Near Affine Hjelmslev Planes

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A study is made of two generalizations of affine Hjelmslev planes in which the parallel axiom is not required to hold. Integer invariants are obtained for the finite planes in these new classes. Formulas are derived which enable one to compute the cardinalities of certain subsets of points and lines in terms of the invariants, and results are obtained on the nonexistence of planes with certain sets of invariants.

INTRODUCTION

It is well known that one may replace the parallel axiom in the standard definition of finite affine planes by a number of different cardinality assumptions [3, p. 139]. For example, one may substitute the assumption that there exists an integer $n > 1$ such that every line possesses $n$ incident points and every point lies on $n + 1$ lines. Somewhat surprisingly, no such substitutions may be made in the definition of finite affine Hjelmslev planes (AH-planes). The preceding assertion is validated by an example which appears in Section 4 of this paper.

We define (see Definition 1.1) a fairly near affine Hjelmslev plane (an FNAH-plane) to be an incidence structure which satisfies those axioms of AH-planes that can be stated in terms of incidence alone; the one axiom excluded is the parallel axiom, for, in general, the parallel relation in an AH-plane is not determined by the incidence relation. Near affine Hjelmslev planes (NAH-planes) may be defined to be finite FNAH-planes which have a common number $s$ of points on each line (see Definition 1.3 and Corollary 2.2). Proposition 1.6 asserts the existence of a second integer invariant $t$ associated with an NAH-plane. A number of facts, known to be true for the invariants $s$ and $t$ of a finite AH-plane (see [5, Satz 2.1 1]), are proved to hold for NAH-planes in general. For example, each point of an NAH-plane has $t^2$ neighbors and $s + t$ incident lines while each line has $s$ incident points. Further if $\alpha$ is an NAH-plane, then the affine plane

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\( \xi \) associated with \( \alpha \) has order \( s/t \); and \( s \) must be \( \leq t^2 \) whenever \( t \neq 1 \). In spite of the fact that \( s \) and \( t \) appear to be as well behaved for NAH-planes as for finite AH-planes, there do exist NAH-planes which cannot be made into AH-planes by any definition of a parallel relation. Such an example with \( (s, t) = (12, 6) \) is presented in Section 4. Besides establishing the independence of the parallel postulate for AH-planes, the example is of interest because its invariant \( s \) is not a power of \( s/t \). The existence of an AH-plane with \( s \) not a power of \( s/t \) is still in doubt. In particular, the existence of AH-planes with \( (s, t) = (12, 6) \) is unknown, though the existence question for AH-planes is completely settled [4] for all other pairs \( (s, t) \) for which \( 1 < t < 7 \).

We prove (Proposition 3.1) that every NAH-plane with \( s = t^2 \) is an AH-plane under some definition of a parallel relation. Proposition 1.3 asserts that all finite FNAH-planes, except those whose associated affine planes have order 2, are NAH-planes. A finite FNAH-plane with associated affine plane of order 2 is called a 2FNAH-plane. Associated with each 2FNAH-plane are four integer invariants: \( t_1, t_2, t_3, t_4 \) (see Proposition 2.1), \( (t_i)^2 \) being the number of points in the \( i \)-th neighbor class of points, \( 1 \leq i \leq 4 \). A 2FNAH-plane is thus an NAH-plane (with \( s = 2t \)) if and only if each \( t_i = t \). Some non-existence results are obtained (Proposition 2.3 and Corollary 2.6), but the existence of 2FNAH-planes which are not NAH-planes is yet in doubt.

1. The Invariants

In an arbitrary incidence structure, one defines points \( P \) and \( Q \) to be neighbor and writes \( P \sim Q \) to mean that \( P \) and \( Q \) have at least two common incident lines. Lines \( g \) and \( h \) are said to be (affinely) neighbor, and one writes \( g \sim h \) to mean that each point incident with either of \( g \) or \( h \) possesses a neighbor point incident with the other of \( g \) or \( h \).

**Definition** 1.1. An incidence structure \( \alpha = (\mathcal{P}, \mathcal{G}, \mathcal{E}) \) is said to be a fairly near affine Hjelmslev plane (an FNAH-plane) provided the following three axioms are satisfied:

Axiom 1. \( P, Q \in \mathcal{P} \) implies the existence of \( g \in \mathcal{G} \) such that \( P, Q \in g \).

Axiom 2. If \( g \nmid h \mid \geq 1 \), then \( g \nmid h \mid \geq 2 \) if and only if \( g \sim h \).

Axiom 3. There exists an epimorphism \( \varphi \) from \( \alpha \) to an ordinary affine plane \( \tilde{\alpha} \) satisfying
We observe that Axiom 3 is purely an incidence axiom, since the parallel relation in $\Xi$ is defined in terms of incidence.

**Definition 1.2.** Let $\parallel$ denote an equivalence relation defined on the line set of an incidence structure $(\mathcal{P}, \mathcal{L}, \epsilon)$. One calls $\alpha = (\mathcal{P}, \mathcal{L}, \epsilon, \parallel)$ an **affine Hjelmslev plane** (an AH-plane) provided that $(\mathcal{P}, \mathcal{L}, \epsilon)$ is an FNAH-plane and the following axiom holds:

**Axiom 4.** Every point of $\mathcal{P}$ lies on exactly one line of each [l-class.

The above definition for AH-planes is due to Lüneburg [5, Satz 2.61. It has been misquoted by Dembowski [3, p. 296], who omits both Axiom 3c and the half of Axiom 2 which asserts that, if $g \cap h \geq 2$, then $g \sim h$.

The independence of Axiom 3c is apparently still in doubt; however, Bacon [2] has found a very simple example to show that Dembowski’s other omission is critical. No systematic study has been made of the independence of the axioms for AH-planes. The only results known seem to be that of Bacon and the independence of Axiom 4 established in Section 4 of this paper. It is, however, easy enough to see that Axiom 3 as a whole is independent of Axioms 1, 2, and 4.

If $(P, g)$ is a flag of a finite incidence structure, one defines $t(P, g)$ to be the number of points on $g$ which are neighbor to $P$.

**Lemma 1.1.** Let $\alpha$ be a finite FNAH-plane. Then if $P$ is any point of $\alpha$ and if $g$ and $h$ are any lines incident with $P$, $t(P, g) = t(P, h)$.

**Proof.** For all points $X$ and all lines $\chi$, let $X'$ denote $\varphi(X)$; $\chi'$ denote $\varphi(\chi)$. Let $P$ be a given point incident with given lines $g$ and $h$. Let $B'$ be any point not incident with either of $g'$, $h'$. Let $B$ be any point such that $\varphi(B) = B'$. Let $P = P_1, P_2, \ldots, P_t$ be all the points neighbor to $P$ which are incident with $g$. Let $j = j_1, j_2, \ldots, j_t$ be the lines joining $P_1, P_2, \ldots, P_t$, respectively, to $B$. Since $P \parallel P_1 \neq B'$, $j_i' = j_i \neq g'$, $h'$ for all $i \leq t$. Then $g \not\sim j_i \not\sim h$ for all $i \leq t$. Thus $|g \cap j_i| = 1$ and $|h \cap j_i| \leq 1$ for all $i$. If $h \cap j_i = 0$, then Axiom 3c implies $h' \parallel j_i'$, a contradiction. Setting $h \cap j_i = Q_i$ for $1 \leq i \leq t$, one sees that $Q_i' \in h' \cap j_i' = \{P_i'\}$; hence $Q_i \sim P$. Since $g \not\sim j_i \not\sim h$, the $j_i$ and the $Q_i$ are all distinct. Then $t(P, g) = t \leq t(P, h)$; equality follows by symmetry.

**Definition 1.3.** A finite FNAH-plane $\alpha$ is called a **near affine Hjelmslev**
plane (an NAH-plane) if there exists an integer \( t \) such that \( t(P, g) = t \) for every flag \( (P, g) \) in \( \alpha \).

**Proposition 1.2** (Lüneburg [5]). Every finite AH-plane is an NAH-plane.

**Proposition 1.3.** Let \( \alpha \) be a finite FNAH-plane such that \( \alpha \) has order \( \neq 2 \). Then \( \alpha \) is an NAH-plane.

Prior to proving Proposition 1.3, we obtain two lemmas:

**Lemma 1.4.** Let \( (P, g) \) be a flag of an FNAH-plane \( \alpha \). Let \( \varphi \) be the map of \( \alpha \) onto \( O_1 \), \( P' = \varphi(P), g' = \varphi(g) \); let \( P' \neq Q' \subseteq g' \). Then there exists a point \( Q \in g \) such that \( \varphi(Q) = Q' \).

**Proof.** Let \( R \) be any point of \( \alpha \) such that \( \varphi(R) = Q' \). Then \( PR \) is a unique line of \( \alpha \); and \( \varphi(PR) = P'Q' = g' = \varphi(g) \), so \( PR \sim g \). Then there exists \( Q \in g \) such that \( Q \sim R \), i.e., such that \( \varphi(Q) = Q' \).

**Lemma 1.5.** Let \( P \) and \( Q \) be non-neighbor points on a line \( g \) of an FNAH-plane \( \alpha \). Let \( x \) be a line through \( Q \) such that \( x \not\parallel g \). Then the number of lines through \( P \) which are neighbor to \( g \) is \( n = t(Q, x) \).

**Proof.** Let \( Q = Q_1, Q_2, \ldots, Q_n \) denote the neighbors of \( Q \) on \( x \); let \( q_i \) denote the unique line joining \( Q_i \) to \( P \) for \( i \leq n \). The \( q_i \) are all neighbor to \( g \), since \( Q' \) and \( P' \) both lie on \( g' \) and on \( q_i' \) for all \( i \). We claim that the \( q_i \) are all distinct. If not, \( x \sim q_i \) for some \( i \); hence \( x \sim g \), a contradiction. Now let \( y \) be any neighbor of \( g \) which is incident with \( P \). Since \( y' = g' \) is not parallel to \( x' \), Axiom 3c implies \( y \cap x \neq 0 \). Let \( W \) be any line through \( y \). Then \( W' \subseteq g' \cap x' = (P') \); hence \( W = P_i \) for some \( i \).

**Proof of Proposition 1.3.** Let \( (P, g) \) and \( (Q, h) \) be any two flags of \( \alpha \). Let \( p \) be any line connecting \( P \) and \( Q \). Let \( R' \) and \( M' \) be points of \( \alpha \) such that \( R' \not\subseteq p', Q' \neq R' \not\subseteq P', M' \not\subseteq p' \). The choice of \( R' \) is possible, because the order of \( \alpha \) is \( \neq 2 \). By Lemma 1.4, one may choose points \( R \) and \( M \) of \( \alpha \) such that \( \varphi(M') = M' \), \( \varphi(R') = R' \), and \( R \not\subseteq p \). Let \( x \) be the line joining \( P \) to \( M \); \( y \), the line joining \( Q \) to \( M \). Since \( M' \not\subseteq p', x \not\subseteq p \not\subseteq y \). By Lemma 1.5, the number of lines through \( R \) which are neighbor to \( p \) is equal to each of \( t(P, x), t(Q, y) \). By Lemma 1.1, \( t(P, g) = t(P, x) = t(Q, y) = t(Q, h) \).

**Proposition 1.6.** Associated with every NAH-plane \( \alpha = (P, G, \epsilon) \) are two invariant integers \( s \) and \( t \) which satisfy all the following conditions:

(a) If \( (P, g) \) is any flag of \( \alpha \), then there are precisely \( t \) neighbors of \( g \) incident with \( P \) and precisely \( t \) neighbors of \( P \) incident with \( g \).
Every line contains $s$ points.

Every point lies on $s + t$ lines.

Each point (and each line) has $t^2$ neighbors.

$P = s^2$.

$G = s^2 + st$.

t $s,$ and $s/t$ is the order of $\alpha$.

Either $t = 1,$ or $s \leq t^2$.

Proof. The definition of NAH-planes assures the existence of an integer $t$ which satisfies the second conclusion of (a). Let a flag $(P, g)$ be given. There exists a point $Q'$ on $g$ with $Q' \neq P'$; hence (by Lemma 1.4), a point $Q$ on $g$ with $Q \not\sim P$. If $M'$ is a point of $\alpha$ not on $g'$ and $M$ is a point of $\alpha$ such that $\varphi(M) = M'$, then there is a unique line $x$ of $\alpha$ which joins $Q$ to $M$; $x \not\sim g$. By Lemma 1.5, the number of neighbors to $g$ which are incident with $P$ is $t(Q, x) = t$. Thus (a) holds.

For each line $g$ of $\alpha$, set $s(g)$ equal to the number of points incident with $g$; for each point $P$, define $s(P)$ to be $t$ fewer than the number of lines incident with $P$. Let $g$ be any line of $\alpha$; $P$ be any point which is neighbor to no point of $g$. Let $Q_1, \ldots, Q_s$ be the points on $g$; let $q_i$ denote the line connecting $Q_i$ and $P$. All $q_i$ are distinct, for otherwise $g$ would be neighbor to some $q_i$, whence $P$ would be neighbor to a point of $g$. Let $h'$ be the unique line which is parallel to $g'$ and incident with $P'$. Let $R'$ be any point on $h'$ other than $P'$, and let $R$ be any point satisfying $\varphi(R) = R'$. Let $h$ be the line which joins $P$ and $R$. Since $P'$ and $R'$ both lie on both of $\varphi(h)$ and $h'$, $\varphi(h) = h'$. Since $h' \cap g' = 0$, $h \cap g = 0$. Let $x$ be any line through $P$ which fails to meet $g$. Then Axiom 3c implies $x' \parallel g'$, hence $x' = h'$ and $x \sim h$. Conversely, if $x \sim h$, then $x' = h'$; hence $x' \cap g' = 0$; hence $x \cap g = 0$. We have established a one-to-one correspondence between the lines through $P$ not neighbor to $h$ and the points of $g$. Then $s(P) = s(g)$.

Now let $P$ and $Q$ be given points. Let $g$ be any line such that neither $P'$ nor $Q'$ lies on $g'$. Then $s(P) = s(g) = s(Q)$, and the existence of an integer $s$ which satisfies (c) is established. Now, if $h$ is an arbitrary line, there exists a point $R$ such that $R' \not\sim h'$; then $s(h) = s(R) = s$, and the proof of (b) is complete.

Let $P$ be any point of $\alpha$, and choose $Q$ so that $Q' \neq P'$. Let $P = P_1, P_2, \ldots, P_n$ denote all the points neighbor to $P$; and let $p_i$ denote the line joining $P_i$ to $Q$. Since every $p_i'$ contains both $P'$ and $Q'$, every $p_i \sim P_1 = p$. Conversely, let $x$ be any neighbor of $p$ which contains $Q$. Let $h$ be any line such that $P'$ lies on $h'$ and $h' \neq p'$. If $x \cap h' = 0$, then Axiom 3c implies $h' \parallel x' = p'$. Since $P' \in h' \cap p'$, $h' = p'$. From the
contradiction, one concludes that \( x \cap h \) contains a point \( X \). Since \( X' \in p' \cap h' = \{P'\} \), \( X \sim P \). Then each of the \( t \) neighbors of \( p \) which contains \( Q \) also contain a neighbor of \( P \) and in fact, by (a), contain precisely \( t \) neighbors of \( P \). Since no neighbor of \( P \) can lie on more than one line through \( Q \) (\( P \not\sim Q \)), \( P \) must have at least \( t^2 \) neighbors. Since \( p_i \sim p \) for each neighbor point \( P_i \) of \( P \), the number of neighbors of \( P \) must be precisely \( t^2 \).

It follows from (a), (b), (c), Axiom 1, and the definition of the neighbor relation for points that the number of points of \( \alpha \) which are not neighbor to a given point \( P \) of \( \alpha \) must be \( (s + t)(s - t) = s^2 - t^2 \). That \( \alpha \) contains \( s^2 \) points follows from the fact that \( P \) has \( t^2 \) neighbors. Since \( \alpha \) possesses at least two points, \( \alpha \) also has a pair of points, and Axiom 1 implies that some line contains at least two points. Then (b) implies that every line is incident with a point of \( \alpha \). We may therefore calculate the number of lines of \( \alpha \) by multiplying the number of points of \( \alpha \) by the number of lines incident with a given point and dividing by the number of points per line. The truth of (f) follows.

Since every point of \( \alpha \) has \( t^2 \) neighbors, (e) and Axiom 3a imply that \( \alpha \) has \( s^2/t^2 \) points, i.e., the order of \( \alpha \) is \( s/t \). The truth of (g) is established. It also follows that \( \alpha \) possesses \( (s^2/t^2) + (s/t) \) lines. It follows from (f) and Axiom 3b that the average number of neighbors for a line of \( \alpha \) is \( t^2 \). Let \( \{g_1, \ldots, g_n\} \) be an arbitrary neighbor class of lines of \( \alpha \), Let \( P' \in g_1 \), and let \( h \) be a line of \( \alpha \) such that \( g_i \cap h' = \{P'\} \). Axioms 3c and 2 imply that \( g_i \cap h = 1 \) for \( i \leq n \). If \( P_i \) is the unique point in \( g_i \cap h \), then \( P_i' \in g_i \cap h' \), so \( P_i' = P' \). Then the \( P_i \) are all neighbors; by (a), there are at most \( t \) distinct \( P_i \) and at most \( t \) distinct \( g_j \) incident with a given \( P_i \). Then \( n \leq t^2 \), hence \( n = t^2 \); the proof of (d) is complete.

To prove (h), we assume \( t \neq 1 \) and count the number \( n \) of neighbors of \( P \) distinct from \( P \), each as many times as there are lines joining the point to \( P \). By (c) and (a), \( n = (s + t)(t - 1) \). By (d), (a), and Axiom 2, \( n \leq t^2 - 1 \). Then \( s + t \leq t^2 + t \) from which (h) follows.

Proposition 1.6(g) implies the existence of an integer \( r \) such that \( s = t \cdot r \); henceforth we shall write \((t, r)\) NAH-plane whenever we wish to make the invariants explicit. The reader is warned that this is a departure from previous usage: in [1], Bacon has written of \((s, t)\) AH-planes.

2. FNAH-PLANES

In view of Proposition 1.3, we now turn our attention to the study of a finite FNAH-plane \( \alpha \) for which \( \alpha \) has order 2. Call such an \( \alpha \) a 2FNAH-plane. Clearly, a 2FNAH-plane \( \alpha \) has precisely four neighbor classes of
points. Let $P_1, P_2, P_3, P_4$ be representatives of the four classes. Let $Q_i \sim P_i$, and let $g$ be the line joining $P_j$ to $Q_i$, $i \neq j$. By Lemma 1.5, the number of lines through $P_j$ which are neighbor to $g = t(Q_i, x)$ where $x$ is any line through $Q_i$ not neighbor to $g$; for example one may take $x = Q_iP_k$ where $i \neq k \neq j$. By Lemma 1.1, $t(Q_i, x)$ is independent of $x$, and thus we denote this integer more briefly by $t(Q_i)$. If we take another neighbor $Q_i'$ of $P_i$ and join it to $P_j$ by $g'$, we see that $t(Q_i', x') = t(Q_i)$, since $g' \sim g$. Then set $t_i = t(Q_i) = t(P_i)$. We see that there are $t_i$ lines neighbor to $g$ which contain the point $P_j$. Each contains $t_i$ points neighbor to $g$.

We collect our results in the following proposition:

**Proposition 2.1.** Let $\alpha$ be a 2FNAH-plane. Let $\{\mathcal{I}_i : 1 \leq i \leq 4\}$ denote the four neighbor classes of points of $\alpha$; let $\mathcal{I}_{ij} = \mathcal{I}_{ji}, 1 \leq i < j \leq 4$, denote the neighbor class of lines whose incident points belong to $\mathcal{I}_i$ and $\mathcal{I}_j$. Associated with $\alpha$ are four positive integers $t_i, 1 \leq i \leq 4$, such that

(a) if $(P, g)$ is a flag with $P \in \mathcal{I}_i$, then the number of points neighbor to $P$ on $g$ is $t_i$;
(b) $\mathcal{I}_i = t_i^2$;
(c) if $P \in \mathcal{I}_i$ and $i \neq j$, then the number of lines of $\mathcal{I}_{ij}$ which contain $P$ is $t_j$.
(d) $\mathcal{I}_{ij} = t_it_j$.

One calls the $t_i$ of Proposition 2.1 the invariants of $\alpha$.

**Corollary 2.2.** Let $\alpha$ be a finite FNAH-plane. Then the following assertions are equivalent:

(a) $\alpha$ is an NAH-plane.
(b) The number of lines through a given point $P$ which are neighbor to a given line $g$ through $P$ is independent of the flag $(P, g)$.
(c) The number of points in a neighbor class is an invariant of $\alpha$.
(d) The number of lines in a neighbor class is an invariant.
(e) The number of points on a line is an invariant.
(f) The number of lines incident with a given point is an invariant.

**Proof.** If $\bar{\alpha}$ has order $\neq 2$, the truth of every condition follows from Propositions 1.3 and 1.6. Then we may assume that $\bar{\alpha}$ has order 2. By
Proposition 2.1, each of the six conditions is equivalent to the requirement that all four \( t_i \) are equal.

Let \( \{i, j, k, l\} = \{1, 2, 3, 4\} = F, t_i \geq t_m \) for \( m \in F \). Then one defines the deviance of \( \alpha \) to be

\[
d(\alpha) = 3t_i - t_j - t_k - t_l.
\]

**Proposition 2.3.** Let \( \alpha \) be a 2FNAH-plane. Then \( d(\alpha) \leq \max\{t_m\} - 2 \) if \( \max\{t_n\} > 1 \); otherwise, \( \alpha \) is the affine plane of order 2.

**Proof.** Without loss of generality, we assume \( t_1 = \max\{t_m\} \). The result follows trivially from Proposition 2.1 if \( t_1 = 1 \), so we assume that \( t_1 > 1 \). By proposition 2.1(d) and (a), the number of lines of \( \alpha \) which contain points of \( \mathcal{S}_1 \) is \( t_1(t_2 + t_3 + t_4) \), and each such line contains \( t_1 \) points of \( \mathcal{S}_1 \). We total the number of pairs of points of \( \mathcal{S}_1 \), counting each pair \((P, Q)\) as many times as there are lines which join \( P \) to \( Q \). The total is exactly

\[
\left( \begin{array}{c} t_1 \\ 2 \end{array} \right) t_1(t_2 + t_3 + t_4).
\]

Since each pair must be joined by at least two lines, the total is at least

\[
2 \cdot \left( \begin{array}{c} t_1^2 \\ 2 \end{array} \right).
\]

The result follows.

We now introduce some useful language. If \( \mathcal{G} \) is a set of lines, we say the points \( P \) and \( Q \) "meet in \( \mathcal{G} \)" to indicate that \( P \) and \( Q \) are joined by at least one line of \( \mathcal{G} \); we say \( P \) and \( Q \) "meet in \( i \) in \( \mathcal{G} \)" to mean that \( P \) and \( Q \) are joined by precisely \( i \) lines of \( \mathcal{G} \). We use the same language for the dual notion.

**Lemma 2.4.** Let \( \alpha \) be a 2FNAH-plane. Let \( \mathcal{G}_{1, t} \) be given, \( t \) denote \( t_1 \). Let \( \mathcal{S} = \{P, \ldots, P_t\} \) be a subset of \( \mathcal{S}_1 \) such that \( P_m \) and \( P_n \) meet in \( \mathcal{G}_{1, t} \) for all distinct \( m, n \leq t \). Let \( \{i, j, k\} = (2, 3, 4) \). Assume that \( t_j + t_k < 2t \) or that \( t_j \) and \( t_k \) are odd and \( t_j + t_k = 2t \). Then every point \( Q \in \mathcal{S}_1 \) meets at least one element of \( \mathcal{G} \) in \( \mathcal{G}_{1, t} \).

**Proof.** If \( m \neq n \) and \( m, n \leq t \), then \( P_m \) and \( P_n \) cannot meet in either of \( \mathcal{G}_{1, t} \) or \( \mathcal{G}_{1, k} \). Each point \( Q \) of \( \mathcal{S}_1 \) must meet each \( P_m \) in at least 2. Then since \( Q \) lies on precisely \( t_j \) lines of \( \mathcal{G}_{1, t} \) (Proposition 2.1(c)), \( Q \) meets at most \( t_j/2 \) points of \( \mathcal{G} \) in \( \mathcal{G}_{1, t} \). Similarly, \( Q \) meets at most \( t_k/2 \) points of \( \mathcal{G} \) in \( \mathcal{G}_{1, k} \). Since \( Q \) must meet all \( t \) points of \( \mathcal{S} \), \( Q \) must meet some point of \( \mathcal{G} \) in \( \mathcal{G}_{1, t} \).

**Lemma 2.5.** Let \( \alpha, \mathcal{G}_{1, t}, \mathcal{S} \) be the objects described in Lemma 2.4.
Let $\mathcal{G} = \{g_1, \ldots, g_k\}$ be the set of all lines of $\mathcal{G}_{1i}$ which are incident with one or more points of $\mathcal{S}$. Under the hypotheses of Lemma 2.4, it follows that $k \geq 2t_i + t_i - 2$.

**Proof.** By Proposition 2.1(a) and (c), the number of incidences of lines of $\mathcal{G}$ with points of $\mathcal{S}_1$ is $kt$, with points of $\mathcal{S}_1$ is $t_1 \cdot t_i$. Then (since neighbor points meet in at least two lines) the number $j$ of points of $\mathcal{S}_1$ which lie on lines of $\mathcal{G}$ is at most $j = (k - t_i) t_i/2$. By Lemma 2.4, $j = t_i^2$, and the conclusion follows.

**Corollary 2.6.** Let $\alpha$ be a 2FNAH-plane, not the affine plane of order 2. Let $t_1, t_2, t_3, t_4$ be the invariants of $\alpha$, $t_1 \geq t_i$ for all $i$.

(a) If $t_1 \leq 5$, then $t_1 = t_2 = t_3 = t_4 = b$ and $b$ is 2 or 4.

(b) If $t_1 = 2n + 1$ for $n \geq 3$, then $t_i \geq 6$ for all $i$.

(c) If $t_1 = 2n$ for $n \geq 3$, then either $t_i \geq 6$ for all $i$; or exactly two of $t_2, t_3, t_4$ are equal to $t_1$.

**Note 2.7.** Corollary 2.6 includes the result that no NAH-planes exist for which $(t, r) = (3, 2)$ or $(5, 2)$. In [4] an alternate proof of this result is given which also excludes the case $(t, r) = (7, 2)$.

**Proof of Corollary 2.6.** If $t_1 = 1$ or 2, the desired conclusion follows from Proposition 2.3. Then let $t = t_1 > 2$. Let $\{i, j, k\} = (2, 3, 4)$. Assume either that $t$ is odd or that $t > one of $t_2, t_k$. Let $g = g_1 \in \mathcal{G}_{1i}$. Let $\mathcal{S} = \{P_1, \ldots, P_t\}$ be the set of points of $\mathcal{S}_1$ incident with $g$. Let $\mathcal{G} = \{g_1, \ldots, g_k\}$ be the set of all lines of $\mathcal{G}_{1i}$ which contain points of $\mathcal{S}$. Since $g \cap g_1 \cap g_2 \cap g_3 \cap g_4$ for $2 \leq \mathcal{m} \leq k$, Proposition 2.1(c) yields $2(k - 1) \leq t_i(t_i - 1)$. By Lemma 2.5, one obtains

\[(2.1) \quad t_1(5 - t_i) + 2t_i \leq 6.\]

The conclusion now follows for $t_1 > 2$.

### 3. Uniform NAH-PLANES

A finite AH-plane is “uniform” if and only if $t = 1$ or $t = r$. (See [5, Definition 2.4, Satz 2.131.]) We now define a $(t, r)$ NAH-plane to be uniform if and only if $t = 1$ or $t = r$.

**Proposition 3.1.** If $\alpha$ is a uniform NAH-plane, then there exists a parallel relation which makes $\alpha$ an AH-plane.
Proof. If \( t = 1 \), Proposition 1.6 says that \( \alpha \) is an affine plane. Then \( \alpha \) is an AH-plane under the parallel relation which \( \alpha \) possesses as an affine plane. Next, assume \( t = r \). By Proposition 1.6(c), (a), (d), the average number of lines which join a point \( P \) to a neighbor distinct from itself is \((s + t)(t - 1)/(t^2 - 1) = t\). If \( P \) and \( Q \) are distinct neighbors, then all lines joining \( P \) and \( Q \) are neighbor by Axiom 2. By Proposition 1.6(a), \( P \) and \( Q \) are joined by at most \( t \) lines and hence by exactly \( t \) lines whenever \( P \) and \( Q \) are distinct neighbors. It follows (again by Proposition 1.6(a)) that each pair of distinct neighbor lines of a intersect in \( t \) points.

Now let \( \mathcal{N} \) denote a complete neighbor class of lines of \( \alpha \). Name the elements of \( \mathcal{N} \): \( g(1), \ldots, g(t^2) \). Let \( \mathcal{M}' \) denote the set of all points of \( \alpha \) which are incident with one or more \( g(i) \). By Proposition 1.6 (b and a), \( \mathcal{M}' = st^2/t = t^3 \). If \( P \in \mathcal{M}' \), then every \( g(i) \) contains \( t \) neighbors of \( P \), and each neighbor of \( P \) lies on at most \( t \) of the \( g(i) \). Then at least \( t^2 \) neighbors of \( P \) lie in \( \mathcal{M}' \). By Proposition 1.6(d), \( \mathcal{M}' \) is composed of \( t \) complete neighbor classes of points of \( \alpha \). Then each point of \( \mathcal{M}' \) lies on the same \( t \) lines of \( \mathcal{N} \) as do \( t - 1 \) other points of \( \mathcal{M}' \). The property of being incident with the same \( t \) lines of \( \mathcal{N} \) thus induces an equivalence relation on \( \mathcal{M}' \) separating \( \mathcal{M}' \) into \( t^3 \) classes of \( t \) points each. Let \( \mathcal{M} \) denote a subset of \( \mathcal{M}' \) containing one representative from each equivalence class.

We now let \( \pi \) denote the incidence structure whose points are the lines of \( \mathcal{N} \), whose lines are the points of \( \mathcal{M} \), and whose incidence relation is the one induced from \( \alpha \). We may represent \( \mathcal{M} \) as a disjoint union of sets, \( \mathcal{M}(1), \ldots, \mathcal{M}(t) \), each consisting of \( t \) neighbor points of \( \alpha \). By choice of \( \mathcal{M} \), no point of \( \pi \) lies on more than one line of each set \( \mathcal{M}(i) \). By counting, each point of \( \pi \) lies on exactly one line of each set \( \mathcal{M}(i) \). Since no line of \( \mathcal{M}(i) \) may intersect any line of \( \mathcal{M}(j) \) for \( i \neq j \), each line of \( \mathcal{M}(i) \) must meet each line of \( \mathcal{M}(j) \) in exactly one point of \( \pi \). Then \( \pi \) is a t-net of order \( t \) in \( \alpha \), and hence may be extended to an affine plane of order \( t \) by adjoining an additional parallel class \( \mathcal{M}(t + 1) \) of lines.

We now define lines \( g(i) \) and \( g(j) \) to be parallel in \( \alpha \) if and only if when regarded as points of \( \pi \), they lie on the same line of \( \mathcal{M}(t + 1) \). Thus \( \mathcal{N} \) is divided into \( t \) disjoint parallel classes of \( t \) lines each. Clearly.

\[
(3.1) \quad \text{each point of } \mathcal{M} \text{ and hence each point of } \mathcal{M}' \text{ lies on precisely one line of each parallel class in } \mathcal{N}.
\]

By the same method, one can define parallel classes of lines within every neighbor class \( \mathcal{N} \) of lines of \( \alpha \). By Proposition 1.6(g), \( \tilde{\alpha} = \varphi(\alpha) \) is an affine plane of order \( t \). Then we may label the neighbor classes of lines \( \mathcal{N}(i,j), 1 \leq i \leq t + 1, 1 \leq j \leq t \), so that a line from \( \mathcal{N}(i,j) \) and a line from \( \mathcal{N}(i',j') \) have parallel images under \( \varphi \) if and only if \( i = i' \). Label the
parallel classes within each $N(i, j)$ by $N(i, j, k)$, $1 \leq k \leq t$. We now define a parallel relation on the lines of $\alpha$ as a whole according to the following rule: a line of $N(i, j, k)$ is parallel to a line of $N(i', j', k')$ if and only if $i = i'$ and $k = k'$. For a given $i$ and a given point $P$ of $\alpha$, there exists at most one $j$ such that $P$ lies on a line of $N(i, j)$. We proved above that there are $t^3$ points incident with the lines of each $N(i, j)$; and, by Proposition 1.6(e), $\alpha$ contains $t^4$ points. Then, for specified $i$, each point of $\alpha$ must lie on lines from precisely one of the classes $N(i, j)$. By (3.1), each point of $\alpha$ lies on precisely one line of each parallel class. Thus it is possible to define a parallelism on $\alpha$ so that Axiom 4 is satisfied, i.e., so that $\alpha$ is an AH-plane.

4. The Example

**Proposition 4.1.** Let $s$, $t, r = s/t$ be positive integers. Let $A_{ij}$ be a $t^2 \times t^2$ $(0, 1)$-matrix for $1 \leq i \leq r^2$, $1 \leq j \leq r^2 + r$. Let $M$ denote the $s^2 \times (s^2 + st) [A_{ij}]$. Define the $t^2 \times (t^2 + r)$ $(0, 1)$-matrix $P = [p_{ij}]$ by the condition that $p_{ij} = 0$ if and only if $A_{ij}$ is the O-matrix. We write $B_{ij}$ to denote $A_{ij}$ if and only if $A_{ij}$ is not the O-matrix. Then $M$ is the incidence matrix for a $(t, r)$ NAH-plane provided the following conditions hold:

(a) $P$ is the incidence matrix of an affine plane.

(b) The inner product of distinct columns from any $B_{ij}$ is $\neq 1$.

(c) The inner product of a column of $B_{ij}$ and a column of $B_{ik}$ is 1 for all $i, j, k$ with $j \neq k$.

(d) The inner product of a row of $B_{ij}$ and a row of $B_{kj}$ is 1 for all $i, j, k$ with $i \neq k$.

(e) Let $n = r^2 + r$. Then for any $i \leq r^2$, the inner product of distinct rows of $A_i := [A_{i1}, A_{i2}, \ldots, A_{in}]$ is $\geq 2$.

Conversely, every $(t, r)$ NAH-plane may be represented by an incidence matrix $M$ which satisfies all the above conditions.

**Proof** Assume the existence of a matrix $M$ which satisfies conditions (a)-(e). Let $\alpha$ be the incidence structure represented by $M$, lines of $\alpha$ being represented by columns of $M$. By (d) and (e), Axiom I is satisfied, i.e., each pair of points of $\alpha$ is joined by a line of $\alpha$. It also follows from (d) and (e) in conjunction with (a) that points of $\alpha$ are neighbor if and only if they are represented by rows from one of the $t^2$ successive sets of $t^2$ rows of $M$. It follows from (c) and (a) that every column of every $B_{ij}$ contains at least one 1. From the existence of these 1’s and another
application of (a), one concludes that lines of $\alpha$ are neighbor if and only if they are represented by columns from one of the $r^2 + r$ successive sets of $r^2$ columns of $M$. The truth of Axiom 2 follows from (a), (b), and (c). If the affine plane represented by $P$ is denoted by $P$, it is clear that there exists an epimorphism $\varphi$ from $\alpha$ to $\beta$ which satisfies Axiom 3(a) and Axiom 3(b). It follows from conditions (a) and (c) that Axiom 3(c) is also satisfied for this $\varphi$. Then $\alpha$ is a finite FNAH-plane. It follows from Corollary 2.2(a) and (c) that $\alpha$ is an NAH-plane. The proof of the converse is also easy and is left to the reader.

Remark 4.2. A result similar to Proposition 4.1 can be proved for 2FNAH-planes. Conditions (a)-(e) remain unchanged, but one must make appropriate changes in the size of $M$ and the $A_{ij}$ as dictated by Proposition 2.1.

Remark 4.3. It is sometimes possible to construct the incidence matrix $M$ of a $(t, r)$ NAH-plane by constructing only a few of the $B_{ij}$. One proceeds in accord with the following recipe. First, find an incidence matrix $P$ for an affine plane of order $r$. Second, find a set of $t^2 \times t^2 (0, 1)$ matrices $C_1, C_2, \ldots, C_r$ which can be used for the $B_{ij}$ in a single row of $A_{ij}$'s. Specifically, conditions (b), (c), and (e) of Proposition 4.1 must be satisfied. Now $P$ has an extension to an incidence matrix $P'$ of a projective plane of order $r$. As such, $P'$ can be represented as a sum of $r + 1$ permutation matrices. (See, e.g. [6, p. 57, Theorem 5.31.] It follows that one may obtain a matrix $N$ from the matrix $P$ by replacing each 0 by a $t^2 \times t^2$ submatrix of O's and each 1 by some $C_i$ and that this may be done in such a fashion that each $C_i$ replaces exactly one 1 of each row and at most one 1 of each column. All the requirements of Proposition 4.1 with the possible exception of (d) are now satisfied by $N = [A_{ij}]$. If $\sigma$ denotes a permutation of the integers $1, 2, \ldots, t^2$, we write $\sigma(A_{ij})$ to denote the matrix obtained from $A_{ij}$ by permuting columns according to the permutation $\sigma$. If one replaces any number of $A_{ij}$ in $N$ by $\sigma(A_{ij})$, the resulting matrix $M$ will still satisfy conditions (a), (b), (c), and (e) of Proposition 4.1. The last step in the construction of the incidence matrix of a $(t, r)$ NAH-plane is thus to select a collection of permutations $\sigma_{ij}$ in such a way that the resultant $M$ satisfies condition (d).

We now proceed to follow the recipe of Remark 4.3 to create a $(6, 2)$ NAH-plane which is not an AH-plane. In Figure 4.1 are displayed three $(0, 1)$ matrices: $A$, $B$, and $C$. In the matrix $A$, the letter Z stands for an $18 \times 18$ matrix of O's; in the matrices $B$ and $C$, the Z's stand for submatrices of O's of various appropriate sizes.

We now define a permutation $\sigma$ on the columns of $B$ to obtain a new $36 \times 18 (0, 1)$-matrix $B'$; we apply the same permutation $\sigma$ to the columns
of C to obtain a matrix C'. In addition, we define permutations \( \tau \) and \( \tau' \) which, when applied to the columns of C, yield respectively the matrices \( C'' \) and \( C''' \).

\[
\sigma = \begin{bmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 \\
\end{bmatrix}
\]

\[
\tau = \begin{bmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18
\end{bmatrix}
\]

\[
\tau' = \begin{bmatrix}
10 & 12 & 18 & 14 & 16 & 19 & 17 & 18 & 13 & 10 & 15 & 12 & 16 & 17 & 18 & 19
\end{bmatrix}
\]

We now set \( B_1 = [B \ B], \ B_2 = [B' \ B'], \ C_1 = [C \ C], \ C_2 = [C' \ C'], \ C_3 = [C'' \ C'''] \). Finally, we let 0 denote the 36 x 36 matrix of 0's. Then each of the following is a 36 x 36 (0, 1) matrix: A, Bi, Cj, 0. We define \( \alpha \) to be the incidence structure whose incidence matrix \( M \) is given in Figure 4.2 (the points of \( \alpha \) are represented by the rows of \( M \); the lines of \( \alpha \), by the columns of \( M \)):

\[
M = \begin{bmatrix}
A & B2 & C2 & 0 & 0 & 0 \\
C2 & 0 & 0 & A & B1 & 0 \\
0 & A & 0 & B2 & 0 & C3 \\
0 & 0 & A & 0 & C3 & B1
\end{bmatrix}
\]
We now proceed to prove that \( \alpha \) satisfies all the conclusions of Proposition 4.1. By tedious inspection, we see that, for \( r \neq s \), the inner product of the \( r \)-th and \( s \)-th rows is \( \geq 2 \) for precisely one of \( A, B_1, C \), and is \( = 0 \) for the other two matrices. One can reduce somewhat the necessary checking by making the following observation: in each of \( A, B, \) and \( C \), the inner product of the \( r \)-th and \( s \)-th rows is the same as the inner product of the \( (r + 18) \)-th and \( (s + 18) \)-th rows provided \( 1 \leq r, s \leq 18 \). In particular, to see that this is so for \( A \), observe that \( A \) is of the form \( \text{diag}[D, D] \). A little ingenuity also makes it quite easy to check the inner product of each of the first 18 rows with each of the last 18. Since \( B_2, C_2, \) and \( C_3 \) are obtained from \( B_1 \) and \( C_1 \) by column permutations, we have the following result.

\[
(4.1) \quad \text{Given } i = 1 \text{ or } 2 \text{ and given } j = 2 \text{ or } 3, \text{ then for all } r \neq s, \text{ the inner product of the } r\text{-th and } s\text{-th rows is } \geq 2 \text{ for precisely one of } A, B_i, C_j, \text{ and is } = 0 \text{ for the other two matrices.}
\]

In particular, condition (e) of Proposition 4.1 is satisfied. Next, we observe that each row of \( A \) has inner product 1 with each row of \( B_2 \). Since \( A \) is of the form \( \text{diag}[D, D] \) and since \( B_2 \) has only 12 distinct rows, one need only check the first 18 rows of \( A \) against 12 of the rows of \( B_2 \). Since the rows of \( C_2 \) are the same as the rows of \( B_2 \), every row of \( A \) also has inner product 1 with every row of \( C_2 \). Last, we observe that every row of \( C_3 \) has inner product 1 with every row of \( B_1 \). To make the comparison between \( C_3 \) and \( B_1 \), it is helpful to observe that, for \( 1 \leq i \leq 9 \), \( T'(i) = \tau(i + 9) = \tau(i) + 9 \) and \( \tau'(i + 9) = \tau(i) \). Putting all these observations together, we see that condition (d) of Proposition 4.1 is satisfied.

By inspection, no two columns of \( B \) (and no two columns of \( C \)) have inner product 1. Then no two columns from any one of \( B_1, B_2, C_2, \) or \( C_3 \) has inner product 1. That no two columns of \( A \) have inner product 1 follows from (4.1) and the observation that \( A \) is symmetric. Then condition (b) of Proposition 4.1 is satisfied.

Now represent \( A, B, C \) by

\[
A = \begin{bmatrix} D & Z \\ Z & D \end{bmatrix}, \quad B = \begin{bmatrix} E \\ F \end{bmatrix}, \quad C = \begin{bmatrix} G \\ H \end{bmatrix},
\]

where \( D, Z, E, F, G, \) and \( H \) are all \( 18 \times 18 \). Observe that the column vectors of \( E, F, \) \( G \) and \( H \) all match row vectors of \( B' \). It has been observed already that row vectors of \( B_2 = [B' B'] \) have inner products \( ] \) with row vectors of \( A \). Since \( Z \) is a matrix of \( 0 \)’s and \( D \) is symmetric, it follows that columns of \( A \) have inner products 1 with columns of \( B \) or \( C \). It follows
that the inner product of a column of A together with a column of \( B_i \) or Cj for any \( i \) or j is 1. By inspection, any column of B has inner product 1 with any column of C, hence any column of \( B_i \) has inner product 1 with any column of Cj for all \( i \) and j. Then condition (c) of Proposition 4.1 is satisfied by \( M \); it is clear that condition (a) is also satisfied. Then \( M \) is the incidence matrix of a (6, 2) NAH-plane \( \alpha \).

(4.2) We now prove that no parallelism can be defined on the lines of \( \alpha \) in such a way as to make \( \alpha \) into an AH-plane.

Assume that \( \alpha \) does possess such a parallelism. Label a point of \( \alpha \) by \( P(i,j) \) if it is represented by the \((36i + j)\)-th row of \( M \), the first row of \( M \) being counted as the 0-th row. Similarly, we label the lines of \( \alpha \) by \( g(i,j) \) where \( 0 \leq i \leq 5 \) and \( 0 \leq j \leq 35 \). Let \( \pi \) be the parallel class containing \( g(0,0) \). Since distinct parallel lines cannot intersect, every point \( P(0,i) \) with \( 0 \leq i \leq 17 \) must lie on some line \( g(0,j) \) of \( \pi \) where \( j \leq 17 \). Clearly there must be precisely three \( g(0,j) \) in \( \pi \) with \( j \leq 17 \). However, \( g(0,0) \) intersects every \( g(0,j) \) with \( 14 \leq j \leq 17 \) in some \( P(1,i) \). The contradiction completes the proof of (4.2).

Remarks

The existence of \( \alpha \) establishes the independence of Axiom 4 in the definition of AH-planes used in this paper (Definition 1.2). One may also define AH-planes as incidence structures with parallelism satisfying axioms which refer neither to an associated affine plane nor to any epimorphism \( \varphi \). (See [5, p. 263].) This requires eight axioms including Axioms 1, 2, and 4 of Definitions 1.1 and 1.2. We use \( \beta \) to denote the incidence structure of \( \alpha \) together with the trivial parallel relation (\( h \parallel g \) if and only if \( h = g \)). Then it is easy to see that \( \beta \) satisfies seven of the eight axioms, all but Axiom 4 of Definition 1.2.

We remark that \( \alpha \) is the first known example of a \((t,r)\) NAH-plane for which \( t \) is not a power of \( r \). There also exist \((4,2)\) NAH-planes which cannot be made into AH-planes by any definition of a parallel relation; we have presented the larger example here, because it yields information on the existence question. Aside from the \((4,2)\) examples, there are no \((t,1)\) NAH-planes with \( t < 6 \) which cannot be made into AH-planes. To see that this is so, one may recall Proposition 3.1 and apply the following result of [4]: all \((t,r)\) NAH-planes with \( 2 \leq t \leq 6 \) satisfy \( t = r \) or \((t, r) = (4, 2) \) or \((6, 2)\). To construct a \((4,2)\) NAH-plane which is not the incidence structure of an AH-plane, one may begin with the Desarguesian
AH-plane $\pi$ defined over the ring $\mathbb{Z}_8$. ($\pi$ is a $\langle 4, 2 \rangle$ AH-plane.) Let $A_4$ be an incidence matrix for $\pi$ which satisfies conditions (a)-(e) of Proposition 4.1. One can permute the columns of any single $B_{ij}$ of $M$, obtaining a new matrix $M'$ which agrees with $M$ except in the replacement of $B_{ij}$. If the permutation is carefully selected, $M'$ will represent an incidence structure which possesses no parallel relation satisfying Axiom 4, but condition (d) of Proposition 4.1 will hold for $M'$. It will follow from Proposition 4.1 that $M'$ is the incidence matrix of a $\langle 4, 2 \rangle$ NAH-plane.

Let $a(P)$ denote the incidence structure induced by $\alpha$ on the 36 points neighbor to $P$ and the 18 lines incident with $P$. If one examines $\alpha$ carefully, he will notice that $a(P)$ is independent of the choice of $P$. We have represented $a(P)$ in Figure 4.3. Every line which descends from $P$ is induced by a single line of $\alpha$; each remaining line is induced by two lines of $\alpha$. In an incidence structure $\pi$, one defines (see [7, p. 94]) a funnel flag to be a flag $(P, g)$ which satisfies the following condition:

$$P, Q, R, \in g; P, Q \in h; P, R \in k \implies Q \in k \quad \text{or} \quad R \in h.$$ 

One calls $P$ a finnelpoint if $(P, g)$ is a funnel flag for all $g$ incident with $P$. It is clear that every point of $\alpha$ pairs with precisely 12 of its 18 incident lines to yield a funnel flag. Then no point of $\alpha$ is a funnel point. On the other hand, it is still an open question whether or not there exist AH-planes with points which are not funnel points.
Acknowledgment

The structure of $\alpha(P)$ can be obtained by taking points from the group $D_6 \oplus D_4$, lines being certain subgroups of order 6. The joint discovery of $\alpha(P)$ in this guise occurred during one of several helpful conversations the author has had with Mark Hale concerning AH-planes.

References

2. P. Y. Bacon, Strongly $n$-uniform and level $\eta$ Hjelmslev planes, Math. Z. 127 (1972), 1-9.