0}^{k} b_i x^i \) have only real zeros. Then \( \sum_{i=0}^{\infty} i! a_i b_i x^i \) has only real zeros.
Statement (i) can be found in [16, Problem V.1.65], or in [15, Theorem 2.4.1], while (ii), known as Schur’s theorem and first proved in [20], appears also in [16, Problems V.2.155–156]

A function \( f(u) \) is called a Pólya frequency (or PF) function if

\[
\det(f(x_i - y_j)) \geq 0
\]

for all \(-\infty < x_1 < \cdots < x_m < +\infty \) and \(-\infty < y_1 < \cdots < y_m < +\infty \). A PF function \( f \) can be characterized in terms of its Laplace transform \( \mathcal{L}[f] \) (see [11, Theorem 3.2, p. 345]) in the following way.

**Proposition 2.** A function \( f : \mathbb{R} \to \mathbb{R} \) with \( \int_{-\infty}^{+\infty} f(u) \, du = 1 \), is a PF function if and only if

\[
\frac{1}{\mathcal{L}[f](s)} = e^{-As^2 + Bs} \prod_{i=1}^{\infty} (1 + \lambda_i s)e^{-\lambda_i s}
\]

for some \( A \geq 0 \), \( B \in \mathbb{R} \), and \( \lambda_i \in \mathbb{R} \), such that \( 0 < A + \sum_{i \geq 0} \lambda_i^2 < +\infty \).

A connection between PF functions and finite PF sequences is given in the following theorem (see [11, Theorem 3.1, p. 345]):

**Proposition 3.** Let \( f : \mathbb{R} \to \mathbb{R} \) be a PF function with \( \int_{-\infty}^{+\infty} f(u) \, du = 1 \), and assume that

\[
\frac{1}{\mathcal{L}[f](s)} = \sum_{m=0}^{\infty} \frac{c_m}{m!} s^m,
\]

\( c_0 = 1 \), converges for \( |s| < A \) (for some \( A > 0 \)). Then

\[
\sum_{m=0}^{n} \binom{n}{m} c_m s^m
\]

has only real zeros.

We will also make use of Hadamard’s factorization theorem and state it here for the convenience of the reader.

**Proposition 4.** (cf. [23, p. 250]) If \( f(z) \) is an integral (entire) function of order \( \rho \), then

\[
f(z) = e^{Q(z)} P(z),
\]

where \( P(z) \) is the canonical product formed with the zeros of \( f(z) \), and \( Q(z) \) is a polynomial of degree not greater than \( \rho \).

3. Hypergeometric polynomials with negative zeros

We begin with a straightforward consequence of a property of PF functions and sequences.

**Lemma 5.** Let \( f : \mathbb{R} \to \mathbb{R} \) be a PF function with \( \int_{-\infty}^{+\infty} f(u) \, du = 1 \), such that

\[
\frac{1}{\mathcal{L}[f](x)} = pF_q \left( \begin{array}{c} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{array}; x \right), \quad a_i, b_j > 0.
\]
Then for every $m, n \in \mathbb{N}$ the hypergeometric polynomial
\[
p_{+2}F_{q}\left(-n, -m, a_1, \ldots, a_p ; b_1, \ldots, b_q \right)
\]
has only real negative zeros.

**Proof.** Let $f : \mathbb{R} \to \mathbb{R}$ be a PF function with $\int_{-\infty}^{+\infty} f(u) du = 1$, such that
\[
\frac{1}{L[f](x)} = pF_{q}\left(a_1, \ldots, a_p ; b_1, \ldots, b_q ; x \right) = \sum_{k=0}^{\infty} \frac{(a_1)_k(a_2)_k \cdots (a_p)_k x^k}{(b_1)_k(b_2)_k \cdots (b_q)_k k!}.
\]
Observing that
\[
\frac{(-n)_k}{k!} = \begin{cases} (-1)^k \binom{n}{k} & \text{for } 0 \leq k \leq n, \\ 0 & \text{for } k > n, \end{cases}
\]
it follows from Proposition 3 that the polynomial
\[
\sum_{k=0}^{n} \binom{n}{k} \frac{(a_1)_k(a_2)_k \cdots (a_p)_k x^k}{(b_1)_k(b_2)_k \cdots (b_q)_k k!} = p+1F_{q}\left(-n, a_1, \ldots, a_p ; b_1, \ldots, b_q ; -x \right)
\]
has only real zeros. If, in addition we assume that $a_i, b_j > 0$, then the zeros must obviously be negative. On the other hand, for any $m \in \mathbb{N}$, it is immediately apparent that the polynomial
\[
(1 + x)^m = \sum_{i=0}^{m} \binom{m}{i} x^i
\]
has only real and negative zeros. An application of Schur’s theorem (Proposition 1(ii)) then shows that
\[
\sum_{i=0}^{\infty} i! \binom{n}{i} \frac{(a_1)_i(a_2)_i \cdots (a_p)_i x^i}{(b_1)_i(b_2)_i \cdots (b_q)_i i!} = \sum_{i=0}^{\infty} \frac{(-n)_i(-m)_i(a_1)_i(a_2)_i \cdots (a_p)_i x^i}{(b_1)_i(b_2)_i \cdots (b_q)_i i!}
\]
has only negative real zeros. \(\square\)

A consequence of this result is that if we can establish $1F_2$ functions that satisfy (6), then we will be able to generate $3F_2$ polynomials that will a priori have only real negative zeros. This problem has been studied, inter alia, by Richards in [18]. In particular he proved the following result.

**Proposition 6.** Let $a_i > 0$ for $i = 1, \ldots, q$, $p \leq q$ and $k_1, \ldots, k_p \in \mathbb{N}$. Then functions
\[
pF_{q}\left(a_1 + k_1, \ldots, a_p + k_p ; z \right)
\]
have only negative real zeros.

Taking into account the fact that functions $pF_q$ for $p < q$ are entire functions of genus 0, using the Hadamard factorization theorem (Proposition 4) and Proposition 2, we see that functions (8) are indeed reciprocals of Laplace transforms of PF functions. Applying Lemma 5 we obtain the following
Theorem 7. Let $a, b > 0$ and let $l, m, n \in \mathbb{N}$. Then polynomial
\[ \binom{3}{2} \binom{-n, -m, a + l}{a, b} ; z \]
has only negative real zeros.

However, we observe that this result does not solve the conjecture of Greene and Wilf about the zeros of the polynomial given in (2). Nevertheless, the following 1-parameter family of $3\binom{F}{2}$ polynomials has the required property:

Theorem 8. Let $m, n \in \mathbb{N}$, and $a > 0$. The polynomial
\[ \binom{3}{2} \binom{-n, -m, a - 1/2}{a, 2a - 1} ; z \]
has only negative real zeros.

Proof. By formula (2) in [10, p. 185],
\[ \binom{0}{1} \binom{-a}{z} = 2 \binom{2}{3} \binom{(a + b)/2, (a + b - 1)/2}{a, b, a + b - 1} ; 4z, \]
so that
\[ \left( \binom{0}{1} \binom{-a}{z} \right)^2 = \binom{1}{2} \binom{a - 1/2}{a, 2a - 1} ; 4z. \] (9)
Since for $a > 0$,
\[ \phi(w) = \Gamma^{-1}(w + a) \]
is an entire function of genus 1, whose zeros are of the form $-a - k, k = 0, 1, 2, \ldots$, by a theorem of Laguerre [23, p. 270, §8.63],
\[ \binom{0}{1} \binom{-a}{z} = \Gamma(a) \sum_{k=0}^{\infty} \frac{1}{\Gamma(k + a) k!} z^k \]
is also an entire function of genus 1, whose zeros are real and negative. Thus, the same property is inherited by the right-hand side of (9), and it remains to apply Lemma 5. \qed

Theorem 9. Let $a, b > 0$ and let $l, m, n \in \mathbb{N}$. Then the polynomial
\[ \binom{3}{2} \binom{-n, -m, -l}{a, b} ; z \]
has only positive real zeros.

Proof. The order of a product of two entire functions cannot be more than the order of the function in the product with the largest order. This means that
\[ \frac{1}{\Gamma(k + a)} \frac{1}{\Gamma(k + b)} \]
satisfies the conditions of Laguerre’s theorem [23, p. 270, §8.63] and hence

\[ 0F_2\left( -a, b; z \right) = \Gamma(a)\Gamma(b) \sum_{k=0}^{\infty} \frac{1}{\Gamma(k+a)\Gamma(k+b) \, k!} z^k \]

is an entire function with only real and negative zeros. It follows from Hadamard’s factorization theorem (Proposition 4) and Proposition 2 that the function (10) may be written as the reciprocal of a Laplace transform of a PF function. Now, applying Lemma 5, it follows that

\[ 2F_2\left( -n, -m; a, b; z \right) \]

has only negative real zeros. Using the fact that polynomial (7) has only negative, real zeros and applying Schur’s theorem (Proposition 1(ii)),

\[ \sum_{i=0}^{\infty} \frac{(-n)_i(-m)_i(-l)_i}{(a)_i(b)_i i!} (-z)^i \]

has only negative real zeros. \( \square \)

4. Asymptotic distribution of negative zeros of \( 3F_2 \) polynomials

In this section we are interested in the asymptotic behavior of the zeros of the families of polynomials, given in Theorems 7 and 8, when their degree tends to infinity.

For that purpose we use the differential equation

\[ \left[ \theta(\theta + b - 1)(\theta + c - 1) - z(\theta - n)(\theta - m)(\theta + a) \right] y = 0, \quad \theta = z \frac{d}{dz}, \]

satisfied by the polynomial (see, e.g., [17])

\[ y = 3F_2\left( -n, -m, a, b, c; z \right). \]

After straightforward simplifications it takes the form

\[ z^2(1-z)y''' + z(1+b+c+z(n+m-3-a))y'' + \left( bc + z(a(n+m-1) - mn + m + n - 1) \right)y' - amny = 0. \]

Without loss of generality let us assume \( m \geq n \), which implies that the degree of \( y \) is \( n \). We reduce the equation above by defining \( h_n = y'/(ny) \). Then

\[ \frac{y''}{ny} = h'_n + nh_n^2 \quad \text{and} \quad \frac{y'''}{ny} = h''_n + 3nh'_n + n^2h_n^3. \]

Dividing (11) by \( n^3y \) and substituting \( y''', y'' \) and \( y' \), we obtain

\[ z^2(1-z)\left( \frac{h''_n}{n^2} + 3h'_n + \frac{h_n^3}{n^2} \right) + \frac{1 + b + c + z(n + m - 3 - a)}{n} \]

\[ \times \left( \frac{h'_n}{n} + h_n^2 \right) + \frac{bc + z(a(n + m - 1) - mn + m + n - 1)}{n^2} \frac{h_n}{n^2} - \frac{am}{n^2} = 0. \]
In order to perform the asymptotic analysis we must impose assumptions on the behavior of the parameters involved; namely, we suppose that they vary with \( n \) in such a way that the following limits exist:

\[
\lim_{n \to \infty} \frac{a_n}{n} = \alpha, \quad \lim_{n \to \infty} \frac{b_n}{n} = \beta, \quad \lim_{n \to \infty} \frac{c_n}{n} = \gamma, \quad \text{and} \quad \lim_{n \to \infty} \frac{m_n}{n} = \lambda \geq 1.
\]

Let \( \Omega \) be a domain in \( \mathbb{C} \) not containing any zero of \( y \) (for all sufficiently large \( n \)), and \( K \) be a compact subset of \( \Omega \). Since \( h_n \) is the Cauchy transform of the normalized zero counting measure of \( y \), by the compactness principle there exists a subsequence \( \Lambda \subset \mathbb{N} \) such that

\[
\lim_{z \in \Lambda} h_n(z) = H(z), \quad z \in K.
\]

Since \( h_n' \) and \( h_n'' \) are bounded on \( K \), taking limits in (12), we obtain an algebraic equation for \( H \):

\[
z^2(1 - z)H^3(z) + z((\lambda + 1 - \alpha)z + \beta + \gamma)H^2(z) + ((\alpha\lambda + \alpha - \lambda)z + \beta\gamma)H(z) - \alpha\lambda = 0.
\]

Taking into account that this equation is independent of \( \Lambda \) and making use of the additional information about the location of the zeros it is possible to recover their asymptotic distribution.

For instance, let us consider the sequence

\[
p_n(z) = \binom{-n, -m, a_n - 1/2}{a_n, 2a_n - 1} : z
\]

with \( m, n \in \mathbb{N} \), \( m \geq n \), and such that limits

\[
\lim_{n \to \infty} \frac{a_n}{n} = \alpha \quad \text{and} \quad \lim_{n \to \infty} \frac{m_n}{n} = \lambda \geq 1
\]

exist. According to Theorem 8, polynomials \( p_n \) have only real and negative zeros. Thus, on any compact set of \( \mathbb{C} \setminus \mathbb{R} \), the limit function \( H \) satisfies the equation

\[
z^2(1 - z)H^3(z) + z((\lambda + 1 - \alpha)z + 3\alpha)H^2(z) + ((\alpha\lambda + \alpha - \lambda)z + 2\alpha^2)H(z) - \alpha\lambda = 0,
\]

that can be factorized as

\[
-(zH(z) + \alpha)\{z(z - 1)H^2(z) - ((\lambda + 1)z + 2\alpha)H(z) + \lambda\} = 0,
\]

with solutions

\[
H(z) = -\frac{\alpha}{z}, \quad H(z) = \frac{(\lambda + 1)z + 2\alpha \pm \sqrt{(\lambda - 1)^2z^2 + 4(\alpha\lambda + \lambda + \alpha)z + 4\alpha^2}}{2z(z - 1)}.
\]

Let \( \lambda > 1 \). Then the discriminant

\[
(\lambda - 1)^2z^2 + 4(\alpha\lambda + \lambda + \alpha)z + 4\alpha^2 = (\lambda - 1)^2(z - \zeta_{-})(z - \zeta_{+}),
\]

where

\[
\zeta_{\pm} = -\frac{2}{(\lambda - 1)^2}(\lambda + \alpha\lambda \pm \sqrt{(1 + 2\alpha)\lambda(2\alpha + \lambda)})
\]

By the arithmetic–geometric mean inequality,

\[
\lambda + \alpha + \alpha\lambda = \frac{\lambda + 2\alpha\lambda + 2\alpha + \lambda}{2} \geq \sqrt{(1 + 2\alpha)\lambda(2\alpha + \lambda)}.
\]
so that \( \zeta_+ < \zeta_- \leq 0 \). Taking into account that \( H \) is the Cauchy transform of a positive measure on \((-\infty, 0] \) with total mass \( \leq 1 \) (it could be \( < 1 \) if some of the zeros of \( p_n \) "escape" to infinity), we see that necessarily

\[
H(z) = \frac{(\lambda + 1)z + 2\alpha - (\lambda - 1)\sqrt{(z - \zeta_-)(z - \zeta_+)}}{2z(z - 1)},
\]

where we fix the branch of the square root in \( \mathbb{C} \setminus (\zeta_+, \zeta_-) \) such that

\[
\lim_{z \to \infty} \frac{\sqrt{(z - \zeta_-)(z - \zeta_+)}}{z} = 1.
\]

Let us assume that the asymptotic distribution of the zeros of \( p_n \) is absolutely continuous measure with density \( w(x) \) on \((-\infty, 0] \). Since

\[
-\frac{1}{2\pi i} H(z) = \frac{1}{2\pi i} \int_{-\infty}^{0} \frac{w(x)}{x - z} \, dx
\]

it follows from the Sokhotsky–Plemelj theorem (cf. [13]) that

\[
w(x) = \left( -\frac{1}{2\pi i} H \right)_+ - \left( -\frac{1}{2\pi i} H \right)_-, \]

where the subindexes \( \pm \) denote here the boundary values of \( H \) on \((-\infty, 0] \) from the upper and lower half planes, respectively. Straightforward computations show that \( w(x) \) is positive only on the interval \( (\zeta_+, \zeta_-) \), support of the limit distribution of zeros of \( p_n \), where

\[
w(x) = \frac{(\lambda - 1)\sqrt{(\zeta_- - x)(x - \zeta_+)}}{2\pi x(x - 1)} > 0.
\]

It is easy to check that

\[
\int_{\zeta_-}^{\zeta_+} w(x) \, dx = 1,
\]

showing that the limiting distribution of the zeros is indeed absolutely continuous measure with the density \( w \).

The case \( \lambda = 1 \) is handled analogously: we obtain now that the asymptotic distribution of zeros is again an absolutely continuous measure with the density

\[
w(x) = \frac{\sqrt{2\alpha + 1}}{\pi} \frac{\sqrt{\zeta - x}}{x(x - 1)}, \quad \zeta = -\frac{\alpha^2}{2\alpha + 1} < 0, \quad x \in (-\infty, \zeta).
\]

It is worth observing that when the parameter \( a_n = a \) is independent of \( n \), we have \( \alpha = 0 \), so that the asymptotic distribution is given by

\[
w(x) = \begin{cases} 
\frac{1}{\pi} \frac{\sqrt{\lambda + (\lambda - 1)^2} x^2}{\sqrt{-x(1-x)}}, & x \in (-\frac{4\lambda}{(\lambda - 1)^2}, 0), \quad \text{if } \lambda > 1, \\
\frac{1}{\pi} \frac{1}{\sqrt{-x(1-x)}}, & x \in (-\infty, 0), \quad \text{if } \lambda = 1.
\end{cases}
\]

Notice that for \( \lambda = 1 \), the measure \( w(x) \, dx \) is invariant under transformation \( x \to 1/x \).
As another example, consider the case of the sequence

\[ k_n(z) = \binom{3}{n - m, a_n + l; z} \]

with \( l \in \mathbb{N} \), fixed and \( m, n \in \mathbb{N}, m \geq n \), such that limits

\[ \lim_n \frac{a_n}{n} = \alpha, \quad \lim_n \frac{b_n}{n} = \beta \quad \text{and} \quad \lim_n \frac{m}{n} = \lambda \geq 1 \]

exist.

In this case, the limit function \( H \) satisfies the equation

\[
\begin{align*}
z^2(1 - z)H^3(z) &+ z((\lambda + 1 - \alpha)z + \alpha + \beta)H^2(z) \\
+ ((\alpha\lambda + \alpha - \lambda)z + \alpha\beta)H(z) - \alpha\lambda &= 0,
\end{align*}
\]

on any compact set of \( \mathbb{C} \setminus \mathbb{R} \) since the zeros are located on the negative real axis by Theorem 7. The equation can be factorized as

\[
-(zH(z) + \alpha)\left\{z(z - 1)H^2(z) - ((\lambda + 1)z + \beta)H(z) + \lambda\right\} = 0
\]

with solutions

\[
H(z) = -\frac{\alpha}{z}, \quad H(z) = \frac{(\lambda + 1)z + \beta \pm \sqrt{(\lambda - 1)^2z^2 + 2(2\lambda + \beta + \lambda\beta)z + \beta^2}}{2z(z - 1)}.
\]

When \( \lambda > 1 \), the discriminant becomes

\[
(\lambda - 1)^2z^2 + 4(\alpha\lambda + \lambda + \alpha)z + 4\alpha^2 = (\lambda - 1)^2(z - \zeta_-)(z - \zeta_+),
\]

with

\[
\zeta_{\pm} = -\frac{2}{(\lambda - 1)^2}\left(\frac{\beta + 2\lambda + \beta\lambda}{2} \pm \sqrt{(\beta + 1)(\beta + \lambda)^2}\right)
\]

and now the analysis is the same as that done for the sequence \( p_n(z) \).

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References