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TOPOLOGICAL QUANTUM FIELD THEORIES DERIVED FROM THE KAUFFMAN BRACKET

C. Blanchet, N. Habegger†, G. Masbaum and P. Vogel

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1. INTRODUCTION: DEFINITIONS, AXIOMS AND STATEMENT OF RESULTS

In [41], Witten has made the remarkable discovery of an intricate relationship between the *Jones polynomial* [15, 16] and *gauge theory.* (See also the prophetical article by Atiyah [2].) Although his approach uses the *Feynman path integral* of *Quantum Field Theory,* Witten gave convincing arguments that a viable combinatorial approach could be made rigorous using the method of *surgery.* His discovery includes new 3-manifold invariants (sometimes called *Jones–Witten invariants*), whose existence was first proven by Reshetikhin and Turaev [30] using *quantum groups* and Kirby's *surgery calculus* [19] (see also [20]). Other combinatorial approaches for related invariants were developed by Kohno [21], Turaev and Viro [36], Lickorish [22, 23], the authors [10], Morton and Strickland [29], Wenzl [40], Turaev and Wenzl [35].

According to Witten, his invariants should belong to a *topological quantum field theory* (*TQFT*). This notion was axiomatized by Atiyah *et al.* [6, 3] (see also [38]). In particular, the states of a manifold, Σ, form a *hermitian vector space*, $V(\Sigma)$ (more generally $V(\Sigma)$ is a module over a *commutative ring* k with unit and involution), and a *cobordism* M from Σ_1 to Σ_2 induces a transition (k-linear map), denoted Z_M, from $V(\Sigma_1)$ to $V(\Sigma_2)$. One has that $V(\emptyset)$ is the ground ring k, so that if $\partial M = \Sigma$ (i.e., M is a cobordism from $\emptyset$ to Σ), one obtains a vector $Z(M)$ in $V(\Sigma)$, given by $Z(M) = Z_M(1)$. Thus, M induces a *state* of ∂M. In particular, if M is closed, $Z(M)$ (also denoted by $\langle M \rangle$ in keeping with the physicists' *expectation value* notation) lies in $V(\emptyset) = k$, so that TQFTs, by their very nature, yield manifold *invariants.*

In this paper, we give a purely topological construction of the TQFTs associated to invariants satisfying the Kauffman bracket relations [17], that is, essentially, of the TQFTs corresponding to Jones' original V-polynomial [15].

We renormalize the invariants θ_p of [10] to construct a series of invariants $\langle\ \rangle_p$ of banded links in closed 3-manifolds, and then use these invariants to define, in a "universal" way, modules $V_p(\Sigma)$ ($p \geq 1$), associated to surfaces Σ (which may have banded links, too). Here, for technical reasons, all manifolds are equipped with a p_1-structure (a weak form of framing, see Appendix B). We prove the finiteness and multiplicativity properties of the $V_p(\Sigma)$, using the language of bimodules over algebroids. It turns out that the ranks of our modules are given by *Verlinde's formula.* Thus, we are led to believe that ours is a rigorous construction of Witten's theory for $SU(2)$ (and in some sense also for $SO(3)$, see Remark 1.17).

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Finally, we describe an action of a Heisenberg type group $\Gamma(\Sigma)$ on the modules $V_{2p}(\Sigma)$. This action is used to obtain a natural decomposition of $V_{2p}(\Sigma)$ into a non-trivial tensor product, or into a direct sum of subspaces (whose ranks are computed explicitly), associated to spin structures or cohomology classes on the surface.

Remark. The invariants θ_p (and $\langle\ \rangle_p$) are constructed from the Kauffman bracket [17] evaluated at a primitive $2p$th root of unity. If $p = 2r$ is even, such invariants were first constructed by Lickorish [22, 23]. They are closely related to the invariants $\tau_r(M)$ constructed by Reshetikhin and Turaev [30] and Kirby and Melvin [20] from the representation theory of the quantum group $U_qSU(2)$ at $q = e^{2\pi i/r}$. If p is odd, the invariant θ_p is related to the refined invariant $\tau_r'(M)$ of [20] (see [9] for details).

The remainder of this section is a detailed introduction to the results of this paper. This introduction has four parts. In Section 1.A, we describe what we mean by a TQFT, and, more generally, by a quantization functor on a cobordism category. In Section 1.B, we state the main theorems of this paper. In Section 1.C, we state additional properties which serve to characterize the theories dealt with in this paper. These are surgery properties and the Kauffman bracket relations. Finally, in Section 1.D we give an overview of the method of proof of our results. This method may be useful in other contexts.

1.A. Manifold invariants, quantization functors, and TQFT

Cobordism categories

Recall that an oriented $(n+1)$-manifold M with boundary decomposed as $\partial M = -\Sigma_1 \coprod \Sigma_2$, where Σ_1, Σ_2 are oriented n-manifolds, and $-\Sigma_1$ means Σ_1 with reversed orientation, is called a *bordism*, or a *cobordism*† from Σ_1 to Σ_2 (see [31]). Given a cobordism M_1, from Σ_1 to Σ, and a cobordism M_2, from Σ to Σ_2, one can *glue* these together along Σ to obtain a cobordism from Σ_1 to Σ_2. In this way, one may define a *category* whose objects are the oriented n-manifolds, whose morphisms are equivalence classes of cobordisms, and where gluing plays the role of composition. Here, two cobordisms from Σ_1 to Σ_2 are called equivalent if they are isomorphic rel. boundary (i.e. the isomorphism is required to be the identity on Σ_1 and Σ_2). Taking equivalence classes ensures that composition is associative, and the product manifold, $[0, 1] \times \Sigma$, plays the role of the identity morphism of Σ. Observe that this category has an involution (given by orientation reversal) and finite sums (given by disjoint union).

In practice, manifolds and bordisms generally have additional *structure* (e.g. they are compact, smooth or piecewise linear, they may be equipped with a tangential structure, such as a framing or spin structure, and they may contain subobjects such as submanifolds or framed submanifolds).

The main example in this paper is the category $C_2^{p_1}$ of smooth closed oriented 2-manifolds with p_1*-structure* (see Appendix B) and containing a *banded link* (i.e. a set of embedded oriented intervals). The bordisms are thus compact smooth oriented 3-manifolds with p_1-structure containing a *banded link.* (That is, a set of embedded oriented surfaces diffeomorphic to the product of a 1-manifold with an interval, meeting the boundary in the product of the boundary of the 1-manifold with the interval. Only the band, and not the 1-manifold, called the *core* of the band, is assumed to be oriented.) We also consider the full subcategory $C_2^{p_1}$ (*even*), where objects are restricted to surfaces having a banded link with an even number of components. The appropriate notion of equivalence on the bordisms is

† We use the prefix co- in cobordism to signify "mutually bordant".

orientation-preserving diffeomorphism (rel. boundary), which restricts to an orientation preserving diffeomorphism of the banded links, and such that on the mapping cylinder of the diffeomorphism, there is a p_1-structure extending the one given on its boundary. Note that, in particular, isotopic links (rel. boundary) and homotopic p_1-structures (rel. boundary) are equivalent.

Remark. The introduction of p_1-structure arises since the invariants we consider turn out to have a *framing anomaly*,† i.e. the invariants themselves depend on the p_1-structure (albeit in a very weak way, see 1.8). (There is in fact another way of resolving the framing anomaly, by explicitly using the signature cocycle; see [38].) See also [4] for an interpretation in terms of *2-framings*.

The main aim of this paper is to construct and describe TQFT-functors on the category $C_2^{p_1}$. The latter notion makes sense in general on any *cobordism category*, i.e. a category together with an empty object $\emptyset$ and a notion of *disjoint union* (denoted by $\coprod$), *orientation reversal* (denoted by a minus sign), and *boundary* (denoted by ∂), satisfying the obvious axioms abstracted from the basic example of manifolds and cobordisms. For the remainder of part A of this introduction, let C be a cobordism category.

Quantization functors

Let k be a commutative ring with unit and conjugation denoted by $\lambda \mapsto \bar{\lambda}$. Consider a functor $V: C \to k$-modules, such that

(Q1) $$V(\emptyset) = k.$$

Notation. If M is a cobordism, the linear map $V(M)$ is denoted by Z_M (for historical reasons). Moreover, if M is (or is considered as) a cobordism from $\emptyset$ to ∂M, we write $Z(M)$ for the element $Z_M(1) \in V(\partial M)$. Denote by $\langle M \rangle$ the element $Z(M) \in k$, if M is a *closed bordism*, i.e. if $\partial M = \emptyset$.

Remark. Since morphisms in C are equivalence classes of bordisms, the function $\langle\ \rangle$ is an *invariant* of closed bordisms.

The functor V is called a *quantization functor* if it satisfies (Q1) above and the following condition (Q2).

(Q2) *There is a non-degenerate‡ hermitian§ sesquilinear form* $\langle\, , \rangle_\Sigma$ *on* $V(\Sigma)$, *such that if* $\partial M_1 = \partial M_2 = \Sigma$, *then*

$$\langle Z(M_1), Z(M_2)\rangle_\Sigma = \langle M_1 \cup_\Sigma (-M_2)\rangle.$$

Remark. Condition (Q1) is understood in the following precise sense: the module $V(\emptyset)$ is equipped with an element $1 \in V(\Sigma)$ which is a k-basis. Note that one has the following:

$$\langle \emptyset \rangle = Z(\emptyset) = Z_\emptyset(1) = 1 \quad \text{and} \quad \langle 1, 1\rangle_\emptyset = 1.$$

† In fact, an early version of [41] inadvertently overlooked this anomaly. Witten's discovery of the anomaly was the cause of some excitement among the experts.

‡ Recall that a form is *nondegenerate* if the adjoint mapping is injective.

§ A sesquilinear form is called hermitian if $\langle y, x\rangle = \overline{\langle x, y\rangle}$.

We say that V is *cobordism generated, or C-generated,* if the following property holds:
(CG) *The elements $Z(M)$, with $\partial M = \Sigma$, generate $V(\Sigma)$.*

We say an invariant $\langle\ \rangle$ is *multiplicative*, if the following property holds:

(m) $$\langle M_1 \coprod M_2 \rangle = \langle M_1 \rangle \langle M_2 \rangle \quad \text{and} \quad \langle \emptyset \rangle = 1.$$

We say an invariant $\langle\ \rangle$ is *involutive*, if the following property holds:

(i) $$\langle -M \rangle = \overline{\langle M \rangle}.$$

The relationship between quantization functors and invariants is contained in the following result.

1.1. PROPOSITION. *If V is a quantization functor on the cobordism category C, then the association $M \mapsto \langle M \rangle$ is a multiplicative and involutive invariant. Conversely, given a multiplicative and involutive invariant on the set of closed bordisms of C, there is a unique cobordism generated quantization functor on C extending it.*

The proof of this result is straightforward from the definitions and will be left to the reader. It uses the *universal construction* described below.

The Universal Construction. Denote by $\mathscr{V}(\Sigma)$ the k-module freely generated by the set of all morphisms M from $\emptyset$ to Σ (i.e. such that $\partial M = \Sigma$). Given an invariant $\langle\ \rangle$, the formula

$$\langle M, M' \rangle_\Sigma = \langle M \cup_\Sigma (-M') \rangle$$

extends to a hermitian sesquilinear form $\langle\ ,\ \rangle_\Sigma$ on $\mathscr{V}(\Sigma)$. Then $V(\Sigma)$ *is the quotient of* $\mathscr{V}(\Sigma)$ by the *radical*† of the form $\langle\ ,\ \rangle_\Sigma$, which descends to a nondegenerate form, still denoted by $\langle\ ,\ \rangle_\Sigma$, on $V(\Sigma)$. If M is a cobordism from Σ_1 to Σ_2, the assignment $M' \mapsto M' \cup_{\Sigma_1} M$ defines a linear map $Z_M : V(\Sigma_1) \to V(\Sigma_2)$, such that (V, Z) is a quantization functor.

The following proposition is easy to prove using the universal construction.

PROPOSITION. *Let V be a cobordism generated quantization functor. Then $V(-\Sigma)$ is the conjugate module of $V(\Sigma)$, and one has a natural mapping $V(-\Sigma) \to V(\Sigma)^*$ (where $V(\Sigma)^*$ denotes the dual module). Furthermore, there is a natural mapping $V(\Sigma_1) \otimes V(\Sigma_2) \to V(\Sigma_1 \coprod \Sigma_2)$.*

We say the quantization functor is *involutive*, if property (I) below holds, and *multiplicative*, if property (M) below holds.

(I) *The map $V(-\Sigma) \to V(\Sigma)^*$ is an isomorphism.*
(M) *The map $V(\Sigma_1) \otimes V(\Sigma_2) \to V(\Sigma_1 \coprod \Sigma_2)$ is an isomorphism.*

We also consider the following finiteness property.
(F) *For all Σ, $V(\Sigma)$ is free of finite rank and the form $\langle\ ,\ \rangle_\Sigma$ is unimodular.*‡

Remark. Of course (F) implies (I). One can easily see that (F) and (CG) imply that the map $V(\Sigma_1) \otimes V(\Sigma_2) \to V(\Sigma_1 \coprod \Sigma_2)$ is an isomorphism onto a direct summand.

† The radical of the form $\langle\ ,\ \rangle$ is the set of x such that $\langle x, y \rangle = 0$ for all y.
‡ A form is *unimodular* if the adjoint mapping is an isomorphism.

Definition. A topological quantum field theory (TQFT) on a cobordism category C is a cobordism generated quantization functor of C satisfying property (F) (and hence also property (I)) and property (M).

Remark. The above definition generalizes the axioms of Atiyah and Segal for TQFT, to more general coefficient rings. Note that if $k = \mathbf{C}$, then we have not required that the vector space $V(\Sigma)$ be a Hilbert space, since the form $\langle\, , \rangle_\Sigma$ is not required to be *positive definite*. In fact, for the V_p theories discussed below, different embeddings of the ground ring k_p in $\mathbf{C}$ lead to forms with different *signatures* (see Remark 4.12).

1.2. *Trace formula for TQFT. Let M be a cobordism from Σ to Σ, and let M_Σ be the closed bordism obtained by identifying the two copies of Σ. Then $\langle M_\Sigma \rangle = \mathrm{trace}_{V(\Sigma)}(Z_M)$. In particular $\langle S^1 \times \Sigma \rangle = \mathrm{rank}\, V(\Sigma)$.*

Proof. For a C-generated quantization functor, gluing along Σ induces maps

$$f: V(-\Sigma \coprod \Sigma) \to k$$

$$\Phi: V(-\Sigma \coprod \Sigma) \to \mathrm{Hom}(V(\Sigma), V(\Sigma)).$$

These satisfy $f(Z(M)) = \langle M_\Sigma \rangle$ and $\Phi(Z(M)) = Z_M$. One can check that $\Phi \circ \mu(x \otimes y)(z) = D(x)(z)y$, where $\mu: V(-\Sigma) \otimes V(\Sigma) \to V(-\Sigma \coprod \Sigma)$ and $D: V(-\Sigma) \to V(\Sigma)^*$ are the natural maps. It follows that $f \circ \mu(x \otimes y) = D(x)(y) = \mathrm{trace}(\Phi \circ \mu(x \otimes y))$. Since μ is surjective (by axiom (M)), we have $f = \mathrm{trace} \circ \Phi$. The result follows.

1.B. Statement of the main results

After appropriately renormalizing the invariant θ_p of [10, 9], and changing coefficients to a ring k_p, defined in Section 2, one obtains the following.

1.3. THEOREM. *There is a series of multiplicative and involutive invariants, $\langle\ \rangle_p$, p a positive integer, defined on the set of closed bordisms of the cobordism category $C_2^{p_1}$ (i.e. on the set of (equivalence classes of) closed oriented 3-manifolds, equipped with a p_1-structure and a banded link), and taking values in the ring k_p.*

Via Proposition 1.1, the invariant $\langle\ \rangle_p$ determines a $C_2^{p_1}$-generated quantization functor V_p. We may now state the main result of this paper.

1.4. MAIN THEOREM (Existence of TQFT). *Let $p \geq 3$. The quantization functor V_p satisfies axiom* (F)*. If p is even, then axiom* (M) *holds and hence V_p satisfies all the axioms of TQFT. If p is odd, then axiom* (M) *holds provided the link is at least one of the Σ_i has an even number of components. In particular, V_p is a TQFT when restricted to the category $C_2^{p_1}$ (even).*

Remark. If $p = 1$ or 2, the quantization functor V_p satisfies axiom (F) but not the multiplicativity axiom (M). However, these theories are useful, since they can be used to relate the p and $2p$ theories, if $p \geq 3$ is odd.

A tensor product formula for odd p

Set $\langle M \rangle_2' = (-2)^{-n} \langle M \rangle_2$, where n is the number of components of the banded link in M. This is a multiplicative and involutive invariant and thus, via Proposition 1.1, it determines a quantization functor V_2'. In Section 6, we prove the following theorem.

1.5. Theorem. *Let $p \geq 1$ be odd. There is a natural isomorphism*

$$f_\Sigma \colon V_{2p}(\Sigma) \xrightarrow{\approx} V_2'(\Sigma) \otimes V_p(\Sigma)$$

such that $f_\Sigma(Z_{2p}(M)) = Z_2'(M) \otimes Z_p(M)$. (Here, all modules are considered to be k_{2p}-modules, via a change of coefficients explained in Section 6.)

Note that the formula for $f_\Sigma(Z_{2p}(M))$ is valid only if M is a manifold with p_1-structure equipped with a banded link, and not a linear combination of banded links.

A decomposition formula for even p

In the last section of this paper, we define a natural action of a Heisenberg type group $\Gamma(\Sigma)$ on the module $V_{2p}(\Sigma)$. Let us denote the banded link contained in a surface Σ by l (of course, l may be empty). Then the group $\Gamma(\Sigma)$ is a central extension of $H_1(\Sigma - l; \mathbf{Z}/2)$ by $\mathbf{Z}/4$. Its action nontrivially decomposes $V_{2p}(\Sigma)$ into subspaces, and we have the following theorem.

1.6. Theorem. (i) *If p is odd, then the natural action of $\Gamma(\Sigma)$ on $V_{2p}(\Sigma)$ factors through an action on $V_2'(\Sigma)$.*

(ii) *If $p \equiv 2 \bmod 4$, the natural action of $\Gamma(\Sigma)$ decomposes $V_{2p}(\Sigma)$ into a direct sum of subspaces $V_{2p}(\Sigma, h)$, canonically associated to* mod 2 *cohomology classes h on $\Sigma - l$.*

(iii) *If $p \equiv 0 \bmod 4$, the natural action of $\Gamma(\Sigma)$ decomposes $V_{2p}(\Sigma)$ into a direct sum of subspaces $V_{2p}(\Sigma, q)$, canonically associated to spin structures q on $\Sigma - l$.*

In fact, the modules $V_{2p}(\Sigma, h)$ and $V_{2p}(\Sigma, q)$ fit into a refined TQFT associated to manifolds equipped with mod 2 cohomology classes or spin structures (see [11]).

1.C. Kauffman bracket relations and surgery axioms

1.7. Theorem. (i) *The quantization functor V_p, on the cobordism category $C_2^{p_1}$, satisfies the Kauffman bracket relations and the surgery axioms described below.*

(ii) *Moreover, every cobordism generated quantization functor, over an integral domain, satisfying the Kauffman bracket relations and the surgery axioms, is obtained from one of the V_p, by a change of coefficients.*

Kauffman bracket relations

Definition. Let M be a compact 3-dimensional manifold and let l be a banded link in ∂M. Let k be a commutative ring containing an invertible element A. Set $\delta = -A^2 - A^{-2}$. The *Jones–Kauffman skein module* $K(M, l)$ (with coefficients in k) is the k-module generated by the set of isotopy classes of banded links L in M, meeting ∂M transversally in l, quotiented by the relations [17] shown in Fig. 1.

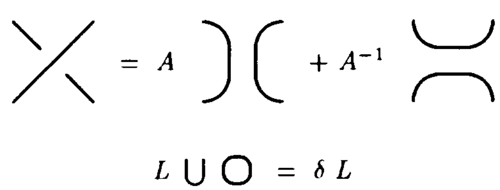


Fig. 1.

Convention. In the figures in this paper, we use the convention that any line is to represent a band parallel to the plane, with orientation compatible with that of the plane.

(In the figure above, the first equation means that a link, which locally (in a ball) is given by the left-hand side, may be replaced by the linear combination given (locally) by the right-hand side. The second equation means that a link having an unknotted component, lying in a disk disjoint from the rest of the link, may be removed at the cost of the factor $\delta = -A^2 - A^{-2}$.)

Note. The universal coefficient ring for Jones–Kauffman modules is $k = \mathbf{Z}[A, A^{-1}]$, the ring of Laurent polynomials in the indeterminate A.

Notation. Assume that M as above is equipped with a p_1-structure. Denote by $\mathscr{L}(M, l)$ the k-module freely generated by the set of (isotopy classes of) banded links L in M, meeting Σ in l.

Definition. Let V denote a quantization functor on the category $C_2^{p_1}$. We say that V satisfies the Kauffman bracket relations (for an element $A \in k$) if for all M, the linear map $\mathscr{L}(M, l) \to V(\Sigma, l)$, $L \mapsto Z(M, L)$, factors through $K(M, l)$.

Note. The induced map $K(M, l) \to V(\Sigma, l)$ may depend on the p_1-structure on M (see 1.8 below).

Surgery axioms

Let V denote a quantization functor on the category $C_2^{p_1}$. We say that V *satisfies the surgery axioms* provided (S0), (S1) and (S2) below are satisfied.

Let S^3 denote the 3-sphere equipped with a standard p_1-structure, i.e. one extending to D^4.

(S0) *$\langle S^3 \rangle$ is invertible in k.*

Assume $S^0 \times D^3$ and $D^1 \times S^3$ are equipped with their product orientations (and some fixed p_1-structure, which is the restriction of a p_1-structure on $D^1 \times D^3$), so that $\partial(S^0 \times D^3) = \partial(D^1 \times S^2) = S^0 \times S^2$.

(S1) (*Index one surgery*) *There is an element $\eta \in k$, such that $Z(S^0 \times D^3) = \eta Z(D^1 \times S^2)$ in $V(S^0 \times S^2)$.*

Assume $S^1 \times D^2$ and $D^2 \times S^1$ are equipped with their product orientations (and some fixed p_1-structure, which is the restriction of a p_1-structure on $D^2 \times D^2$), so that $\partial(-(S^1 \times D^2)) = \partial(D^2 \times S^1) = S^1 \times S^1$.

(S2) (*Index two surgery*) *The element $Z(D^2 \times S^1)$ in $V(S^1 \times S^1)$ lies in the submodule generated by banded links in the solid torus $-(S^1 \times D^2)$.*

Remark. In the above, surgery means surgery in the category of manifolds with p_1-structure. However, for index 1 or 2 surgery, this essentially makes no difference (see Appendix B).

Remark. Since S^3 is obtained from $S^3 \coprod S^3$ by index one surgery, we have $\langle S^3 \coprod S^3 \rangle = \eta \langle S^3 \rangle$. By (S0) and (m), it follows that

$$\langle S^3 \rangle = \eta.$$

Let $S^2 \times S^1$ denote $S^2 \times S^1$ equipped with a p_1-structure with σ-invariant zero (see Appendix B). Applying (S1) to the above, we find

$$\langle S^2 \times S^1 \rangle = 1.$$

Remark. The surgery axioms can be expressed in terms of the invariant $\langle\ \rangle$ on closed 3-manifolds (equipped with p_1-structures and banded links) as follows. We assume axioms (CG) and (m).

1. Using (CG), we see that axiom (S1) is equivalent to the following: if M' is obtained from M by index one surgery, then

$$\langle M \rangle = \eta \langle M' \rangle.$$

2. Axiom (S2) is equivalent to the existence of a linear combination $\omega = \sum_i \lambda_i L_i$ of banded links in the solid torus $-(S^1 \times D^2)$ such that the following holds. Let $\phi\colon -(S^1 \times D^2) \to M$ be an embedding corresponding to a framed knot $K \subset M$ (disjoint from the given banded link in M). Let M' be the result of index two surgery along K (equipped with the same banded link as M). Then

$$\langle M' \rangle = \sum_i \lambda_i \langle M_i \rangle$$

where M_i is the manifold M with $\phi(L_i)$ adjoined to the banded link in M.

1.8. *Dependence on the p_1-structure.* If the quantization functor V satisfies the surgery axioms, then the associated invariant $\langle\ \rangle$ depends affinely on the p_1-structure in the following sense. There is a **Z**-valued homotopy invariant, $\sigma(\alpha)$, of p_1-structures α on closed 3-manifolds (see Appendix B). Let S^3_1 denotes the 3-sphere with a p_1-structure with σ-invariant 1. Axioms (m), (S0), (S1) imply that taking connected sum with S^3_1 (which increases $\sigma(\alpha)$ by 1) multiplies the invariant by

$$\kappa = \frac{\langle S^3_1 \rangle}{\langle S^3 \rangle} = \eta^{-1} \langle S^3_1 \rangle.$$

It follows that if M_1 and M_2 differ only by their p_1-structures, α_1 and α_2, then

$$\langle M_2 \rangle = \kappa^{\sigma(\alpha_2) - \sigma(\alpha_1)} \langle M_1 \rangle.$$

If M_1 and M_2 are as above, but have boundary Σ, an analogous formula holds for the elements $Z(M_i) \in V(\Sigma)$. As for the module $V(\Sigma)$, it is independent of the p_1-structure up to a *noncanonical* isomorphism.†

Remark. If $\kappa \neq 1$, one says that the quantization functor V has a *framing anomaly.* This is the case for the functors V_p (except for some low values of p). The uniqueness part of Theorem 1.7 shows that a quantization functor satisfying the surgery axioms and the Kauffman relations must generally have a framing anomaly.

The surgery axioms serve to reduce the universal construction (Proposition 1.1) to a more manageable setting. We need the following lemma.

† Indeed, since any two p_1-structures on Σ are homotopic, the identity cobordism, equipped with some p_1-structure, induces an isomorphism between the modules associated to different p_1-structures. Note, however, that the isomorphism, induced by a nontrivial self-homotopy of a p_1-structure, induces the multiplication by a power of κ. This shows that (in general) only the *projectivization* of $V(\Sigma)$ is canonical.

LEMMA. *Let M_0 and M_1 be compact oriented 3-manifolds with boundary Σ (not necessarily connected). Assume M_1 is connected. Then M_1 can be obtained from M_0 (up to oriented diffeomorphism rel. Σ), by index* 1 *and* 2 *surgeries.*

After connecting up the components of M_0 by index 1 surgeries, the proof proceeds by attaching standard handlebodies to the components of Σ to obtain closed manifolds, and then using the *Kirby moves* to show that such manifolds are attainable by index 2 surgery on the complement of a standard collection of embedded handlebodies in the 3-sphere.

Using this lemma (and the fact that one may change the p_1-structure up to homotopy by connected sum with a 3-sphere, with appropriate p_1-structure), one easily obtains the following proposition.

1.9. PROPOSITION. *Let M be a compact oriented 3-manifold with p_1-structure and with boundary Σ. If V satisfies the surgery axioms, and M is connected, then the natural map $\mathscr{L}(M,l) \to V(\Sigma,l)$, $L \mapsto Z(M,L)$, is surjective.*

Moreover, if M' denotes another connected compact oriented 3-manifold with boundary Σ (with a p_1-structure on M' inducing the same p_1-structure on Σ), then the kernel of the above map is the left kernel of the sesquilinear form $\langle\, ,\rangle_{(M,M')}\colon \mathscr{L}(M,l) \times \mathscr{L}(M',l) \to k$ given by

$$\langle L, L'\rangle_{(M,M')} = \langle (M \cup_\Sigma - M', L \cup_l - L')\rangle.$$

Remark. This result is a key property of a quantization functor satisfying the surgery axioms, since it says that the module $V(\Sigma)$ can be computed using banded links in *any* two *connected* 3-manifolds, M and M' with boundary Σ. If V satisfies also the Kauffman bracket relations, then one may replace $\mathscr{L}(M,l)$ by $K(M,l)$ in 1.9.

For example, if Σ is *connected*, we may take for M and M' two handlebodies H, H', such that $S^3 = H \cup_\Sigma - H'$. Assume Σ is equipped with the empty link. Then the module $V(\Sigma)$ is the quotient of the module $K(H)$ by $\mathscr{N}$, where $\mathscr{N}$ denotes the left kernel of the pairing $K(H) \times K(H') \to k$ given by 1.9. If the skein variable A is a primitive $2p$th root of unity for some $p \geq 1$, in an integral domain k, then the quantization functor V is V_p (up to change of coefficients), and it follows from the Main Theorem 1.4 that $K(H)/\mathscr{N} = V(\Sigma)$ is free of finite rank (given in 1.16 below). (A different proof of the finite-dimensionality of $K(H)/\mathscr{N}$ at $4r$th roots of unity has been given by Lickorish [24]. In the case where Σ has genus one, this result goes back to [22, 23, 10].)

1.D. The splitting theorem and the introduction of colors

In the remainder of this section we will discuss certain aspects of the proof of the Main Theorem 1.4. In doing so, we will also discuss general methods involving the decomposition of a TQFT, which may be of use when studying TQFT in other contexts.

The proof of the finiteness and tensor product axioms rely heavily on certain properties of the *Jones–Wenzl idempotents* in the *Temperley–Lieb algebra.* (See Section 3 for a discussion of these notions.) In the algebra we develop to decompose the quantization functors V_p, these idempotents play a central role. Geometrically, this involves splitting surfaces along curves and 3-manifolds along surfaces. Algebraically, this leads naturally to the notion of an algebroid.

Definition. An *algebroid* is by definition a *k-linear category*, i.e. the morphism sets are k-modules and composition is k-bilinear.

We choose to use the terminology algebroid instead of k-linear category, because we wish to stress the analogy with algebras.† In particular, we shall use the notions of *left and right modules* over an algebroid, *tensor product* of modules over an algebroid, and *Morita equivalence* of algebroids. (These notions are defined in Appendix A.)

As a particular example of our methods, consider the multiplicative axiom (M). The map $V(\Sigma_1)\otimes V(\Sigma_2)\to V(\Sigma_1\coprod\Sigma_2)$ is an isomorphism if and only if $V(\Sigma_1\coprod\Sigma_2)$ is generated by *split* objects $M_1\coprod M_2$. Reducing a given bordism M, with $\partial M=\Sigma_1\coprod\Sigma_2$, to a split object, involves first splitting M along a surface, say Σ, reducing the genus of Σ via surgery (this uses axiom (S2)) to obtain a sphere, which may have bands running through it, reducing the number of bands to zero (this is where the idempotents play a role, and where the p even and odd theories differ) and finally using axiom (S1) to obtain the splitting.

This reduction process can be expressed algebraically as a Morita equivalence. Let $\Delta(\emptyset)$ be the algebroid whose objects are the objects of the category $C_2^{p_1}$ (i.e. closed surfaces equipped with p_1-structures and banded links), and whose set of morphisms from Σ to Σ' is the module $V(-\Sigma\coprod\Sigma')$. Then one has an isomorphism (this is a special case of the general splitting theorem 1.12 below)

$$V(\Sigma_1)_- \underset{\Delta(\emptyset)}{\otimes} {}_-V(\Sigma_2)\overset{\approx}{\to} V(\Sigma_1\coprod\Sigma_2)$$

where $V(\Sigma_1)_-$ (resp. ${}_-V(\Sigma_2)$) is considered as a right (resp. left) module over the algebroid $\Delta(\emptyset)$. Here, if a denotes an object Σ of $\Delta(\emptyset)$, then $V(\Sigma_1)_a$ (resp. ${}_aV(\Sigma_2)$) denotes the module $V(\Sigma_1\coprod\Sigma)$ (resp. $V(-\Sigma\coprod\Sigma_2)$). (Note that one has $V(\Sigma_1)_\emptyset=V(\Sigma_1)$ and ${}_\emptyset V(\Sigma_2)=V(\Sigma_2)$.)

Now since tensor products of modules are *preserved* under Morita equivalence (see Appendix A) axiom (M) is seen to be a consequence of the following property:

(ME) *The algebroid $\Delta(\emptyset)$ is Morita equivalent to k.*

(Here k denotes the k-algebroid consisting of one object and whose morphism set is k.)

Using this algebraic language, we have that the multiplicativity property claimed in the Main Theorem 1.4 is a consequence of the following.

1.10. THEOREM. *Axiom* (ME) *holds for the V_p theory on the category $C_2^{p_1}$, if $p\geq 4$ is even, and on the category $C_2^{p_1}$ (even), if $p\geq 3$.*

The above result is a particular case of a more general result concerning splitting surfaces along curves. We now describe this in more detail.

Gluing along objects with boundary

In the following, we will assume the objects and bordisms of the cobordism category can be *split along subobjects*. (This holds for the category $C_2^{p_1}$ ($C_2^{p_1}$ (*even*)) studied in this paper.) This means that one has a more general gluing operation, where now one can glue bordisms along pieces of their boundary, such that the pieces themselves are allowed to have boundary. The objects with boundary will be viewed as cobordisms between subobjects, and we assume they can be glued together in the usual way. For example, an object of the cobordism category will be viewed as a cobordism from the empty subobject to itself.

† Just as a groupoid may be thought of as a "group with many objects", an algebroid may be thought of as an "algebra with many objects".

Associated algebroids and bimodules

In what follows, we will study a cobordism generated quantization functor V, such that the objects and bordisms of the category can be split along subobjects. To a subobject Γ we associate a category $\Delta(\Gamma)$ as follows. The objects of $\Delta(\Gamma)$ are the cobordisms from Γ to $\emptyset$. For example, in the case of $C_2^{p_1}$, the subobject Γ is closed 1-manifold (with p_1-structure), and an object of the category $\Delta(\Gamma)$ is a surface (with p_1-structure and a banded link) with boundary $-\Gamma$. If $a = \Sigma_1$ and $b = \Sigma_2$ are two objects of $\Delta(\Gamma)$, the set of morphisms from a to b, denoted by ${}_a\Delta(\Gamma)_b$, is by definition the module $V(-\Sigma_1 \cup_\Gamma \Sigma_2)$.

Suppose that Σ is an object with boundary, such that $\partial\Sigma = -\Gamma_1 \coprod \Gamma_2$, and let $a = \Sigma_1$ (resp. $b = \Sigma_2$) be an object of $\Delta(\Gamma_1)$ (resp. $\Delta(\Gamma_2)$). We set

$${}_aV(\Sigma)_b = V(-\Sigma_1 \cup_{\Gamma_1} \Sigma \cup_{\Gamma_2} \Sigma_2).$$

One has the following easy consequence of the universal construction of 1.1.

1.11. Proposition. *Let Σ_1 and Σ_2 denote objects with boundary with $\partial\Sigma_1 = -\Gamma_1 \coprod \Gamma$ and $\partial\Sigma_2 = -\Gamma \coprod \Gamma_2$. Then one has bilinear gluing maps*

$${}_aV(\Sigma_1)_b \times {}_bV(\Sigma_2)_c \to {}_aV(\Sigma_1 \cup_\Gamma \Sigma_2)_c$$

$${}_a\Delta(\Gamma)_b \times {}_b\Delta(\Gamma)_c \to {}_a\Delta(\Gamma)_c$$

$${}_a\Delta(\Gamma)_b \times {}_bV(\Sigma_2)_c \to {}_aV(\Sigma_2)_c$$

$${}_aV(\Sigma_1)_b \times {}_b\Delta(\Gamma)_c \to {}_aV(\Sigma_1)_c$$

such that ${}_-\Delta(\Gamma)_-$ is an algebroid and ${}_-V(\Sigma_1)_-$ is a $\Delta(\Gamma_1) \times \Delta(\Gamma)$-bimodule.

The following result is an almost formal consequence of the definitions.

1.12. General Splitting Theorem. *Let Σ_1 and Σ_2 denote objects with boundary, with $\partial\Sigma_1 = -\Gamma_1 \coprod \Gamma$ and $\partial\Sigma_2 = -\Gamma \coprod \Gamma_2$. Then the natural map*

$${}_-V(\Sigma_1)_- \underset{\Delta(\Gamma)}{\otimes} {}_-V(\Sigma_2)_- \to {}_-V(\Sigma_1 \cup_\Gamma \Sigma_2)_-$$

is an isomorphism of $\Delta(\Gamma_1) \times \Delta(\Gamma_2)$-bimodules.

Proof. Let $\alpha = \Sigma_1'$ and $\beta = \Sigma_2'$ denote objects, respectively, in $\Delta(\Gamma_1)$ and $\Delta(\Gamma_2)$. We let $\delta = -\Sigma_1 \cup_{\Gamma_1} \Sigma_1'$, and we denote by ε the element of ${}_\alpha V(\Sigma_1)_\delta$ induced by the bordism $(-\Sigma_1' \cup_{\Gamma_1} \Sigma_1) \times [0,1]$, where the points of $\Gamma \times [0,1]$ are identified with Γ via the projection.

One has a canonical isomorphism $f: {}_\alpha V(\Sigma_1 \cup_\Gamma \Sigma_2)_\beta \mapsto {}_\delta V(\Sigma_2)_\beta$, and one checks that the map $\varepsilon \otimes f$ is an inverse to the natural map

$${}_\alpha V(\Sigma_1)_- \underset{\Delta(\Gamma)}{\otimes} {}_-V(\Sigma_2)_\beta \to {}_\alpha V(\Sigma_1 \cup_\Gamma \Sigma_2)_\beta. \qquad \square$$

Reducing the algebroid $\Delta_p(\Gamma)$

In what follows, we will work with the quantization functor V_p on the category $C_2^{p_1}$ associated to the invariant $\langle\ \rangle_p$ of Theorem 1.3. The algebroid of a closed 1-manifold Γ with p_1-structure will be denoted by $\Delta_p(\Gamma)$.

We say that an algebroid Δ is *completely reduced*, if the set of morphisms between an object and itself is a free module of rank 1, and the set of morphisms between different objects is the zero module.

Our study of the V_p theories boils down to the following.

1.13. Morita Reduction Theorem. *Let Γ be a closed 1-manifold with m components, $m \geq 0$. Suppose $p \geq 3$. Then the algebroid $\Delta_p(\Gamma)$ is Morita equivalent to the completely reduced algebroid on n^m (resp. $2n^m$) objects, if p is even (resp. odd), where $n = [(p-1)/2]$.*

If $p \geq 3$ is odd, the algebroid $\Delta_p(\Gamma)$ breaks up into a disjoint union of even and odd pieces, $\Delta_p^0(\Gamma)$ and $\Delta_p^1(\Gamma)$, each of which is Morita equivalent to the completely reduced algebroid on n^m objects.

Note that if $\Gamma = \emptyset$, then 1.13 says that axiom (ME) holds, if $p \geq 4$ is even, and the cobordism category is $C_2^{p_1}$. Axiom (ME) also holds if $p \geq 3$ is odd, and the cobordism category is $C_2^{p_1}$ *(even)*, because the even part, $\Delta_p^0(\Gamma)$, is the algebroid associated to the quantization functor V_p restricted to the cobordism category $C_2^{p_1}$ *(even)*. Hence, in these two cases, the quantization functor V_p satisfies the tensor product axiom (M).

However, if $p \geq 3$ is odd and the cobordism category is $C_2^{p_1}$, then 1.13 says that axiom (ME) *fails* to hold. In fact, the tensor product axiom (M) also *fails* to hold. Here is an example. Let Σ be the 2-sphere equipped with a banded link with $p-2$ components. Since $p-2$ is odd, $V_p(-\Sigma)$ and $V_p(\Sigma)$ are zero. But $V_p(-\Sigma \coprod \Sigma) \approx k_p$ is nonzero (see 3.9).

Remark. The objects of the completely reduced algebroids correspond, in the V_p theory, to *Jones–Wenzl idempotents*, and the above theorem is a reformulation of certain of their properties. We think that the above theorem should generalize to other theories based on other *specializations* of the 2-variable *Jones–Conway* (*HOMFLY*) *polynomial* and the 2-variable *Kauffman polynomial*, using appropriate Jones–Wenzl idempotents.

Morita reduction of the splitting theorem

Let us now apply the Morita reduction theorem to the splitting theorem. A Morita equivalence between algebroids Δ and $\tilde{\Delta}$ is a functor, from Δ-modules to $\tilde{\Delta}$-modules, which is an equivalence of categories (see Appendix A). Thus, if $\partial\Sigma_1 = -\Gamma_1 \coprod \Gamma$, the Morita equivalence of 1.13, applied to the algebroids $\Delta_p(\Gamma_1)$ and $\Delta_p(\Gamma)$, sends the bimodule ${}_{-}V_p(\Sigma_1)_{-}$ to a bimodule ${}_{-}\tilde{V}_p(\Sigma_1)_{-}$ over the "Morita-reduced" algebroids. Similarly, if $\partial\Sigma_2 = -\Gamma \coprod \Gamma_2$, we have a bimodule ${}_{-}\tilde{V}_p(\Sigma_2)_{-}$. Again, since tensor products are preserved by Morita equivalence, for $p \geq 3$ the splitting theorem gives natural isomorphisms

$${}_i\tilde{V}_p(\Sigma_1 \cup_\Gamma \Sigma_2)_k \approx \bigoplus_j {}_i\tilde{V}_p(\Sigma_1)_j \otimes {}_j\tilde{V}_p(\Sigma_2)_k.$$

(Here, i (resp. j, k) are objects of the completely reduced algebroids associated to Γ_1 (resp. Γ, Γ_2), and the right hand side is precisely the tensor product of the modules ${}_i\tilde{V}_p(\Sigma_1)_{-}$ and ${}_{-}\tilde{V}_p(\Sigma_2)_k$ over the completely reduced algebroid associated to Γ.)

Colored links

The above form of the splitting theorem (valid for $p \geq 3$) leads naturally to the notion of *colored links*, where the colors are the finitely many objects of the completely reduced algebroid of S^1.

Let $l \subset \Sigma$ be a banded link. Let l_0 be a sublink. By a *coloring* c of l_0, we mean the assignment of a number between 0 and $n-1$ (resp. $2n-1$), if $p = 2n+2$ is even (resp. $p = 2n+1$ is odd), to the components of l_0. Given such a coloring, we will define a module $V_p(\Sigma, l, c) \subset V_p(\Sigma, l_c)$, where l_c denotes the link obtained from l by replacing each component of l_0 by the number of parallel copies assigned by the coloring. (One may as well color all uncolored components with the color 1, and remove all components with the color 0, since this yields the same link l_c and the same module $V_p(\Sigma, l, c)$.) These submodules are defined by projectors obtained from the Jones–Wenzl idempotents, and we will show that these projectors are orthogonal. Moreover, the induced forms on these modules remain unimodular.

The reason for introducing colored links is that they allow us to state the Morita-reduced version of the splitting theorem without explicitly mentioning the Morita equivalence. Indeed, if $\partial\Sigma = -\Gamma_1 \coprod \Gamma$, we have an isomorphism

$$_i\tilde{V}_p(\Sigma)_j \approx V_p(_i\Sigma_j)$$

where the left hand side is the module obtained by Morita reduction, the right hand side is the V_p-module associated to the closed surface $_i\Sigma_j$ obtained as follows. We "cap off" the boundary circles of Σ by gluing in standard disks, containing standard colored banded links, whose colors are indicated by i and j.

Let Γ be a simple closed curve on the surface disjoint from the (colored) link in Σ. Consider the result of doing surgery on Γ, i.e. cutting Σ along Γ and gluing in standard disks along the boundary curves. Suppose each disk contains a standard 1-component banded link. We denote by $_i\Sigma(\Gamma)_j$ the surface (containing the original link) obtained by coloring one of the new components with the color i and the other with the color j.

1.14. Colored Splitting Theorem. *Let $\Gamma \subset \Sigma$ be as above. The natural gluing map induces an orthogonal decomposition*

$$\bigoplus_i V_p(_i\Sigma(\Gamma)_i) = V_p(\Sigma)$$

where the sum is over all colors i given below. Moreover, the sesquilinear form on each factor is the form induced on that factor, multiplied by the invertible scalar $\langle S^3\rangle_p\langle i\rangle$. (Here $\langle i\rangle = (-1)^i[i+1]$, where $[n] = (A^{2n} - A^{-2n})/(A^2 - A^{-2})$.)

If $p = 2n+2$ is even, the sum is over all colors i, with $0 \le i \le n-1$.

If $p = 2n+1$ is odd, the sum is over all colors i, with $0 \le i \le 2n-1$, and parity given as follows. The colors are even, except if $\Sigma(\Gamma)$ breaks up into a disjoint union $\Sigma' \coprod \Sigma''$, with Γ as the boundary of each, such that both Σ' and Σ'' contain an odd link (i.e. the sum over all colors is odd). In this case the colors are odd.

Using the above result, one can decompose Σ into *elementary cobordisms*. The unraveling of the V_p theories is thus completed once we have established the following result.

1.15. Theorem. *Let $S^2(i,j)$ (resp. $S^2(i,j,k)$) be the 2-sphere with a 2-component (resp. 3-component) banded link, colored with the colors i, j (resp. i, j and k), where the colors are $<n$, if $p = 2n+2$, and are $<2n$, if $p = 2n+1$. Then the modules $V_p(S^2(i,j))$ and $V_p(S^2(i,j,k))$ are free of rank given below and the sesquilinear forms on these modules are unimodular.*

1. rank $V_p(S^2(i,j)) = 0$, *if $i \ne j$.*
2. rank $V_p(S^2(i,i)) = 1$.
3. rank $V_p(S^2(i,j,k)) = 1$, *if (i,j,k) is p-admissible (i.e. $i+j+k$ is even and the triangle inequality $|i-j| \le k \le i+j$ holds and $i+j+k < p-2$, if p is even, and $i+j+k < 2p-2$, if p is odd.)*
4. rank $V_p(S^2(i,j,k)) = 0$, *otherwise.*

Proof of the Main Theorem 1.4. By 1.10 (which is a special case of 1.13), Axiom (ME) is satisfied and hence Axiom (M) holds. Using 1.14 repeatedly, we obtain a finite orthogonal decomposition of $V(\Sigma)$ into pieces, which by 1.15 are free of rank 1, and on which the induced form is unimodular. Hence, $V(\Sigma)$ is free of finite rank, and the form $\langle\,,\rangle_\Sigma$ is unimodular.

1.16. COROLLARY. *Let $p \geq 3$. Set $d_g(p) = \operatorname{rank} V_p(\Sigma_g)$, where Σ_g is a closed surface of genus g equipped with the empty link.*

(i) *For $g \geq 1$, one has*

$$d_g(p) = \left(\frac{p}{4}\right)^{g-1} \sum_{j=1}^{[(p-1)/2]} \left(\sin\frac{2\pi j}{p}\right)^{2-2g}$$

(ii) *Moreover, for $g \geq 2$, one has $d_g(p) = (-p)^g C_g(p)$, if p is even, and $d_g(p) = (-p)^g C_g(2p)$, if p is odd, where*

$$C_g(p) = \text{the coefficient of } t^{2g-2} \text{ in } \left(\frac{t}{2\sinh t}\right)^{2g-2} \frac{t}{e^{pt}-1}.$$

Remark. (i) If $p = 2k + 4$, then 1.16(i) is Verlinde's formula [37] for the dimension of a certain vector space, denoted by $Z_k(\Sigma_g)$ in [32], arising from the $SU(2)$ Wess Zumino Witten model at level k. In fact, one may conjecture the existence of a natural isomorphism $V_p(\Sigma) \otimes \mathbf{C} \approx Z_k(\Sigma)$.

(ii) Note that $\operatorname{rank}(V_{2p}(\Sigma_g)) = 2^g \operatorname{rank}(V_p(\Sigma_g))$, if p is odd. (This is not a coincidence. Indeed, this follows from 1.5 and the fact that $\operatorname{rank}(V_2'(\Sigma_g)) = 2^g$, see Section 6.)

(iii) We have $d_0(p) = 1$ and $d_1(p) = [(p-1)/2]$. For fixed parity of p and $g \geq 2$, $d_g(p)$ is a polynomial in p of degree $3g - 3$, as follows easily from 1.16(ii).

(iv) In 4.11 and 4.14, we shall describe a basis of $V_p(\Sigma)$ in terms of colorings of certain trivalent graphs (compare [21]). We shall also give a formula which allows one to compute the signature of the hermitian form $\langle\,,\rangle_\Sigma$ on $V_p(\Sigma)$, in the case where coefficients are extended from k_p to $\mathbf{C}$.

Remark. We also give a "Verlinde formula" (see 7.16), for the ranks of the modules associated to surfaces with spin structure (in the case where the link is empty). For example, in the case of the V_8-theory, the modules $V_8(\Sigma_g, q)$ of Theorem 1.6(iii) have rank one, if the spin structure q has Arf invariant zero, and rank zero otherwise. In particular, $\operatorname{rank}(V_8(\Sigma_g)) = 2^{g-1}(2^g + 1)$ is the number of spin structures on Σ_g with Arf invariant zero.

Remark. These theorems, taken together, show that the modules $V_p(\Sigma)$ form part of a *modular functor*, in the sense of Segal. In fact, a decomposition of Σ into *pairs of pants*, together with the colored splitting theorem, yields a decomposition of the module $V_p(\Sigma)$ as a direct sum of tensor products of elementary modules. Other authors (e.g. [21, 28, 12]) use such a decomposition as the *definition* of the V-modules. In such an approach, the difficulty is to show that the modules are well defined, i.e. independent of the particular decomposition of Σ.

1.17. *Remark.* In the Witten–Reshetikhin–Turaev theory, one decorates (or colors) the links with representations. In their language, our colors correspond to irreducible representations of $SU(2)$. Note that if p is odd, the multiplicativity axiom of TQFT only holds when restricted to the category $C_2^{p_1}$ (*even*). This means that in this case, for a TQFT, we must restrict to even colors, which correspond to the representations which lift to $SO(3)$. In this sense, we think we have constructed Witten's theory for $SU(2)$ ($p \geq 4$ even) and $SO(3)$ ($p \geq 3$ odd).

2. THE INVARIANTS $\langle\ \rangle_p$

The universal ring k_p. We will denote by k_p ($p \geq 1$) the following ring:

$$k_p = \mathbf{Z}[A, \kappa, d^{-1}]/(\varphi_{2p}(A), \kappa^6 - u)$$

where $\varphi_{2p}(A)$ is the $2p$th (reduced) cyclotomic polynomial in the indeterminate A, where $d = p$ for $p \notin \{1,3,4,6\}$, $d = 1$ for $p \in \{1,3,4\}$, $d = 2$ for $p = 6$, and where $u = A^{-6-p(p+1)/2}$ for $p \notin \{1,2\}$, $u = 1$ for $p = 1$, and $u = A$ for $p = 2$. Thus, A is a primitive $2p$th root of unity. Note that κ is determined by the choice of A up to multiplication by a 6th root of unity. The ring k_p has an involution defined by sending A to A^{-1} and κ to κ^{-1}.

Define $\eta \in k_p$ by $\eta = \kappa^3$, if $p = 1$, and $\eta = (1 - A)\kappa^3/2$, if $p = 2$, and, if $p \geq 3$,

$$\eta = (A\kappa)^3(A^2 - A^{-2})p^{-1}\overline{g(p,1)}$$

where $g(p,1) = \frac{1}{2}\sum_{m=1}^{2p}(-1)^m A^{m^2}$. (If $A = e^{\pi i/p}$, then $g(p,1) = \sqrt{p}\, e^{\pi i(1-p)/4}$ (see [7]).) Using $g(p,1)\overline{g(p,1)} = p$ and $\overline{g(p,1)} = A^{p(p-1)/2}g(p,1)$, one checks that η is invertible in k_p, and $\bar{\eta} = \eta$.

The invariant θ_p. Let M be a connected oriented 3-dimensional closed manifold, and let K be a banded link in M. We recall the definition of the invariant $\theta_p(M, K)$ [10, 9]. Recall that M is orientation-preserving diffeomorphic to a manifold $S^3(L)$ obtained from S^3 by surgery on a framed link $L \subset S^3$. Moreover, up to isotopy, we may as well suppose that the link K is contained in $S^3 - L \subset M$. Let W_L denote the four-ball with two-handles attached along L. Then we have $M = S^3(L) = \partial W_L$. Let $b_+(L)$, $b_-(L)$, $b_0(L)$, denote the number of positive, negative, zero eigenvalues of the framing matrix of L. Then the signature of W_L is given by $\mathit{signature}(W_L) = b_+(L) - b_-(L)$, and the first Betti number of $S^3(L)$ is given by $b_1(S^3(L)) = b_0(L)$.

In [10, 9], a certain element Ω_p in the Jones–Kauffman module of the solid torus was defined. (It is described in 5.8.) We denote by $L(\Omega_p) \cup K$ the element of $K(S^3)$ obtained from $L \cup K$ by inserting a copy of Ω_p in a neighborhood of each component of the framed link L. The Kauffman bracket is an isomorphism $\langle\ \rangle : K(S^3) \xrightarrow{\approx} k_p$. (Here $\langle\ \rangle$ is normalized so that $\langle \emptyset \rangle = 1$, where $\emptyset$ denotes the empty link.) Let U_ε denote the unknot with framing ε.

The invariant $\theta_p(S^3(L), K)$ is defined by the expression

$$\theta_p(S^3(L), K) = \frac{\langle K \cup L(\Omega_p) \rangle}{\langle U_1(\Omega_p) \rangle^{b_+(L)} \langle U_{-1}(\Omega_p) \rangle^{b_-(L)}}.$$

(In [10, 9], $\langle K \cup L(\Omega_p) \rangle$ is denoted by $\langle \Omega_p, \ldots, \Omega_p, z, \ldots, z \rangle_{L \cup K}$, and $\langle U_\varepsilon(\Omega_p) \rangle$ is denoted by $\langle t^\varepsilon \Omega_p \rangle$.) One has (see [10])

$$\langle U_\varepsilon(\Omega_p) \rangle = \eta^{-1}\kappa^{3\varepsilon} \quad (\varepsilon = \pm 1) \tag{$*$}$$

$$\theta_p(S^1 \times S^2) = \langle U_0(\Omega_p) \rangle = \langle U_1(\Omega_p) \rangle \langle U_{-1}(\Omega_p) \rangle. \tag{$**$}$$

Definition of $\langle\ \rangle_p$

In this and subsequent sections, an oriented manifold, of dimension less than or equal to three with p_1-structure and banded link, will simply be called a ***manifold with structure***. (A manifold with structure of dimension less than or equal to one has no link.) Note that the boundary of a manifold with structure is again a manifold with structure.

Let $M = (M, \alpha, K)$ be a closed 3-manifold with structure (where α is a p_1-structure and K is a banded link). Let $(M, K) = \coprod_{i=1}^{n} (M_i, K_i)$, where M_i are the connected components of M. We define

$$\langle M \rangle_p = \eta^{b_0(M) + b_1(M)} \kappa^{\sigma(\alpha)} \prod_{i=1}^{n} \theta_p(M_i, K_i)$$

where $b_i(M)$ is the ith Betti number, and $\sigma(\alpha) \in \mathbf{Z}$ is defined in Appendix B. The invariant $\langle\ \rangle_p$ is multiplicative, and lies in the ring k_p defined above. It is also involutive, since $\eta = \bar{\eta}$ and $\theta_p(-M, -K) = \overline{\theta_p(M, K)}$, see [10, 9].

Notation. The quantization functor corresponding to the invariant $\langle M \rangle_p$ will be denoted by V_p. The sesquilinear forms $\langle\, ,\rangle_\Sigma$ on the modules $V_p(\Sigma)$ will simply be denoted by $\langle\, ,\rangle_p$, as Σ is usually clear from the context.

Proof of 1.7 (i). It is clear that V_p satisfies the Kauffman relations. Axiom (S0) holds because $\langle S^3 \rangle_p = \eta$. For (S2), we put

$$\omega_p = \eta \Omega_p.$$

We claim that $Z_p(D^2 \times S^1) \in V_p(S^1 \times S^1)$ is the same as the image of the element ω_p of $K(-S^1 \times D^2)$. Indeed, using $(*)$, one has

$$\langle (S^3(L), \alpha, K) \rangle_p = \eta^{1 + b_1(S^3(L))} \kappa^{\sigma(\alpha)} \theta_p(S^3(L), K) = \eta \kappa^{-\langle p_1(W_L, \alpha), [W_L] \rangle} \langle L(\omega_p) \cup K \rangle$$

which implies that for all M with $\partial M = S^1 \times S^1$,

$$\langle Z_p(D^2 \times S^1), Z_p(M) \rangle_p = \langle Z_p(-S^1 \times D^2, \omega_p), Z_p(M) \rangle_p.$$

Since the form $\langle\, ,\rangle_p$ is nondegenerate, it follows that axiom (S2) is satisfied.

Let us now show (S1). It is sufficient to show index one surgery on M multiplies the invariant by η^{-1}. Performing a p_1-surgery of index one on M means either replacing two components of M by their connected sum, or replacing a component M_i by $M_i \# (S^2 \times S^1)$. Since the invariant θ_p is multiplicative under connected sums, it is clear that in the first case, the invariant $\langle\ \rangle_p$ picks up a factor η^{-1} as claimed. In the second case, it gets multiplied by $\eta \theta_p(S^2 \times S^1)$. By $(*)$ and $(**)$, we have $\theta_p(S^2 \times S^1) = \eta^{-2}$, hence the invariant $\langle\ \rangle_p$ is multiplied by η^{-1} in this case also.

Proof of 1.7 (ii). Suppose we are given a cobordism generated quantization functor V over an integral domain k, satisfying the surgery axioms and the Kauffman relations for an element $A \in k$. As in 1.C, observe $\eta = \langle S^3 \rangle$, and define $\kappa = \eta^{-1} \langle S^3_1 \rangle$. Recall that the surgery axioms imply affine dependance on the p_1-structure (see 1.8). By the uniqueness result of [10, 9], axiom (S2) together with the Kauffman relations imply that the skein variable $A \in k$ is a primitive $2p$th root of unity for some p. Moreover, $\theta_p(M, K)$ lies in a subring $\Lambda'_p \subset k_p$ (defined in [10, p. 697]), and there is a ring homomorphism $f: \Lambda'_p \to k$ (satisfying $f(A) = A$) and a $\lambda \in k$ such that if M is connected then

$$\langle (M, \alpha, K) \rangle = \eta \kappa^{\sigma(\alpha)} \lambda^{b_1(M)} f(\theta_p(M, K)).$$

It also follows from [10] that for ω, the linear combination of banded links implementing (S2), we may take

$$\omega = \lambda \Omega_p.$$

(In fact, the uniqueness result of [10, 9] needs the hypothesis that $\langle (S^3, U_\varepsilon(\omega))\rangle$ be invertible for $\varepsilon = \pm 1$. But this follows from $\kappa^{3\varepsilon} = \eta^{-1}\langle (S^3, U_\varepsilon(\omega))\rangle$. To see this equality, perform p_1-surgery on S^3 along U_ε and note that the result is the 3-sphere with a p_1-structure with σ-invariant 3ε.)

Since $\kappa^6 = \langle (S^3, U_1(\omega))\rangle / \langle (S^3, U_{-1}(\omega))\rangle = \langle U_1(\Omega_p)\rangle / \langle U_{-1}(\Omega_p)\rangle$, it follows from the formula for $\langle U_\varepsilon(\Omega_p)\rangle$ given in [10] that $\kappa^6 = u$, where u is as in the definition of k_p. Next, recall that $S^2 \times S^1$ can be obtained from S^3 both by index one surgery and by index two surgery (on U_0, the unknot with framing zero.) Using formula (**) above, this implies that η is related to the skein variable A as in the definition of k_p, and that $\lambda = \eta$. Finally, since the number called d in the definition of k_p is invertible in Λ'_p, it has to be invertible in k also. This shows that f extends to a ring homomorphism $f: k_p \to k$ satisfying $f(\kappa) = \kappa$, and one has $\langle M\rangle = f(\langle M\rangle_p)$ for all connected M. By multiplicativity, this formula extends to nonconnected M. Since the quantization functor V is determined by the invariant $\langle\ \rangle$, this completes the proof of the uniqueness part of Theorem 1.7.

3. THE TEMPERLEY–LIEB ALGEBROID, THE JONES–WENZL IDEMPOTENTS, AND MORITA REDUCTION OF THE ALGEBROID $\Delta_p(\Gamma)$

Our goal in this section is to prove the Morita Reduction Theorem 1.13.

Notation. For $n \geq 0$, let l_n be a standard banded link with n components in the standard disk D^2 (with boundary S^1, and a standard p_1-structure), and let $a_n = (-D^2, -l_n)$ be the corresponding object in $\Delta_p(S^1)$.†

The Temperley–Lieb algebroid. Let k be a commutative ring endowed with an invertible element $A \in k$. The *Temperley–Lieb algebroid*, T with coefficients in k, is defined as follows:

— the objects of T are the nonnegative integers;

— for $m, n \geq 0$, the k-module ${}_mT_n$ is the Jones–Kauffman module $K(D^2 \times \mathrm{I}, -l_m \times 0 \cup l_n \times 1)$ with coefficients in k, i.e. the k-module generated by banded (m, n)-tangles in the box $D^2 \times I$ meeting the boundary in a standard link with m components in $D^2 \times 0$ and a standard link with n components in $D^2 \times 1$, quotiented by isotopy and the Kauffman bracket relations.

The product ${}_aT_b \otimes {}_bT_c \to {}_aT_c$ is given by the standard product of tangles: one puts the tangle u over the tangle v and one gets a new tangle uv.

Remark. An (m, n)-tangle is represented by a diagram in the square $I \times I$ with m standard points in the top edge and n standard points in the bottom edge. For $n \geq 0$, the algebra ${}_nT_n$ is the classical Temperley–Lieb algebra, as considered by Lickorish [22, 23].

The sesquilinear form $\langle\, ,\rangle$. Let T be the Temperley–Lieb algebroid with coefficients in k. There is a sequilinear form $\langle\, ,\rangle$ defined on ${}_mT_n$ as follows. Let u and v be represented by diagrams U and V. Let $\bar{V}$ be the image of V by a reflection along a vertical axis. Then $\langle u, v\rangle$ is the Kauffman bracket of the diagram obtained by connecting U and $\bar{V}$ with $m + n$ extra strands without crossing. Figure 2 gives an example.

The form $\langle\, ,\rangle$ is hermitian and satisfies

$$\langle u \times u', v \times v'\rangle = \langle u, v\rangle \langle u', v'\rangle$$

† Recall that an object of $\Delta_p(S^1)$ is a cobordism from S^1 to $\emptyset$, whence the minus signs.

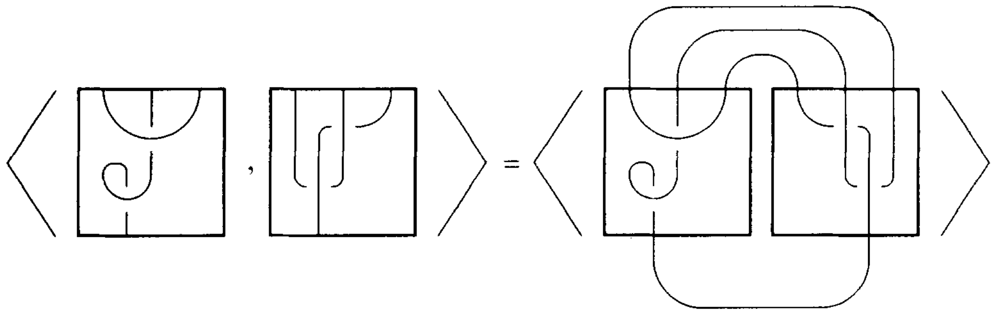


Fig. 2.

where u and v are elements of ${}_iT_j$, u' and v' are elements of ${}_{i'}T_{j'}$, and where the symbol $\times$ is the juxtaposition map from ${}_iT_j \otimes {}_{i'}T_{j'}$ to ${}_{i+i'}T_{j+j'}$.

The Jones–Wenzl idempotents. The module ${}_mT_n$ has a standard basis given by diagrams without crossings and closed loops. There is a standard augmentation character $\varepsilon : {}_nT_n \to k$, whose value is 1 on the identity element and 0 on the other basis elements. Note that ε is a ring homomorphism. If k is good enough, for instance, if k is a field and A is not a root of unity, the algebra ${}_nT_n$ is semisimple, and the character ε is represented by an augmentation idempotent f_n, i.e. for all $x \in {}_nT_n$

$$f_n x = x f_n = \varepsilon(x) f_n.$$

These idempotents were first discovered by Jones [14]. There is a recursive formula for these idempotents, due to Wenzl [39] (see also [22]):

$$f_{n+1} = f_n \times 1_1 + \frac{[n]}{[n+1]}(f_n \times 1_1)(1_{n-1} \times e)(f_n \times 1_1)$$

for all $n \geq 0$, where 1_q denotes the identity of ${}_qT_q$, e is the unique $(2,2)$-tangle distinct from 1_2 without any crossing ($e = \boxtimes$), and $[n] = (A^{2n} - A^{-2n})/(A^2 - A^{-2})$.

3.1. If f_n exists (with coefficients in k), then it is easily established by induction that $\langle f_n, 1_n \rangle = (-1)^n [n+1]$. It follows that for all $u \in {}_nT_n$, one has

$$\langle f_n, u \rangle = (-1)^n [n+1] \overline{\varepsilon(u)}.$$

3.2. Lemma. *Let T be the Temperley–Lieb algebroid with coefficients in the ring k_p.*

(i) *If $p \leq 2$, all idempotents f_n exist in the ring $T \otimes Q$.*
(ii) *If $p = 2$, f_0, f_1, f_2 exist in T.*
(iii) *If $p \geq 3$ is odd, then $f_0, f_1, \ldots, f_{p-1}$ exist in T.*
(iv) *If $p \geq 4$ is even, then $f_0, f_1, \ldots, f_{(p-2)/2}$ exist in T.*

Proof. This follows from Wenzl's recursion formula. It suffices to check that $[n]$ is invertible in the required range.

Note. In the ring k_p, one has $[n] = n$ if $p = 1$, $[n] = (-1)^n n$ if $p = 2$, and if $p \geq 3$, then $[n]$ is invertible in k_p except if $2n$ is divisible by p, in which case $[n] = 0$.

Convention. For the remainder of this section, we assume $p \geq 3$. (The cases $p = 1$ and 2 are different and will be treated in Section 6.) We denote by q the number n, if $p = 2n + 2$ is even, and $2n$, if $p = 2n + 1$ is odd.

Since the invariant $\langle\ \rangle_p$ satisfies the Kauffman bracket relations, we have an obvious morphism

$$\Phi: T \to \Delta_p(S^1), \qquad \Phi_{m,n}: {}_mT_n \to {}_{a_m}\Delta_p(S^1)_{a_n}.$$

By 1.9, the map $\Phi_{m,n}$ is surjective for all $m, n \geq 0$. Moreover, we have that

$$\langle \Phi_{m,n}(u), \Phi_{m,n}(v) \rangle_p = \langle S^3 \rangle_p \langle u, v \rangle$$

where $\langle\, , \rangle_p$ is the canonical sesquilinear form on ${}_{a_m}\Delta_p(S^1)_{a_n} = V_p(-a_m \cup a_n)$.

By 3.2, we have the idempotents f_i, $0 \leq i \leq q$ in T. For simplicity of notation, we will continue to denote by f_i the image $\Phi(f_i) \in {}_{a_i}\Delta_p(S^1)_{a_i}$ of the idempotent $f_i \in {}_iT_i$.

Notation. Let $s_i = -a_0 \cup a_i = (S^2, l_i)$ be the sphere S^2 with a standard banded link with i components. (This is an object of $\Delta_p(\emptyset)$.)

For $0 \leq i \leq q$, let $f_i' \in K(S^2 \times \mathbf{I}, -l_i \times 0 \cup l_i \times 1)$ be obtained in the obvious way from f_i, i.e. f_i' is the image of f_i under the inclusion $D^2 \times \mathbf{I} \subset S^2 \times \mathbf{I}$. The induced element of ${}_{s_i}\Delta_p(\emptyset)_{s_i} = V_p(-s_i \coprod s_i)$ will again be denoted simply by f_i'.

For $0 \leq i \leq q$, let $e_i' \in K(S^2 \times S^1)$ be the "closure" of f_i', i.e. the element obtained from f_i' by identifying both copies of s_i.

3.3. Proposition. (i) *The idempotent f_q is zero in ${}_{a_q}\Delta_p(S^1)_{a_q}$.*

(ii) $\langle (S^2 \times S^1, e_i') \rangle_p = 1$, *if either $i = 0$, or if p is odd and $i = p - 2$, and* $\langle (S^2 \times S^1, e_i') \rangle_p = 0$ *otherwise.*

Proof. (i) Since $[q + 1] = 0$ in k_p, we have, by 3.1, that $\langle f_q, u \rangle = 0$, for all $u \in {}_qT_q$. Hence $\langle f_q, u \rangle_p = 0$, for all $u \in {}_{a_q}\Delta_p(S^1)_{a_q}$. Since the form $\langle\, , \rangle_p$ is nondegenerate, the result follows.

The proof of (ii) is deferred to 5.9.

Notation. Let Γ be a nonempty closed 1-manifold with structure, with m components $\Gamma_1, \dots, \Gamma_m$. For each component Γ_j, choose an annulus (with p_1-structure) with boundary $-\Gamma_j \coprod S^1$. Then for each multi-index $i = (i_1, \dots, i_m)$, with all $i_j \geq 0$, the objects a_{i_j} of $\Delta_p(S^1)$, defined above, induce (glue the annuli to the disks) an object $a_i = a_i(\Gamma)$ in $\Delta_p(\Gamma)$.† The object a_i is the disjoint union of m disks (with boundaries $-\Gamma_1, \dots, -\Gamma_m$), equipped with standard banded links with $i_1, \dots, i_m$ components. Moreover, if $i_1, \dots, i_m < q$, the elements $f_{i_1}, \dots, f_{i_m}$ induce an idempotent, denoted by ε_i, in the algebra ${}_{a_i}\Delta_p(\Gamma)_{a_i}$.

Let $\Delta_p^\varepsilon(\Gamma)$ be the full subcategory of $\Delta_p(\Gamma)$ generated by the set of objects Σ, where the number of components of the link in Σ is congruent to $\varepsilon \bmod 2$. The algebroid $\Delta_p(\Gamma)$ is the disjoint union of $\Delta_p^0(\Gamma)$ and $\Delta_p^1(\Gamma)$.

The following result says that these algebroids are all finitely generated.

3.4. Theorem. (i) *If p is even, then the algebroid $\Delta_p(\emptyset)$ is generated as a two-sided ideal by the identity element $1_\emptyset$ of the empty surface. If p is odd, then the algebroid $\Delta_p^0(\emptyset)$ is generated as a two-sided ideal by the element $1_\emptyset$, and $\Delta_p^1(\emptyset)$ is generated as a two-sided ideal by the idempotent $f_{p-2}' \in {}_{s_{p-2}}\Delta_p(\emptyset)_{s_{p-2}}$.*

(ii) *Let Γ be a closed 1-manifold with structure with $m > 0$ components. Then the algebroid $\Delta_p(\Gamma)$ is generated as a two-sided ideal by the idempotents ε_i, such that $i = (i_1, \dots, i_m)$ satisfies $i_j < q$.*

† Strictly speaking, the object $a_i(\Gamma)$ depends on the choice of the annuli. We suppress this from our notation.

Moreover, if p is odd, then for any m-tuple $k=(k_1,\dots,k_m)\in\{0,1\}^m$, *such that* $\sum k_j\equiv\varepsilon \bmod 2$, *the algebroid* $\Delta_p^\varepsilon(\Gamma)$ *is generated as a two-sided ideal by the idempotents* ε_i, *where* $i=(i_1,\dots,i_m)$ *satisfies* $0\le i_j<q$ *and* $i_j\equiv k_j \bmod 2$ *for all j.*

The proof uses the following general lemma.

3.5. LEMMA. *Let* Γ *be a closed* 1*-manifold with structure, and let* Σ *be any nonempty surface with* p_1*-structure and boundary* Γ. *Let I be a collection of banded links in* Σ, *such that each banded link* $l\subset\Sigma$ *is isotopic to a member of I. For* $i\in I$, *denote by* α_i *the corresponding object of* $\Delta(\Gamma)$. *Then* $\Delta(\Gamma)$ *is generated, as a two-sided ideal, by the identity elements* 1_{α_i}.

Remark. (i) Let $\Sigma_1,\dots,\Sigma_m$ be the components of Σ. Then the hypothesis means that for all $n_1,\dots,n_m\ge 0$, I contains a banded link having precisely n_j components in Σ_j.

(ii) Let $\Delta'(\Gamma)$ be the algebroid defined by setting

$$Obj(\Delta'(\Gamma))=I,\quad \forall i,j\in I\colon\ {}_i\Delta'(\Gamma)_j={}_{\alpha_i}\Delta(\Gamma)_{\alpha_j}$$

In view of the theorem in Appendix A, the lemma implies that the algebroids $\Delta(\Gamma)$ and $\Delta'(\Gamma)$ are Morita equivalent.

Proof of 3.5. Let $\alpha=\Sigma'$ and $\beta=\Sigma''$ be two objects of $\Delta(\Gamma)$. The surface with p_1-structure $-\Sigma'\cup\Sigma$ bounds a connected 3-manifold with p_1-structure M and $-\Sigma\cup\Sigma''$ bounds a connected 3-dimensional manifold with p_1-structure M'. Since Σ is nonempty, the manifold $W=M\cup_\Sigma M'$ is connected, and by 1.9, every element of $V(-\Sigma'\cup_\Gamma\Sigma'')$ is represented by a linear combination of banded links in W. Such a banded link L will meet Σ in a certain banded link l, which we may isotope to a banded link in the set I. Hence, every element of ${}_\alpha\Delta(\Gamma)_\beta$ is a linear combination of elements of the form $\lambda 1_{\alpha_i}\mu$. This proves the lemma. □

Proof of Theorem 3.4. We first give the proof of the first part of (ii) in case Γ is S^1 (cf. [23]). Let 1_i be the identity element of ${}_{a_i}\Delta_p(S^1)_{a_i}$. Since D^2 is nonempty, we know by Lemma 3.5 that $\Delta_p(S^1)$ is generated as a two-sided ideal by the set $\{1_i\colon i\ge 0\}$. Note that $1_0=f_0$ and $1_1=f_1$. Next, for $2\le i<q$, the element 1_i-f_i is a linear combination of tangles of the form uv, where u is an $(i,i-2)$-tangle and v an $(i-2,i)$-tangle, and for $i\ge q$, the same statement holds for $1_i-f_q\times 1_{i-q}$. But $f_q\times 1_{i-q}$ is zero for $i\ge q$. (Indeed, f_q is zero by 3.3(i), and the proof generalizes: for $i\ge q$ and $w\in{}_{a_i}\Delta_p(S^1)_{a_i}$, there is a $w'\in{}_{a_q}\Delta_p(S^1)_{a_q}$ such that $\langle f_q\times 1_{i-q},w\rangle=\langle f_q,w'\rangle=0$. Hence $f_q\times 1_{i-q}$ is zero.) By induction on i, it follows that all 1_i lie in the two-sided ideal generated by $f_0,\dots,f_{q-1}$. This proves the case where $\Gamma=S^1$.

We now prove (ii). By Lemma 3.5, we know that $\Delta_p(\Gamma)$ is generated as a two-sided ideal by the identities of the objects a_i (with all $i_j\ge 0$). Hence, the first part of the result follows from the case $\Gamma=S^1$.

In case p is odd, we improve this as follows. By the above, we know that $\Delta_p^\varepsilon(\Gamma)$ is generated by the idempotents ε_i where $i=(i_1,\dots,i_m)$ satisfies $0\le i_j<q$ and $\sum i_j\equiv\varepsilon \bmod 2$. We claim that it is sufficient to prove the theorem in the case $m=2$. Indeed, let us define the *defect* of ε_i to be one-half the number of indices j such that $i_j\not\equiv k_j \bmod 2$. We must show that all ε_i lie in the ideal generated by those with defect zero. But the result in the case $m=2$ implies that for all $d>0$, the idempotents with defect d lie in the ideal generated by those with defect $d-1$, and the theorem follows by induction on the defect.

Thus assume $m=2$, and consider an idempotent $\varepsilon_{(i_1,i_2)}$ such that $i_1\not\equiv k_2$ and $i_2\not\equiv k_2$. This is an element of ${}_{a_{(i_1,i_2)}}\Delta_p(\Gamma)_{a_{(i_1,i_2)}}$. We think of $\varepsilon_{(i_1,i_2)}$ as the disjoint union $f_{i_1}\coprod f_{i_2}$, because it is represented by the disjoint union of two tubes $D^2 \times \mathbf{I}$, one of which is equipped with f_{i_1}, and the other with f_{i_2}. Now $\langle (S^2 \times S^1, e'_{p-2}) \rangle_p = 1$ by 3.3(ii), hence $\varepsilon_{(i_1, i_2)}$ is also represented, by the disjoint union of the above two tubes with $S^2 \times S^1$ equipped with e'_{p-2}. We now cut this into two parts, such that the first half is an element of

$$_{a_{(i_1,i_2)}}\Delta_p(\Gamma)_{a_{(i_1,i_2)} \amalg s_{p-2} \amalg s_{p-2}}.$$

This shows that $\varepsilon_{(i_1, i_2)}$ lies in the two-sided ideal generated by the identity $1_{i_1} \amalg 1_{i_2} \amalg 1'_{p-2} \amalg 1'_{p-2}$ (where $1'_{p-2}$ is the identity of s_{p-2}). But we may view $1_{i_v} \amalg 1'_{p-2}$ as lying in $\Delta_p(\Gamma_v)$ ($v = 1, 2$). Applying the first part of part (ii), we see that $1_{i_v} \amalg 1'_{p-2}$ lies in the ideal generated by the f_j, and since $p - 2$ is odd, we only need the f_j, such that $j \equiv k_v \bmod 2$. Thus, we have shown that $\varepsilon_{(i_1, i_2)}$ lies in the ideal generated by the $\varepsilon_{(j_1, j_2)}$, with $j_v \equiv k_v \bmod 2$. This completes the proof of (ii).

We now give the proof of (i). It follows from Lemma 3.5 and (ii) that $\Delta_p(\emptyset)$ is generated as a two-sided ideal by $f'_0, \ldots, f'_{q-1}$. Note that f'_0 lies in the ideal generated by $1_\emptyset$. We claim that $f'_i = 0$, unless p is even and $i = 0$, or unless p is odd and either $i = 0$ or $i = p - 2$. Indeed, for all $u \in {}_{s_i}\Delta_p(\emptyset)_{s_i} = V_p(-s_i \amalg s_i)$, we have

$$\langle f'_i, u \rangle_p = \langle f'_i, 1_{s_i} \rangle_p \overline{\varepsilon(u)} = \langle (S^2 \times S^1, e'_i) \rangle_p \overline{\varepsilon(u)}.$$

Since $\langle\, , \rangle_p$ is nondegenerate, the claim follows from proposition 3.3(ii). This shows (i). □

3.6. *Definition of the reduced algebroid* $\tilde{\Delta}_p(\Gamma)$. Let Γ be a closed 1-manifold with structure, with components $\Gamma_1, \ldots, \Gamma_m$. Recall that for $i = (i_1, \ldots, i_m)$, we have defined an object a_i of $\Delta_p(\Gamma)$, and if all $i_k < q$, we have an idempotent ε_i in the algebra ${}_{a_i}\Delta_p(\Gamma)_{a_i}$. If Γ is empty, we define $a_\emptyset = \emptyset$, $\varepsilon_\emptyset = 1_\emptyset$, and, if p is odd, $a_{\emptyset'} = s_{p-2}$ and $\varepsilon_{\emptyset'} = f'_{p-2}$.

The elements of the algebroid $\tilde{\Delta}_p(\Gamma)$ are the elements of the modules

$${}_i\tilde{\Delta}_p(\Gamma)_j = \varepsilon_i\, {}_{a_i}\Delta_p(\Gamma)_{a_j}\, \varepsilon_j$$

where the sets of objects are given as follows.

Let p be even. If $\Gamma = \emptyset$, there is a single object $\emptyset$. If Γ has $m > 0$ components, the sets of objects is the set of all m-tuples $i = (i_1, \ldots, i_m)$ satisfying $0 \le i_k < q$, for all k.

Let p be odd. The algebroid $\tilde{\Delta}_p(\Gamma)$ breaks up into the disjoint union of even and odd pieces, $\tilde{\Delta}_p^0(\Gamma)$ and $\tilde{\Delta}_p^1(\Gamma)$. If $\Gamma = \emptyset$, there is a single even object $\emptyset$, and a single odd object $\emptyset'$. If $\Gamma \ne \emptyset$, choose an m-tuple $k = (k_1, \ldots, k_m) \in \{0, 1\}^m$ such that $\sum k_j \equiv \varepsilon \bmod 2$. The objects of $\tilde{\Delta}_p^\varepsilon(\Gamma)$ are the m-tuples $i = (i_1, \ldots, i_m)$, with $0 \le i_j < q$ and $i_j \equiv k_j \bmod 2$, for all j.

Note. The algebroid $\tilde{\Delta}_p(\Gamma)$ has n^m (resp. $2n^m$) objects, if p is even (resp. odd), where $n = [(p-1)/2]$. For simplicity, our notation ignores the dependence on the choice of the two m-tuples $k \in \{0, 1\}^m$ in the p-odd case.

3.7. THEOREM. *Suppose $p \ge 3$. Let Γ be a closed 1-manifold with structure. The algebroids $\Delta_p(\Gamma)$ and $\tilde{\Delta}_p(\Gamma)$ are Morita equivalent. If p is odd, then for $\varepsilon \in \{0, 1\}$, the algebroids $\Delta_p^\varepsilon(\Gamma)$ and $\tilde{\Delta}_p^\varepsilon(\Gamma)$ are Morita equivalent.*

Proof. It is a general fact that whenever an algebroid Δ is generated by a set of idempotents ε_i, then it is Morita-equivalent to the algebroid $\tilde{\Delta}$ defined as above. This fact is proven in Appendix A. Thus, the theorem follows directly from 3.4.

A k-algebroid Δ is called *completely reduced* if ${}_i\Delta_i = k$, for all objects i of Δ, and ${}_i\Delta_j = 0$, for all $i \ne j$.

The following result completes the proof of Theorem 1.13.

3.8. THEOREM. *Let Γ be a closed* 1-*manifold with structure, and let $p \geq 3$. Then the algebroid $\tilde{\Delta}_p(\Gamma)$ is completely reduced.* (*In particular, if p is odd, then both $\tilde{\Delta}_p^0(\Gamma)$ and $\tilde{\Delta}_p^1(\Gamma)$ are completely reduced.*)

Proof. Suppose $\Gamma = S^1$. The module ${}_i\tilde{\Delta}_p(S^1)_j = f_i\, {}_i\Delta_p(S^1)_j f_j$ is the quotient of $f_i\, {}_iT_j f_j$ by the radical of the sesquilinear form $\langle\, ,\rangle$. If $i \neq j$, then $f_i\, {}_iT_j\, f_j$ is zero, since the f_i are augmentation idempotents. If $i = j$, then $f_i\, {}_iT_i\, f_i$ is a free k_p-module generated by f_i, and $\langle f_i, f_i\rangle = (-1)^i[i+1]$. Since this is invertible in k_p, we have ${}_i\tilde{\Delta}_p(S^1)_i \simeq k_p$ as required. Thus the theorem holds if $\Gamma = S^1$.

It follows from the above and 3.3(ii) that ${}_i\tilde{\Delta}_p(\emptyset)_i \simeq k_p$, for $i = \emptyset$ or $i = \emptyset'$. Thus the theorem is clear for $\Gamma = \emptyset$.

By 3.7 and the computation of $\tilde{\Delta}_p(\emptyset)$, the tensor product axiom (M) holds (over the category $C_2^{p_1}$ (*even*) if p is odd). Let i and j be objects. The object a_i is the disjoint union of objects $a_{i_1}, \ldots, a_{i_m}$ in $\Delta_p(\Gamma_1), \ldots, \Delta_p(\Gamma_m)$, and a_j is the disjoint union of $a_{j_1}, \ldots, a_{j_m}$. Applying the tensor product axiom (if p is odd, we may assume that the objects i and j have the same parity, in which case, by assumption, the parity of i_k and j_k are the same, so that $-a_{i_k} \cup a_{j_k}$ has an even link), we have

$$_{a_i}\Delta_p(\Gamma)_{a_j} = V_p(-a_i \cup a_j) \simeq V_p(-a_{i_1} \cup a_{j_1}) \otimes \cdots \otimes V_p(-a_{i_m} \cup a_{j_m}).$$

The theorem now follows from the case where $\Gamma = S^1$. □

3.9. *Remark.* We have shown that the tensor product axiom (M) holds, if $p \geq 4$ is even and the cobordism category is $C_2^{p_1}$, or if $p \geq 3$ is odd and the cobordism category is $C_2^{p_1}$ (*even*). Here is an example showing that it does not hold if $p \geq 3$ is odd and the cobordism category is $C_2^{p_1}$.

Let Σ be the 2-sphere equipped with a banded link with $p-2$ components. Since $p-2$ is odd, $V_p(-\Sigma)$ and $V_p(\Sigma)$ are zero. We claim that $V_p(-\Sigma \coprod \Sigma) \approx k_p$. This can be seen as follows. Note that Σ is the object s_{p-2} of $\Delta_p(\emptyset)$. We have shown that the idempotent f'_{p-2} is the generator of ${}_{s_{p-2}}\Delta_p(\emptyset)_{s_{p-2}}$ which by definition is $V_p(-\Sigma \coprod \Sigma)$. Hence $V_p(-\Sigma \coprod \Sigma) \approx k_p$ as claimed.

4. COLORED STRUCTURES

In this section, we prove the Colored Splitting Theorem 1.14, and Theorem 1.15. Taken together, these theorems allow one to totally decompose the V_p theory into its elementary building blocks.

Convention. In this section, we again suppose $p \geq 3$. All colors are assumed $<q$, where $q = n$, if $p = 2n+2$, and $q = 2n$, if $p = 2n+1$.

We begin by enlarging the category of manifolds with structure and we extend the invariants $\langle\ \rangle_p$ to this larger setting. This will make it possible for us to interpret the module $\varepsilon_i\, {}_{a_i}V_p(\Sigma)_{a_j}\, \varepsilon_j$ as the V_p-module of a colored object. It will also make it possible for us to describe an explicit basis of the modules V_p.

4.1. *Notation and definitions.* Let P denote the *pair of pants* surface, i.e. P is a compact connected surface of genus 0 with three boundary components, i.e. $\partial P = S^1 \coprod S^1 \coprod S^1$. (There is no banded link in P.) A triple $i = (i_1, i_2, i_3)$ of colors is said to be an *admissible triple* if $i_1 + i_2 + i_3$ is even and $|i_1 - i_2| \leq i_3 \leq i_1 + i_2$ (the triangle inequality) is satisfied. If $i = (i_1, i_2, i_3)$ is an admissible triple, let u_i denote the element of ${}_0T_{i_1+i_2+i_3}\, f_{i_1} \times f_{i_2} \times f_{i_3} \subset {}_0T_{i_1+i_2+i_3}$, depicted in Fig. 3.

(In Fig. 3, the numbers α, β, γ of connecting strands are determined by $i_1 = \beta + \gamma$, $i_2 = \gamma + \alpha$, $i_3 = \alpha + \beta$.)

We may also view u_i as an element of $K(B, -(l_{i_1} \cup l_{i_2} \cup l_{i_3}))$, where B is the 3-ball with boundary decomposed as $\partial B = P \cup -(D^2 \cup D^2 \cup D^2)$. Let us denote by ${}_\emptyset\tilde{V}_p(P)_i$ the submodule

$$ {}_\emptyset V_p(P)_{a_i}\, \varepsilon_i \subset V_p(P \cup a_{i_1} \cup a_{i_2} \cup a_{i_3}). $$

The induced element (where the colors are $<q$) in ${}_\emptyset\tilde{V}_p(P)_i$ will again be denoted by u_i.

Since the f_{i_ν} are augmentation idempotents, it is clear that ${}_\emptyset\tilde{V}_p(P)_i$ is generated by u_i.

Set

$$ \langle i_1, i_2, i_3 \rangle = (-1)^{\alpha+\beta+\gamma} \frac{[\alpha+\beta+\gamma+1]![\alpha]![\beta]![\gamma]!}{[i_1]![i_2]![i_3]!} $$

where α, β, γ are determined as above, and $[n]!$ is the quantum factorial $[n]! = [1]\,[2]\ldots[n]$.

For a color i, we set $\langle i \rangle = \langle f_i, f_i \rangle = (-1)^i[i+1]$.

4.2. Lemma. $\langle u_i, u_i \rangle_p = \langle S^3 \rangle_p \langle i_1, i_2, i_3 \rangle$.

Proof. We have $\langle u_i, u_i \rangle_p = \langle S^3 \rangle_p \langle u_i, u_i \rangle$ where $\langle\, , \rangle$ is the sesquilinear form defined in Section 3. By Theorem 1 of [26], one has $\langle u_i, u_i \rangle = \langle i_1, i_2, i_3 \rangle$. □

4.3. Corollary. *An admissible triple $i = (i_1, i_2, i_3)$ has the property that $\langle u_i, u_i \rangle_p$ is nonzero in k_p if and only if $i_1 + i_2 + i_3 < 2q$. Moreover, if $\langle u_i, u_i \rangle_p$ is nonzero, then it is invertible.*

Proof. This follows from the previous lemma, since in k_p, we have that $[i]$ is invertible if $0 \le i \le q$, and $[q+1] = 0$. □

Definition. An admissible triple $i = (i_1, i_2, i_3)$ is called *p-admissible* if $i_1 + i_2 + i_3 < 2q$.

Since the form $\langle\, , \rangle_p$ is nondegenerate, we thus have the following.

4.4. Theorem. *Let P be a pair of pants surface. Let $i = (i_1, i_2, i_3)$ be an admissible triple of colors (assumed $<q$). Then the module ${}_\emptyset\tilde{V}_p(P)_i$ is free of rank one, generated by u_i, whenever i is p-admissible. Otherwise, ${}_\emptyset\tilde{V}_p(P)_i = 0$.*

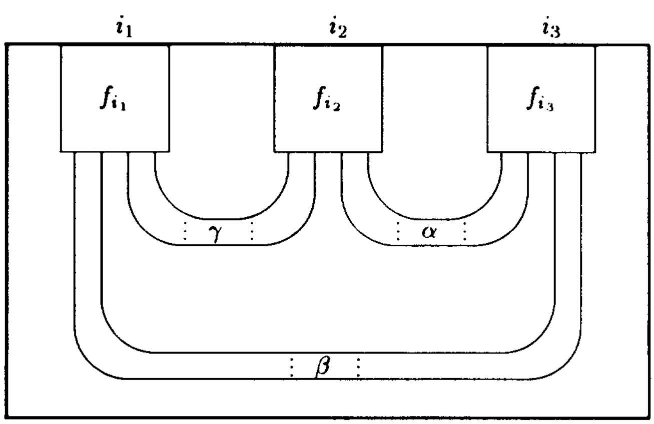


Fig. 3.

4.5. Definition. A *banded 3-valent graph* in a 3-manifold M is a graph G, contained in an oriented surface $SG \subset M$, such that

(i) G meets ∂M transversally in the set of vertices of G of degree 1.

(ii) every vertex of G contained in the interior of M is of degree 2 or 3.

(iii) the surface SG is a regular neighborhood of G in SG, and $SG \cap \partial M$ is a regular neighborhood of $G \cap \partial M$ in $SG \cap \partial M$. (Note that $SG \cap \partial M$ is a banded link in ∂M.)

The set of vertices of G of degree 1 is called the *boundary* of G and denoted by ∂G.

An *admissible coloring* σ of G is a function from the set of edges of G to the set of colors such that the colors of the edges meeting at each 2-valent vertex coincide, and the colors of the edges meeting at each 3-valent vertex form an admissible triple (see 4.1).

A *colored graph* is a banded 3-valent graph with an admissible coloring.

Note. In this paper, we will consider colored graphs only through their *expansions* (see below). However, one could also have built the theory directly out of colored graphs using *colored Kauffman relations* (see [26] for a derivation of these relations).

4.6. *Definition.* A *surface with colored structure* (Σ, l, i) is a surface with structure (Σ, l) such that each component of l is labelled by a color. (Here l is a banded link in Σ and i is a coloring of the components of l.)

A 3-*manifold with colored structure* $M = (M, \alpha, G, \sigma)$ is a 3-manifold M equipped with a p_1-structure α and a colored graph (G, σ).

The cobordism category $C_{2,q}^{p_1,c}$. The boundary of a 3-dimensional manifold with colored structure is a surface with colored structure. As in the case of manifolds with structure, one can cut and paste colored manifolds. We denote by $C_{2,q}^{p_1,c}$ the cobordism category of surfaces with colored structure such that the colors satisfy $0 \leq i < q$.

Remark. A manifold with structure is a particular case of a manifold with colored structure, by making the color everywhere equal to one. Conversely, colored structures may be "expanded", in the following way.

4.7. *Definition.* The *expansion* of a colored link (l, c) in a surface Σ is the link l_c which is obtained from l by replacing each component by as many parallel copies as indicated by the color of that component. If Σ denotes the manifold with structure and c is a coloring of its link, we will denote by Σ_c the manifold with structure obtained by expanding its link.

The *expansion* of a colored graph (G, σ) in M is the element of $K(M, l_c)$, where $l \subset \partial M$ is the link $SG \cap \partial M$, defined as follows.

Choose a homeomorphism from a regular neighborhood N of G in M to $SG \times \mathbf{I}$ such that SG is sent to $SG \times \{\frac{1}{2}\}$. Consider first the case where G is a Y-shaped graph with exactly one trivalent vertex and three boundary vertices. Then we may identify $SG \times \mathbf{I}$ (hence N) in a standard way with the 3-ball $B = D^3$, whose boundary is decomposed as $\partial B = P \cup -(D^2 \cup D^2 \cup D^2)$, and we define the expansion of G, with edges colored by i_1, i_2, i_3, to be the linear combination of banded tangles in N corresponding to the element $u_{(i_1, i_2, i_3)} \in K(B, -(l_{i_1} \cup l_{i_2} \cup l_{i_3}))$. (Think of $u_{(i_1, i_2, i_3)}$ as "drawn" on the surface SG.) The expansion of an I-shaped graph with just one edge colored by i and two boundary vertices is defined in a similar way, so as to correspond to $f_i \in K(D^2 \times \mathbf{I}, -l_i \times 0 \cup l_i \times 1)$, and the expansion of an O-shaped graph with just one edge colored by i and one 2-valent vertex is defined so as to correspond to the "closure" of f_i in $K(D^2 \times S^1)$. Now any banded trivalent graph G in M is covered by a union of Y-, I-, and O-shaped pieces, and since the f_i are idempotents, the above (well-)defines an element of $K(M, l_c)$, for every admissible coloring of G.

Remark. In the above, we have used the fact that the f_i are left fixed by the orientation preserving homeomorphism of $D^2 \times \mathbf{I}$ which reverses the orientation of both factors.

4.8. *Definition.* Let $M = (M, \alpha, G, \sigma)$ be a closed 3-manifold with colored structure. Assume the colors are $< q$. We define $\langle M \rangle_p$ to be the $\langle\ \rangle_p$-invariant of (M, α) equipped with the expansion of (G, σ).

The extended invariant $\langle\ \rangle_p$ on manifolds with colored structure is a multiplicative and involutive invariant on the set of closed bordisms of the cobordism category $C_{2,q}^{p_1,c}$. Hence it determines, by Proposition 1.1, a cobordism generated quantization functor on $C_{2,q}^{p_1,c}$. Since the invariant of a closed 3-manifold with colored structure is the same as that of its expansion, the associated module $V_p^c(\Sigma)$ for a surface with structure (considered as a surface with colored structure) is the same as the module $V_p(\Sigma)$. Hence, the superscript c is superfluous and will be omitted in what follows. Again we will write the associated hermitian form, $\langle\, ,\rangle_\Sigma$, simply as $\langle\, ,\rangle_p$. Every 3-dimensional manifold M with colored structure induces an element $Z_p(M) \in V_p(\partial M)$.

For each 1-manifold with structure Γ, there is an algebroid $\Delta_p^c(\Gamma)$ whose objects are surfaces with colored structure and boundary Γ, and to each cobordism with colored strucure Σ from Γ_1 to Γ_2 there is a $\Delta_p^c(\Gamma_1) \times \Delta_p^c(\Gamma_2)$-bimodule ${}_-V_p(\Sigma)_-$.

Remark. Let $\Sigma = (\Sigma, l, i)$ be a closed surface with colored structure, where $i = (i_1, i_2, \dots)$ are the colors of the components of l. Let Σ_c be the expansion of Σ, i.e. the surface with structure obtained by replacing, for all j, the jth component of l by i_j parallel copies. Then $V_p(\Sigma)$ is canonically isomorphic to a submodule of $V_p(\Sigma_c)$.

There is a colored trivalent graph (Fig. 4) in the manifold $D^2 \times I$ which is a cobordism from D^2 with a standard 1-component banded link, colored with the color i_j, to its expansion, i.e. i_j parallel copies of this link, and whose expansion is the element f_{i_j}. Embedding a copy of this graph in $\Sigma \times \mathbf{I}$, for each component of l, one obtains a colored graph G and hence a (colored) cobordism W from Σ to its expansion Σ_c, whose expansion ε_i is an idempotent. This ε_i induces a projector Z_{ε_i} (considered as a homomorphism of $V_p(\Sigma_c)$). Note that the reflection (in the I factor) of this graph yields a graph G' and hence a cobordism W', whose expansion is also the idempotent ε_i. It is easy to check that the induced map $Z_{W'}$ is a surjection, the induced map Z_W is a section and the composite

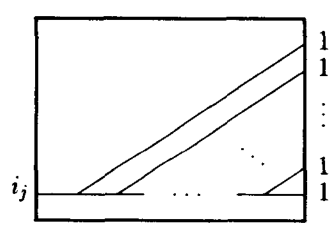


Fig. 4.

mapping $Z_{W' \cup W}$ is the projector Z_{ε_i} onto the module $\varepsilon_i V_p(\Sigma_c)$ (since the expansion of these graphs is ε_i). Thus, the map Z_W provides an isomorphism from $V_p(\Sigma)$ onto the submodule $\varepsilon_i V_p(\Sigma_c)$.

Notation. Let Γ be a 1-manifold with structure. For every coloring $i = (i_1, \ldots, i_m)$ of the components of Γ, we define an object $b_i = b_i(\Gamma)$ of $\Delta_p^c(\Gamma)$ to be the object $a_{(1,\ldots,1)}$ of $\Delta_p(\Gamma)$ (i.e. each component of Γ bounds a disk with a 1-component banded link) and whose link is labelled with the coloring i. If $\Gamma = \emptyset$, we set $b_\emptyset = \emptyset$, and, if p is odd, we let $b_{\emptyset'}$ be the 2-sphere equipped with a 1-component banded link colored by $p-2$.

Note that the expansion of b_i is the surface a_i, defined in Section 3.

4.9. Proposition. *The algebroid $\Delta_p^c(\Gamma)$ is Morita-equivalent to the completely reduced algebroid $\tilde{\Delta}_p(\Gamma)$.*

Proof. Using 3.7 and the graphs G and G', one sees that $\Delta_p^c(\Gamma)$ is generated as a two-sided ideal by the identities of the objects $b_i(\Gamma)$, where i runs through the objects of $\tilde{\Delta}_p(\Gamma)$. The result now follows from 3.8. □

Proof of Theorem 1.15. Since $V_p(S^2(i,j)) \simeq {}_i\Delta_p^c(S^1)_j \simeq {}_i\tilde{\Delta}_p(S^1)_j$, parts 1 and 2 of 1.15 follow, since the algebroid $\tilde{\Delta}_p(S^1)$ is completely reduced. Since $V_p(S^2(i,j,k)) \simeq {}_\emptyset\tilde{V}_p(P)_{(i,j,k)}$, parts 3 and 4 of 1.15 follow from 4.4. □

Proof of the Colored Splitting Theorem 1.14. First suppose that the surface $\Sigma(\Gamma)$, obtained from Σ by cutting along Γ, breaks up into a disjoint union $\Sigma' \coprod \Sigma''$, with Γ as the boundary of each. Applying the Splitting Theorem, Morita equivalence, and the fact that tensor products are preserved under Morita equivalence, one obtains

$$V_p(\Sigma) = \bigoplus_i V_p(\Sigma_i') \otimes V_p({}_i\Sigma'')$$

where the sum is over all colors $<q$.

If p is even, then $V_p(\Sigma_i') \otimes V_p({}_i\Sigma'')$ is isomorphic to $V_p({}_i\Sigma(\Gamma)_i)$, by the tensor product formula. Now suppose p is odd. If the links in Σ' and Σ'' have different (total) parity, then all terms of the expression are zero. Otherwise the links in Σ' and Σ'' have the same parity, and the sum is only over the i terms of that same parity (the terms involving the other parity vanish). Since the links in ${}_i\Sigma'$ and Σ_i'' are now even, we again have $V_p(\Sigma_i') \otimes V_p({}_i\Sigma'') \approx V_p({}_i\Sigma(\Gamma)_i)$.

Now suppose that Γ does not separate Σ. We proceed as follows. Let Γ' be a parallel copy of Γ in Σ. Then Γ and Γ' cut Σ into two surfaces, $\Sigma(\Gamma)$ and $\Gamma \times \mathbf{I}$. We may consider $\Sigma(\Gamma)$ as a cobordism from $\emptyset$ to $\Gamma \coprod -\Gamma$, and $\Sigma \times \mathbf{I}$ as a cobordism from $\Gamma \coprod -\Gamma$ to $\emptyset$. The splitting theorem for $\Gamma \coprod -\Gamma$ yields the decomposition

$$V_p(\Sigma) = \bigoplus_{(i,j)} V_p(\Sigma(\Gamma)_{(i,j)}) \otimes V_p({}_{(i,j)}(\Gamma \times \mathbf{I}))$$

where the sum is over all colorings (i,j), if p is even, and over all *even* colorings, if p is odd. (We have applied 3.7 and 3.8, where in Definition 3.6, the 2-tuple k is taken to be $(0,0)$.) Now $V_p({}_{(i,j)}(\Gamma \times \mathbf{I}))$ is isomorphic to ${}_i\Delta(\Gamma)_j$ which is zero if $i \neq j$, and isomorphic to the ground ring k_p if $i = j$. But it is clear that

$$V_p(\Sigma(\Gamma)_{(i,i)}) = V_p({}_i\Sigma(\Gamma)_i).$$

(On the left hand side, $\Sigma(\Gamma)$ is considered as a cobordism from $\emptyset$ to $\Gamma \coprod -\Gamma$, whereas on the right hand side, $\Sigma(\Gamma)$ is considered as a cobordism from Γ to Γ.)

This establishes the decomposition.

To prove the remainder of the theorem, consider elements $u \in V_p({}_i\Sigma(\Gamma)_i)$, $v \in V_p({}_j\Sigma(\Gamma)_j)$. Let u (resp. v) be represented by a manifold M (resp. N). The image u' of u (resp. v' of v) in $V_p(\Sigma)$ is represented by the manifold M' (resp. N') obtained from M (resp. N) by identifying two disks each containing a 1-component link colored by i (resp. j). The product $\langle u', v' \rangle_p$ of the images is the $\langle\ \rangle_p$-invariant of the manifold with colored structure $M' \cup_\Sigma -N'$. This manifold contains the surface with colored structure $S^2(i, j)$, which is the union of the two disks each with a 1-component banded link colored by i, resp. j. Since, by 1.15, $V_p(S^2(i, j)) = 0$ if $i \neq j$, it follows that the product $\langle u', v' \rangle_p$ is zero. Thus, the decomposition is orthogonal.

Now suppose $i = j$. Then $V_p(S^2(i, i))$ is free of rank 1. The product $\langle u, v \rangle_p$ is represented by the manifold $M \cup_{{}_i\Sigma(\Gamma)_i} -N$. Note that this can be obtained from $M' \cup_\Sigma -N'$ by doing surgery on the sphere $S^2(i, i)$, i.e. one removes $X_0 = S^2(i, i) \times I$ and replaces it with $X_1 = b_i \times I \times S^0$. Now an easy calculation shows that $Z(X_1) = \langle S^3 \rangle_p \langle i \rangle Z(X_0)$. This shows that the form induced on each factor corresponds, up to the unit $\langle S^3 \rangle_p \langle i \rangle$ in k_p, with the usual form. This proves 1.14. □

4.10. Corollary. *Let Γ be a closed 1-manifold with structure. Then $V_p(S^1 \times \Gamma)$ has a canonical basis $\{e_i\}$ (where the colors are $< q$ and, in addition, are even if p is odd), where e_i is represented by the 3-manifold with colored structure $S^1 \times b_i(\Gamma)$. Moreover, this basis is orthonormal with respect to $\langle\, , \rangle_p$.*

Proof. This follows directly from the Colored Splitting Theorem 1.14. □

Remark. In the language of Section 3, e_i is represented by the closure of the idempotent ε_i. In particular, if $\Gamma = S^1$, then e_i is represented by the closure of f_i.

4.11. Theorem. *Let $\Sigma = (\Sigma, l, i)$ be a connected closed surface with colored structure. Assume all colors are $< q$, and, in addition, are even, if p is odd. Let H be a handlebody whose boundary is Σ, and let G be a banded 3-valent graph in H such that $\partial G = l$ and such that H is a tubular neighborhood of G. For each p-admissible coloring σ of G, compatible with the coloring i of ∂G, let u_σ denote the element induced by the manifold with colored structure (H, α, G, σ). Then the elements u_σ, where all colors are $< q$, and, in addition, are even, if p is odd, form an orthogonal basis of $V_p(\Sigma)$. Moreover, one has*

$$\langle u_\sigma, u_\sigma \rangle_p = (\langle S^3 \rangle_p)^{\#v - \#e} \frac{\prod_v \langle \sigma(v) \rangle}{\prod_e \langle \sigma(e) \rangle}$$

where v runs through the set of vertices of G, and e runs through the set of edges of G.

Here, we denote by $\sigma(v)$ the (set of) color(s) of the edge(s) meeting at the vertex v. (The notation $\langle i \rangle$ and $\langle i_1, i_2, i_3 \rangle$ were defined in 4.1. A 2-valent vertex has only one color.)

Proof. This follows from 1.14 by cutting and pasting. The formula for $\langle u_\sigma, u_\sigma \rangle_p$ follows from the following four facts:

— It is true for the $Y-$, $I-$, and O-shaped graphs (see 4.7).

— It is multiplicative for disjoint unions.

— By 1.14, it is preserved if we identify two 1-valent vertices to obtain a 2-valent vertex.

— Finally, if two different edges meet at a 2-valent vertex, then we can suppress the vertex by identifying the two edges, and the formula remains valid. □

4.12. *Remark on signatures.* Using the formula given in 4.11, one can compute the *signature* of the form $\langle\,,\rangle_p$, in the case where coefficients are extended from k_p to $\mathbf{C}$. While the dimension of $V_p(\Sigma)$ clearly depends only on Σ and p, the signature of $\langle\,,\rangle_p$ also depends on the given homomorphism $k_p \to \mathbf{C}$. Actually, it depends only on Σ, the sign of $\langle S^3\rangle_p = \eta$ and the root of unity $A^2 \in \mathbf{C}$.

If Σ is a torus $S^1 \times S^1$ without link, the rank of $V_p(\Sigma)$ is equal to $n = [(p-1)/2]$, and the form $\langle\,,\rangle_p$ is positive definite. But, if Σ is a surface of genus 2 without link, the signature of the form $\langle\,,\rangle_p$ takes, for $p = 5$, the values ± 5, ± 3, depending on the sign of η and the sign of the real part of A^2.

4.13. *More on the p-odd case.* If p is odd, it remains to describe a basis of the modules V_p associated to surfaces $\Sigma = (\Sigma, l, i)$ with colored structure, and colors $< q = 2n$ which are not necessarily even. In analogy with Theorem 4.11, this can be done in the following way. Choose a banded trivalent graph G such that Σ is the boundary of a tubular neighborhood of G. For each p-admissible coloring σ of G, compatible with the given coloring of $\partial G = l$, and with colors $< q$, we have the element $u_\sigma \in V_p(\Sigma)$ represented by the colored graph (G, σ) in its tubular neighborhood. We associate to σ a cellular 1-chain $\gamma(\sigma) \in C_1(G; \mathbf{Z}/2)$ by setting $\gamma(\sigma) = \sum_e \sigma(e)e$ (the sum is over all edges e of G). The boundary of $\gamma(\sigma)$ is the 0-chain $\partial\gamma(\sigma) = \sum_{v \in l} i_v v \in C_0(G; \mathbf{Z}/2)$.

4.14. THEOREM. *Assume p is odd. Let $\Sigma = (\Sigma, l, i)$ be a surface with colored structure (with colors $< q$), and choose G as above. Let $\gamma \in C_1(G; \mathbf{Z}/2)$ such that $\partial\gamma = \sum_{v \in l} i_v v$. Then the u_σ, with $\gamma(\sigma) = \gamma$, form a basis of $V_p(\Sigma)$.*

Proof. Theorem 1.15 implies the result in the case where Σ is S^2 equipped with a 3-component colored link (with colors $< q$). In the general case, we proceed as follows.

Denote by H the regular neighborhood of the graph G, with $\partial H = \Sigma$. Let $B = B_1 \coprod \cdots \coprod B_m \subset H$ be a separating surface such that each B_i is a 2-disk meeting transversally an edge a_i of G, and such that $\partial B_i = B_i \cap \Sigma$ is a 1-manifold $\Gamma_i \subset \Sigma - l$. Set $\nu_i = 1$, if the edge a_i is contained in the chain γ, and $\nu_i = 0$, otherwise. Set $\varepsilon = \nu_1 + \cdots + \nu_m \in \mathbf{Z}/2$. We apply the Splitting Theorem 1.16 to cut Σ along $\Gamma = \Gamma_1 \cup \cdots \cup \Gamma_m$. For parity reasons, the tensor product is only over the subalgebroid $\Delta_p^{c,\varepsilon}(\Gamma)$. But this algebroid is Morita-equivalent to the completely reduced algebroid $\tilde{\Delta}_p^{\varepsilon}(\Gamma)$ (where the m-tuple k of Definition 3.6 is chosen to be $k = (\nu_1, \ldots, \nu_m)$). The theorem now follows by induction. □

4.15. *Remark.* Call two colors i_v, i'_v *complementary* if $i_v + i'_v = p - 2$. The theorem implies that the rank of $V_p(\Sigma, l, i)$ does not change if one replaces two colors by their complementary colors.

5. THE VERLINDE FORMULA

We again suppose $p \geq 3$, and we set $q = n$, if $p = 2n + 2$, and $q = 2n$, if $p = 2n + 1$.

5.1. PROPOSITION. *Let Σ be a compact surface with structure with boundary $-\Gamma \coprod \Gamma'$. Then the morphism $Z_{S^1 \times \Sigma}$ from $V_p(S^1 \times \Gamma)$ to $V_p(S^1 \times \Gamma')$ induced by the cobordism $S^1 \times \Sigma$ is given by*

$$Z_{S^1 \times \Sigma}(e_i) = \sum_j (\text{rank } V_p({}_i\Sigma_j))e_j$$

where $\{e_i\}$ and $\{e_j\}$ denote the orthonormal bases as given in 4.10.

Proof. Denote by ${}_i\Sigma_j$ the closed surface with colored structure $-b_i(\Gamma) \cup \Sigma \cup b_j(\Gamma')$. Then

$$\langle Z_{S^1 \times \Sigma}(e_i), e_j \rangle_p = \langle S^1 \times {}_i\Sigma_j \rangle_p = \operatorname{rank} V_p({}_i\Sigma_j)$$

by 1.2. Since the e_j are an orthonormal basis, the result follows. □

Note also the following "curve pinching" theorem.

5.2. Corollary. *Let Γ be a closed 1-manifold on the cobordism with structure Σ, and let ${}_i\Sigma(\Gamma)_i$ denote the result of replacing $\Gamma \times I$ by $b_i \times \partial I$. Then*

$$Z_{S^1 \times \Sigma} = \sum_i Z_{S^1 \times {}_i\Sigma(\Gamma)_i}$$

(where the sum is over all colors $< q$, if p is even, and over all even colors $< q$, if p is odd).

Proof. Applying 1.14 to ${}_kV_p(\Sigma)_j$ yields

$${}_kV_p(\Sigma)_j = \bigoplus_i {}_kV_p({}_i\Sigma(\Gamma)_i)_j,$$

whence the result follows from 5.1. □

5.3. *Hochschild homology.* (See Appendix A). Let Γ be a closed 1-manifold with structure. By 1.14 and Morita equivalence, we have

$$V_p(S^1 \times \Gamma) \simeq \begin{cases} H_0(\tilde{\Delta}_p(\Gamma)) \simeq H_0(\Delta_p^c(\Gamma)) & \text{if } p \text{ is even} \\ H_0(\tilde{\Delta}_p^0(\Gamma)) \simeq H_0(\Delta_p^{c,0}(\Gamma)) & \text{if } p \text{ is odd.} \end{cases}$$

Under this isomorphism, the basis element e_i of $V_p(S^1 \times \Gamma)$ corresponds to the identity of the object $b_i(\Gamma)$. Now if Σ is a cobordism from a closed 1-manifold Γ with structure to a closed 1-manifold Γ' with structure, the morphism Z_Σ from $\Delta_p^c(\Gamma)$ to $\Delta_p^c(\Gamma')$ induces a morphism $H_0(Z_\Sigma)$ between H_0 modules. It is easy to see that under the above isomorphism, $H_0(Z_\Sigma)$ corresponds to $Z_{S^1 \times \Sigma}$. Using the description of $Z_{S^1 \times \Sigma}$ in Corollary 5.1, we may thus restate the main result of Theorem 1.14 as follows.

5.4. Corollary. *Let Γ be a closed 1-manifold with structure contained in a closed surface Σ with colored structure. Let $\Sigma(\Gamma)$ be the surface obtained by cutting Σ along Γ. Then $V_p(\Sigma)$ is isomorphic to $H_0(\Delta_p(\Gamma), {}_-V_p(\Sigma(\Gamma))_-)$, if p is even, and to $H_0(\Delta_p^0(\Gamma), {}_-V_p(\Sigma(\Gamma))_-)$, if p is odd. Moreover, its rank is the trace of the morphism $H_0(Z_{\Sigma(\Gamma)})$.*

5.5. *Notation.* In the remainder of this section, we abbreviate the module $V_p(S^1 \times S^1)$ by V_p. Recall from 4.10 that V_p has a basis $\{e_i\}$, where the colors i are $< q$, if p is even, and are even and $< q$, if p is odd.

Let $P_\succ$ (resp. $P_\prec$) be the *pair of pants* surface considered as a cobordism from $S^1 \coprod S^1$ to S^1 (resp. from S^1 to $S^1 \coprod S^1$). The surface $P_\prec \cup_{S^1 \coprod S^1} P_\succ$, is a cobordism from S^1 to S^1 of genus one, which will be denoted simply by $P_\prec \cup P_\succ$.

The "multiplication map" $Z_{S^1 \times P_\succ}: V_p \otimes V_p \to V_p$ will simply be denoted by $(a, b) \mapsto ab$.

Let $\Sigma(k)$ denote the cylinder $S^1 \times \mathbf{I}$ equipped with a one-component banded link colored by k. ($\Sigma(k)$ is a cobordism from S^1 to S^1.)

5.6. Proposition. *In the module V_p, one has*

$$Z_{S^1 \times P_\succ}(e_i \otimes e_j) = \sum_{(i,j,k)\ p\text{-admiss.}} e_k$$

$$Z_{S^1 \times P_<}(e_i) = \sum_{(i,j,k)\ p\text{-admiss.}} e_j \otimes e_k$$

$$Z_{S^1 \times \Sigma(k)}(x) = e_k x, \qquad Z_{S^1 \times (P_< \cup P_>)}(x) = Kx,$$

where $K = \sum e_j^2 \in V_p$. *(Here the colors are* $<q$, *if p is even, and are even and* $<q$, *if p is odd.)*

Proof. The first three equations follow immediately from 1.15, together with 5.1. For the last equation, one may apply 5.2 to show

$$Z_{S^1 \times (P_< \cup P_>)} = \sum_j Z_{S^1 \times (\Sigma(j) \cup \Sigma(j))}$$

and then use the third equation. □

5.7. Corollary. *Let* $K = \sum e_j^2 \in V_p$ *(the sum being over all colors* $<q$, *if p is even, and over all even colors* $<q$, *if p is odd). Let* $\Sigma = (\Sigma, l, i)$ *be a closed connected surface with colored structure, such that the components of l are colored by* $i = (i_1, \ldots, i_m)$. *Then*

$$\operatorname{rank} V_p(\Sigma) = \operatorname{trace}_{V_p(\Sigma)}(e_{i_1} \ldots e_{i_m} K^{g-1})$$

where g is the genus of Σ.

Proof. This follows from 5.4 and 5.6, since we can cut the surface into pieces isomorphic to $P_< \cup P_>$ and to $\Sigma(i_j)$. □

Remark. An alternative proof is to apply 4.11. Think of $\Sigma = (\Sigma, l, i)$ as the boundary of a tubular neighborhood of the graph shown in Fig. 5.

Let N_l be the matrix of the multiplication by e_l on V_p, with respect to the basis given by the e_i. It follows from 5.6 that $(N_l)_{ij}$ is equal to 1, if (i, j, l) is p-admissible (with colors $<q$, if p is even and even colors $<q$ if p is odd), and zero otherwise. An elementary argument (see [1, formula (5.8)]) shows that the number of such colorings, compatible with the given coloring of the boundary, is equal to the trace of $N_{i_1} \ldots N_{i_m}(\sum (N_i)^2)^{g-1}$. The result follows.

5.8. *Some results from* [10]. Before giving the proof of 1.16, we describe the module V_p in more detail. The canonical surjection

$$K(D^2 \times S^1) \to V_p = V_p(S^1 \times S^1)$$

is actually a ring homomorphism, the multiplication on $K(D^2 \times S^1)$ again being induced by $S^1 \times P_>$. (This multiplication is the same as the one studied in [10]. Also, the sesquilinear form on $K(D^2 \times S^1)$ becomes the bilinear form on $K(S^1 \times D^2)$, considered in [10], after identifying these modules and their conjugates. Hence, the module V_p is, up to change of coefficients, the V_p of [10].) Let $z \in K(D^2 \times S^1)$ be represented by a standard band. Then z^n

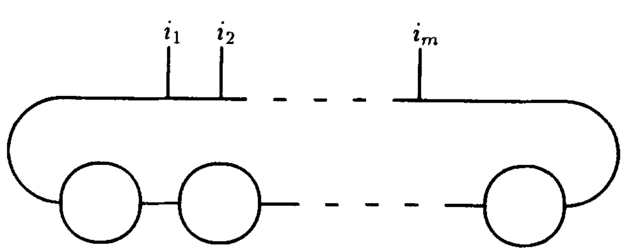


Fig. 5.

means n parallel standard bands, and it is well known [33] that $K(D^2 \times S^1)$ is isomorphic to the polynomial algebra $\mathbf{Z}[A, A^{-1}][z]$. It has a basis of monic polynomials e_i of degree i in z which satisfy $e_0 = 1$ (the empty link), $e_1 = z$ and $ze_j = e_{j+1} + e_{j-1}$. (If one substitutes $z = -y - y^{-1}$, then $e_{i-1} = (-1)^{i-1}(y^i - y^{-i})/(y - y^{-1})$.)

Note. The element e_i is represented in $K(D^2 \times S^1)$ by the closure of the idempotent f_i, provided the latter exists. (This follows by induction from Wenzl's formula.) Thus, the notation is consistent: the image of e_i in the module V_p is the previously defined e_i (see 4.10 and the remark following it.)

In the module V_p, the element Ω_p of [10] can be written

$$\Omega_p = \sum_{i=0}^{n-1} \langle e_i \rangle e_i = \frac{1}{4} \sum_{i=1}^{2p} \langle e_i \rangle e_i.$$

Here $n = [(p-1)/2]$ is the rank of the module V_p. The notation $\langle x \rangle$, for $x \in K(D^2 \times S^1)$, means $\langle U_0(x) \rangle$, where U_0 is the framed unknot with zero framing in S^3, and $U_0(x)$ denotes the result of putting a copy of x in a neighborhood of U_0. We have $\langle e_i \rangle = (-1)^i [i+1]$.

Let t be the self-map of $K(D^2 \times S^1)$ induced by one positive twist, and let c be the result of adding a meridinal band. In [10], the e_i were constructed as an eigenbasis of $K(D^2 \times S^1)$ for both t and c, with eigenvalues $\mu_i = (-1)^i A^{i^2+2i}$ under t, and $\lambda_i = -(A^{2i+2} + A^{-2i-2})$ under c. There is a bilinear form, denoted by $\langle , \rangle$ in [10], on $K(D^2 \times S^1)$, given by

$$\langle x, y \rangle = \langle H(x, y) \rangle$$

where $H(x, y)$ means the Hopf link (with each component framed with zero framing), where, the first component is replaced by x, and the second component is replaced by y. One has $\langle c(u), v \rangle = \langle u, zv \rangle$ for all $u, v \in K(D^2 \times S^1)$. Since $\langle , \rangle$ induces a nondegenerate bilinear form on V_p [10], it follows that the self-map of V_p given by multiplication by $e_1 = z$ has eigenvalues $\lambda_0, \ldots, \lambda_{n-1}$.

Here is an eigenbasis for this self-map. Set $v_j = \frac{1}{4} \sum_{i=1}^{2p} \langle e_j, e_i \rangle e_i$. Then (using $ze_j = e_{j+1} + e_{j-1}$ and $e_{i+2p} = e_i$), we have

$$zv_j = \frac{1}{4} \sum_{i=1}^{2p} \langle e_j, ze_i \rangle e_i = \frac{1}{4} \sum_{i=1}^{2p} \langle c(e_j), e_i \rangle e_i = \lambda_j v_j.$$

5.9. *Proof of* 3.3(ii). We have $\langle (S^2 \times S^1, e_0') \rangle_p = 1$ since e_0' is represented by the empty link. We can obtain $S^2 \times S^1$ by surgery on the framed unknot in S^3 with framing zero. It follows that

$$\langle (S^2 \times S^1, e_i') \rangle_p = \eta \langle H(\omega_p, e_i) \rangle = \eta^2 \langle \Omega_p, e_i \rangle.$$

But $\langle \Omega_p, e_i \rangle = \langle v_i \rangle$. Now $\lambda_0 \langle v_i \rangle = \delta \langle v_i \rangle = \langle zv_i \rangle = \lambda_i \langle v_i \rangle$, and since $\lambda_0, \ldots, \lambda_{n-1}$ are all distinct and nonzero, this implies $\langle \Omega_p, e_i \rangle = \langle v_i \rangle = 0$, for $i = 1, \ldots, n-1$. If $p \geq 4$ is even, this is exactly what was to be shown. If $p = 2n+1$ is odd, then $e_{n+i} = e_{n-1-i}$ in V_p (see [10]), hence $\langle (S^2 \times S^1, e_{n+i}') \rangle_p = \langle (S^2 \times S^1, e_{n-1-i}') \rangle_p$. The result follows. □

5.10. *Proof of the Verlinde Formula* 1.16. By 5.7, we have $\operatorname{rank}(V_p(\Sigma_g)) = \operatorname{trace}_{V_p}(K^{g-1})$. This is easy to compute in terms of the eigenbasis $v_0, \ldots, v_{n-1}$.

Note that in the module V_p, we may write $K = \frac{1}{4} \sum_{i=0}^{2p-1} e_i^2$. Since $zv_j = \lambda_j v_j$, we have $Kv_j = K(\lambda_j) v_j$, where we think of K as a polynomial in z. Now

$$K(\lambda_{j-1}) = \frac{1}{4} \sum_{i=1}^{2p} (e_{i-1}(\lambda_{j-1}))^2 = \frac{1}{4} \sum_{i=1}^{2p} \left(\frac{A^{2ij} - A^{-2ij}}{A^{2j} - A^{-2j}} \right)^2 = \frac{-p}{(A^{2j} - A^{-2j})^2}.$$

Thus

$$d_g(p) = \mathrm{tr}_{V_p}(K^{g-1}) = \sum_{j=1}^{n} K(\lambda_{j-1})^{g-1} = (-p)^{g-1} \sum_{j=1}^{n} \frac{1}{(A^{2j} - A^{-2j})^{2g-2}}.$$

Since A is a primitive $2p$th root of unity, and $n = [(p-1)/2]$, this is precisely formula 1.16(i).

By the residue theorem (cf. [42, p. 159], [32, p. 140]), one has (for $2g - 2 > 0$)

$$\sum_{j=1}^{n} \frac{1}{(A^{2j} - A^{-2j})^{2g-2}} = -p\,\mathrm{Res}_{t=0} \frac{1}{(2\sinh(t))^{2g-2}} \begin{cases} \dfrac{dt}{e^{pt}-1} & \text{if } p \text{ is even} \\ \dfrac{dt}{e^{2pt}-1} & \text{if } p \text{ is odd.} \end{cases}$$

(Use the form $dz/(z(z^p - 1)(z - z^{-1})^{2g-2})$ and the change of variable $z = e^t$.) This shows 1.16(ii). □

5.11. *Remark.* Let $p = 2r$ and let $\Sigma_g(r-2)$ be Σ_g equipped with a 1-component link colored by $r - 2$. If r is odd, then $V_p(\Sigma_g(r-2)) = 0$. If r is even, then

$$\mathrm{rank}\, V_p(\Sigma_g(r-2)) = \mathrm{trace}_{V_p}(e_{r-2}K^{g-1}) = \left(\frac{r}{2}\right)^{g-1} \sum_{j=1}^{r-1} (-1)^{j-1} \left(\sin\frac{\pi j}{r}\right)^{2-2g}.$$

If $r = k + 2$, this is precisely Thaddeus' formula for the dimension of a certain vector space, denoted by $\hat{Z}_k(\Sigma_g)$ in [32], arising from Thaddeus' "twisted version" of the $SU(2)$ Wess Zumino Witten model at level k.

6. A TENSOR PRODUCT FORMULA FOR ODD p

In this section we assume $p \geq 1$ is odd, and we study the relationship between the V_p and the V_{2p} theories. In particular, we give the proof of Theorem 1.5. We also study the V_p theories for $p = 1$ and $p = 2$.

Convention. For the rest of this section, the coefficient ring will be k_{2p}. Modules over k_2 and k_p will be considered as modules over k_{2p} using the ring homomorphisms i_p and j_p, defined below.

The following is easily verified.

6.1. Lemma. *There are well-defined homomorphisms $i_p\colon k_2 \to k_{2p}$, $j_p\colon k_p \to k_{2p}$, p odd, such that $i_1 = id$, $j_1(\kappa) = 1$ and for $p > 1$, $i_p(A) = A^{p^2}$, $i_p(\kappa) = \kappa^{-p}$, $j_p(A) = A^{1+p^2}$, and $j_p(\kappa) = \kappa^{1+p}$.*

Definition. Define an invariant $\langle M \rangle_2' \in k_2$ as follows. If $M = (M, \alpha, K)$ is a closed 3-manifold with structure, we put

$$\langle M \rangle_2' = \frac{\langle M \rangle_2}{(-2)^{\sharp K}}$$

where $\sharp K$ is the number of components of K. As usual, we extend the definition linearly to the case where K is a linear combination of banded links in M. Note, however, that $\langle\ \rangle_2'$ does not satisfy the Kauffman bracket relations.

6.2. PROPOSITION. *If $M = (M, \alpha, K)$ is a closed 3-manifold with structure, then*

$$i_p(\langle M\rangle_2')j_p(\langle M\rangle_p) = \langle M\rangle_{2p}.$$

Proof. It is shown in [9] that

$$i_p(\theta_2(M,K))j_p(\theta_p(M,K)) = \theta_1(M,K)\theta_{2p}(M,K).$$

The result follows because $\theta_1(M,K) = (-2)^{\sharp K}$, $i_p(\kappa)j_p(\kappa) = \kappa$, and $i_p(\eta)j_p(\eta) = \eta$. (The last equality follows from $i_p(\langle U_\varepsilon(\Omega_2)\rangle)j_p(\langle U_\varepsilon(\Omega_p)\rangle) = \langle U_\varepsilon(\Omega_{2p})\rangle$ (see [9, p. 50]) and formula $(*)$ of Section 2.) □

Remark. If we define $j_3': k_2 \to k_6$ by $j_3'(A) = A^9$ and $j_3'(\kappa) = \kappa$, then $j_3'(\langle\ \rangle_2') = \langle\ \rangle_6$.

6.3. Clearly, the invariant $\langle\ \rangle_2'$ is multiplicative and involutive, and hence, by 1.1, for every surface Σ with structure, we have a k_2-module $V_2'(\Sigma)$ (with $V_2'(\Sigma)\otimes_{k_2} k_6 \approx V_6(\Sigma)$). Moreover, the same holds for surfaces with colored structure, where a color is an element of $\{0,1\}$. The basis elements of $V_6(\Sigma)$ given by admissible colorings of a certain graph G by 0 or 1 (see 4.11) are represented by manifolds with structure, and it is easy to see that they also give a basis of $V_2'(\Sigma)$. If $\Sigma_{g,2n}$ is a surface of genus g equipped with a link having $2n$ components, then

$$\operatorname{rank} V_2'(\Sigma_{g,2n}) = 2^g.$$

Indeed, an admissible coloring of G (with colors in $\{0,1\}$) with value one on $l = \partial G$ can be identified with a cellular 1-chain $\gamma \in C_1(G;\mathbf{Z}/2)$ with $\partial\gamma = \sum_{v\in l} v \in C_0(G;\mathbf{Z}/2)$, and the set of such 1-chains is affinely isomorphic to $H_1(G;\mathbf{Z}/2)$, which has 2^g elements.

The proof of Theorem 1.5 uses the following lemma whose proof is elementary.

6.4. LEMMA. *Let $\mathscr{V}$, W be free modules equipped with hermitian sesquilinear forms $\langle\,,\rangle_{\mathscr{V}}$, $\langle\,,\rangle_W$, and let $f: \mathscr{V} \to W$ be a form-preserving linear map. Let $(V, \langle\,,\rangle_V)$ be the quotient of $\mathscr{V}$ by the radical of $\langle\,,\rangle_{\mathscr{V}}$. Suppose that $\langle\,,\rangle_W$ is nondegenerate.*

Suppose either that f is surjective, or that V and W are free of finite rank and $\langle\,,\rangle_V$ is unimodular and furthermore that $\operatorname{rank}(W) \le \operatorname{rank}(V)$.

Then f induces an isometry $(V, \langle\,,\rangle_V) \xrightarrow{\approx} (W, \langle\,,\rangle_W)$.

Proof of Theorem 1.5. We apply the lemma to $\mathscr{V} = \mathscr{V}_{2p}(\Sigma, l)$, the k_{2p}-module freely generated by the set of manifolds with structure M with $\partial M = (\Sigma, l)$, so that $V = V_{2p}(\Sigma, l)$. We set $W = V_2'(\Sigma, l)\otimes V_p(\Sigma, l)$, and define f by $f(M) = Z_2'(M)\otimes Z_p(M)$. The form on W is nonsingular, and f is form-preserving by 6.2.

In case $p = 1$, we will show that f is surjective, so that the first part of the lemma applies. In case $p \ge 3$, we will show that $\operatorname{rank}(W) = \operatorname{rank}(V)$. Since $\langle\,,\rangle_V$ is unimodular, the second part of the lemma applies, and the theorem follows in that case as well.

The case $p \ge 3$. By 4.11, a basis $\mathscr{B}_{2p}$ (resp. $\mathscr{B}_2'$) of $V_{2p}(\Sigma, l)$ (resp. $V_2'(\Sigma, l)$) is given by the $2p$-admissible colorings with colors $< p - 1$ (resp. with colors in $\{0,1\}$) of a certain banded trivalent graph. The map $\sigma \mapsto \gamma(\sigma)$ (see 4.13) is a surjection $\mathscr{B}_{2p} \to \mathscr{B}_2'$, and by 4.14, each fiber of this map corresponds to a basis of $V_p(\Sigma, l)$. This implies

$$\operatorname{rank}(V_{2p}(\Sigma, l)) = \operatorname{rank}(V_2'(\Sigma, l))\operatorname{rank}(V_p(\Sigma, l))$$

as required.

The case $p = 1$. We need the following lemma.

6.5. LEMMA. *Let T be the Temperley–Lieb algebroid with coefficients in k_1. Then for all i, $j \geq 0$, the hermitian sesquilinear form $\langle\, , \rangle$ on ${}_iT_j$, defined in Section 3, is nondegenerate.*

Proof. It suffices to prove the lemma after extending coefficients to $k_1 \otimes \mathbf{Q}$. Then the idempotents f_i exist for all $i \geq 0$, and T is generated by the f_i as a 2-sided ideal (see the proof of 3.4). Hence, T is Morita equivalent to the completely reduced algebroid $\tilde{T}$ given by

$$_i\tilde{T}_j = f_i \, {}_iT_j f_j \approx \begin{cases} k_1 \otimes \mathbf{Q} & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

Now let ${}_iT_j'$ denote the quotient of ${}_iT_j$ by the radical of the sesquilinear form $\langle\, , \rangle$. Since the family of radicals is a 2-sided ideal I of T, the modules ${}_iT_j'$ give rise to an algebroid $T' = T/I$. Again, T' is generated by the f_i, and since $\langle f_i, f_i \rangle \neq 0$, T' is also Morita equivalent to $\tilde{T}$. More precisely, the quotient map $T \to T'$ is a Morita equivalence, that is, an equivalence of categories $\{T - \mathit{modules}\} \tilde{\to} \{T' - \mathit{modules}\}$. But this equivalence sends the map $0 \to I$ to an isomorphism. Hence $I = 0$ and $T = T'$, as required. □

6.6. PROPOSITION. *If (Σ, l) is a closed surface with structure, and the link $l \subset \Sigma$ has m components, then $V_1(\Sigma, l)$ is isomorphic to the Temperley-Lieb module ${}_0T_m$ (with coefficients in k_1). (Of course, this is zero, if m is odd.)*

Proof. Let M be a connected 3-manifold with structure with boundary (Σ, l). Let $D \subset \Sigma$ be a (collection of) disk(s) containing l, and let $B \subset M$ be a 3-ball such that $B \cap \Sigma = D$. Then we have an obvious map $\Phi: {}_0T_m \approx K(B, l) \to K(M, l) \to V_1(\Sigma, l)$. This map is surjective, since the invariant $\langle\ \rangle_1$ on closed 3-manifolds with structure depends (up to an invertible scalar) only on the number of components of the banded link. Also, Φ transforms the form $\langle\, , \rangle$ on ${}_0T_m$, into the nondegenerate form $\langle\, , \rangle_1$ on $V_1(\Sigma, l)$. Since $\langle\, , \rangle$ is nondegenerate, by Lemma 6.5, Φ is an isomorphism. □

Remark. The proof also shows that if Γ is a closed 1-manifold with structure, then the algebroid $\Delta_1(\Gamma)$ is Morita equivalent to the Temperley–Lieb algebroid T (with coefficients in k_1).

Proof of 1.5 *in the case* $p = 1$. We must show that the map $f: \mathcal{V}_2(\Sigma) \to V_2'(\Sigma) \otimes V_1(\Sigma)$ is surjective. We distinguish 3 cases.

Suppose Σ is a surface of genus g with empty link. Then $V_1(\Sigma)$ has rank one, and we can choose a generating set of structured manifolds M_i for $V_2'(\Sigma)$ such that $Z_1(M_i)$ is invertible. (For example, the basis described in 6.3 will do.) Then the M_i also generate $V_2'(\Sigma) \otimes V_1(\Sigma)$.

Similarly, if (Σ, l) is a surface of genus 0 with link l (with an even number, say $2n$, of components), then $V_2'(\Sigma, l)$ has rank one, and we can choose manifolds M_j (with $Z_2'(M_j)$ invertible) which generate $V_1(\Sigma, l)$ and hence $V_2'(\Sigma, l) \otimes V_1(\Sigma, l)$. (For example, the basis elements of $V_1(\Sigma, l)$ corresponding, as in 6.6, to the standard basis of the Temperley–Lieb module ${}_0T_{2n}$, will do. To see that Z_2' of such an element is invertible, one may take its double, which is S^3 with a bunch of unlinked and unknotted circular bands, and the $\langle\ \rangle_2'$-invariant of that double is invertible.)

Finally, if (Σ, l) is a surface of genus g with link, then (Σ, l) is the connected sum of Σ_1 and (Σ_2, l), where Σ_1 has genus g and no link, and Σ_2 has genus 0. Since $V_2'(\Sigma, l) = V_2'(\Sigma_1) \otimes V_2'(\Sigma_2, l)$ and $V_1(\Sigma, l) = V_1(\Sigma_1) \otimes V_1(\Sigma_2, l)$, which is easily established, we see that the manifolds $M_{i,j}$, the boundary connected sum of M_i with M_j, are generators for $V_2'(\Sigma, l) \otimes V_1(\Sigma, l)$.

6.7. *The case* $p = 2$. The V_2-modules can be computed from the formula $V_2 = V_2' \otimes V_1$ using the calculation of V_2' (see 6.3) and of V_1 (see 6.6). One can show that if Γ is a 1-manifold with structure with m components, then the algebroid $\Delta_2(\Gamma)$ is Morita equivalent to the disjoint union of 2^m algebroids $T(i)$, indexed by the colorings $i \in \{0, 1\}^m$ of Γ, and $T(i)$ is isomorphic to the even or odd part of the Temperley–Lieb algebroid T (with coefficients in k_2) according to the parity of $i_1 + \cdots + i_m$.

7. A NATURAL DECOMPOSITION OF THE MODULES $V_{2p}(\Sigma)$

In this section, we prove Theorem 1.6 and we calculate the ranks of $V_{8k-4}(\Sigma, h)$ and $V_{8k}(\Sigma, q)$.

7.1. *Definition.* Let Σ be an oriented surface. Denote the (antisymmetric) intersection form on $H_1(\Sigma; \mathbf{Z})$ by $(x, y) \mapsto x \cdot y$. The *Heisenberg group* $H(\Sigma)$ is defined as follows. The underlying set is $\mathbf{Z} \times H_1(\Sigma; \mathbf{Z})$, with multiplication given by $(n, x)(m, y) = (n + m + x \cdot y, x + y)$. We will denote the element $(1, 0)$ by u, and for $x \in H_1(\Sigma; \mathbf{Z})$, we write $[x] = (0, x)$. Thus u is central, and $[x][y] = u^{x \cdot y}[x + y]$.

Let $\Gamma(\Sigma)$ denote the quotient of $H(\Sigma)$ by the subgroup generated by u^4 and the elements $[2x] = [x]^2$, where $x \in H_1(\Sigma; \mathbf{Z})$.† The following is easily verified.

7.2. Proposition. *There is a commutative diagram of short exact sequences*

$$\begin{array}{ccccccccc} 0 & \to & \mathbf{Z} & \to & H(\Sigma) & \to & H_1(\Sigma; \mathbf{Z}) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \mathbf{Z}/4 & \to & \Gamma(\Sigma) & \to & H_1(\Sigma; \mathbf{Z}/2) & \to & 0. \end{array}$$

We will see that the group $\Gamma(\Sigma)$ acts on $V_{2p}(\Sigma)$. For this, we need a description of $\Gamma(\Sigma)$ in terms of banded links.

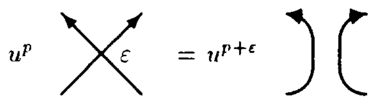
7.3. *Definition.* Let $\mathscr{L}^+(\Sigma)$ be the set of framed links (i.e. banded links with oriented cores) in $\Sigma \times \mathbf{I}$. Elements of $\mathscr{L}^+(\Sigma)$ are represented by oriented link diagrams on Σ. Putting one diagram above the other gives $\mathscr{L}^+(\Sigma)$ the structure of a monoid, with the empty link as identity element. Let $\mathscr{E}^+(\Sigma)$ be the quotient of $\mathbf{Z} \times \mathscr{L}^+(\Sigma)$ by isotopy and the following skein relations. We write $(n, E) \in \mathbf{Z} \times \mathscr{L}^+(\Sigma)$ as $u^n E$, and employ the equality as shown in Fig. 6. Then the relations are as shown in Figs 7 and 8. Here, the links are supposed to be identical except where depicted, and in Fig. 8, the orientations are arbitrary.

Note that $\mathscr{E}^+(\Sigma)$ is again a monoid. (We will soon see that it is, in fact, a group isomorphic to the Heisenberg group $H(\Sigma)$.)

Let $\mathscr{E}(\Sigma)$ be the quotient of $\mathscr{E}^+(\Sigma)$ by the relations $u^4 = 1$ and $u^n E = u^n E'$, if E and E' are framed links with the same underlying banded link (i.e. E and E' are the same up to changing some of the orientations of the cores of the bands). $\mathscr{E}(\Sigma)$ is again a monoid (which will turn out to be isomorphic to $\Gamma(\Sigma)$).

† If Σ has strictly positive genus, then u^4 is already contained in the subgroup generated by the elements $[2x]$.

Fig. 6.



(for $\varepsilon = \pm 1$)

Fig. 7.

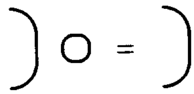


Fig. 8.

7.4. Proposition. *Define $\Phi: \mathbf{Z} \times \mathscr{L}^+(\Sigma) \to H(\Sigma)$ by setting $\Phi(u^n E) = u^{n+N(E)}[e]$, where $N(E)$ is the algebraic number of crossings of E, and e is the class of E in $H_1(\Sigma; \mathbf{Z})$. Then Φ induces a commutative diagram of morphisms of monoids*

$$\begin{array}{ccc} \mathscr{E}^+(\Sigma) & \overset{\approx}{\to} & H(\Sigma) \\ \downarrow & & \downarrow \\ \mathscr{E}(\Sigma) & \overset{\approx}{\to} & \Gamma(\Sigma) \end{array}$$

Proof. It is easy to see that Φ is well defined and surjective. Next, assume $\Phi(u^n E) = \Phi(u^{n'} E')$. Applying the first relation, we may assume E, E' are embedded in Σ, and (hence) $n = n'$. Since E, E' are embedded, and represent the same class in $H_1(\Sigma; \mathbf{Z})$, they are cobordant in $\Sigma \times \mathbf{I}$, and we can go from one to the other by a sequence of surgeries. Thus, one verifies that E, E' represent the same object in $\mathscr{E}^+(\Sigma)$. Hence, Φ induces an isomorphism $\mathscr{E}^+(\Sigma) \overset{\approx}{\to} H(\Sigma)$. We leave it to the reader to show that Φ also induces an isomorphism $\mathscr{E}(\Sigma) \overset{\approx}{\to} \Gamma(\Sigma)$.

Consider a surface with structure $\Sigma = (\Sigma, l)$, or surface with colored structure $\Sigma = (\Sigma, l, i)$. Assume $p \geq 2$. Here is how $\Gamma(\Sigma - l)$ acts on $V_{2p}(\Sigma)$. Set $b = (-1)^p e_{p-2}$. If E is a banded link in $(\Sigma - l) \times \mathbf{I}$, set

$$M(E) = (\Sigma \times \mathbf{I}, l \times \mathbf{I} \cup E(b))$$

where $E(b)$ is E with all components cabled by b. This is a linear combination of cobordisms with structure from Σ to itself, and it induces an endomorphism $\varphi(E) = Z_{M(E)}$ of $V_{2p}(\Sigma)$. □

7.5. Proposition. *Assume $p \geq 2$. Then φ induces an isometric action of $\Gamma(\Sigma - l)$ on $V_{2p}(\Sigma)$, with the central element u acting as $\varphi(u) = (-1)^{p+1} A^{p^2}$*

Proof. We first show that φ induces an action of $\mathscr{E}^+(\Sigma - l) \approx H(\Sigma - l)$. If E is a framed link, we define $\varphi(E)$ as above, forgetting the orientation of the core of E. It is clear that this is an isotopy invariant. Since the V_{2p}-module of a 2-sphere with four points colored by $p - 2$ has rank one, the equality shown in Fig. 9 holds.

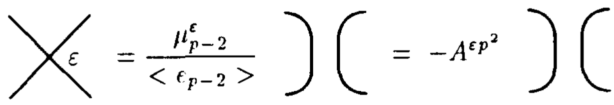


Fig. 9.

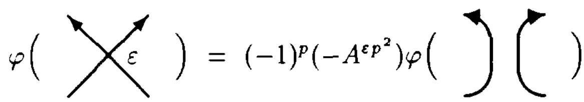


Fig. 10.

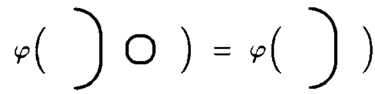


Fig. 11.

In the figure all bands are colored by $p - 2$. (The coefficient is the ratio of the brackets of the closures of the two links.) Now if E has m components, then $E(b) = (-1)^{pm}E(e_{p-2})$. Hence, the above implies the equality shown in Fig. 10 since this move changes the number of components by ± 1. Also, we have the equality given in Fig. 11 because $\langle b \rangle = (-1)^p \langle e_{p-2} \rangle = 1$. Thus, we have verified relations (i) and (ii) of 7.3, hence φ induces an action of $\mathscr{E}^+(\Sigma - l) \approx H(\Sigma - l)$. It is clear that this is a group action. If $[x] \in H(\Sigma - l)$ is represented by an embedded link E in Σ, then the double of the cobordism $M(E)$ acts as the identity on $V_{2p}(\Sigma)$, because $e_{p-2}^2 = e_0 = 1$ in $V_{2p}(S^1 \times S^1)$. This implies $\varphi([2x]) = \varphi([x]^2) = id$, hence the action factors through $\Gamma(\Sigma - l)$ as asserted. Finally, it is clear that $Z_{M(E)} = Z_{-M(E)}$, hence $\varphi([x]) = \varphi([x])^*$. Since $\varphi([x])$ has order two, this implies that $\varphi([x])$ is an isometry. This completes the proof. □

7.6. *Remark.* (i) This action may also be described as follows. Assume $x \in H_1(\Sigma - l; \mathbf{Z})$ is represented by an oriented simple closed curve γ on $\Sigma - l$. Choose a handlebody H with boundary Σ such that γ bounds a disk $D \subset H$, and construct a basis (u_σ) of $V_{2p}(\Sigma)$, as in 4.11, by admissible colorings σ of a banded graph G. We may assume that G meets D transversally in an edge e. Then

$$\varphi([x])(u_\sigma) = (-1)^{\sigma(e)} u_\sigma.$$

Indeed, $\varphi([x])(u_\sigma)$ is represented by a colored graph G', which is the union of G, colored by σ as before, and a meridinal circle, colored by $p - 2$, up to sign, around the edge e. Since the expansion of the edge e is $f_{\sigma(e)}$, which is an augmentation idempotent, $\varphi([x])(u_\sigma)$ is simply u_σ, multiplied by a coefficient C depending only on the color $\sigma(e)$. We may compute C in the special case where G itself is a circle. Thus,

$$C = (-1)^p \frac{\langle e_{\sigma(e)}, e_{p-2} \rangle}{\langle e_{\sigma(e)} \rangle} = (-1)^p e_{\sigma(e)}(\lambda_{p-2}) = (-1)^{\sigma(e)}.$$

(Here, $\langle \, , \rangle$ is the bilinear form defined in 5.8. See the computation in 5.10 and recall that $A^{2p} = -1$.)

(ii) The curve γ also induces a Dehn twist t_γ in the extended mapping class group $\tilde{\mathscr{M}}(\Sigma)$ (see Appendix B), and using the equation $te_i = \mu_i e_i$ (see 5.8), one finds that

$$t_\gamma(u_\sigma) = \mu_{\sigma(e)} u_\sigma.$$

But $\mu_i^{2p} = (-1)^i$ (since $A^{2p} = -1$). Hence, in $End(V_{2p}(\Sigma))$, we have

$$\varphi([x]) = t_\gamma^{2p}.$$

Proof of Theorem 1.6. We begin with a study of the action φ of $\Gamma(\Sigma - l)$ on $V_{2p}(\Sigma)$. Note that $\varphi(u)$ may have order four, two, or one, according to the value of p.

The case $p \equiv 1 \bmod 2$. Then $\varphi(u)$ has order four. By Theorem 1.5, we have a natural isomorphism $V_{2p}(\Sigma) \stackrel{\approx}{\to} V_2'(\Sigma) \otimes V_p(\Sigma)$. If E is a framed link in $(\Sigma - l) \times \mathbf{I}$, we set $\varphi'(E) = Z_{M'(E)}$, where $M'(E) = (\Sigma \times \mathbf{I}, l \times \mathbf{I} \cup E(-z))$. One verifies that this induces an action φ' of $\Gamma(\Sigma - l)$ on $V_2'(\Sigma)$.

7.7. Proposition. *Let $\Sigma = (\Sigma, l)$ be a surface with structure. If $p \geq 3$ is odd, then the action φ of $\Gamma(\Sigma - l)$ on $V_{2p}(\Sigma) \approx V_2'(\Sigma) \otimes V_p(\Sigma)$ is of the form $\varphi' \otimes id$.*

Proof. Let E be a banded link in $(\Sigma - l) \times \mathbf{I}$. If we assume, for simplicity, that E is connected, then the complement of a tubular neighborhood of E is a cobordism M from $S^1 \times S^1$ to $-\Sigma \cup \Sigma$. By naturality, we have a commutative diagram

$$\begin{array}{ccc} V_{2p}(S^1 \times S^1) & \stackrel{\approx}{\to} & V_2'(S^1 \times S^1) \otimes V_p(S^1 \times S^1) \\ \downarrow Z_M & & \downarrow Z_M \otimes Z_M \\ V_{2p}(-\Sigma \cup \Sigma) & \stackrel{\approx}{\to} & V_2'(-\Sigma \cup \Sigma) \otimes V_p(-\Sigma \cup \Sigma). \end{array}$$

The isomorphism in the top row sends e_{p-2} to $z \otimes 1$, as is easily established using the facts that $z^2 = 1$ in V_2' and that $e_{p-2} = e_0 = 1$ in V_p, if p is odd. Since $\varphi(E) = (-1)^p Z_M(e_{p-2}) = -Z_M(e_{p-2})$ in $End(V_{2p}(\Sigma)) = V_{2p}(-\Sigma \cup \Sigma)$, and $\varphi'(E) = -Z_M(z)$, we have $\varphi(E) = \varphi'(E) \otimes id$. The result follows. □

7.8. *Remark.* (i) In the preceding discussion, the case $p = 1$ was excluded for simplicity only. Obviously, we may define an action of $\Gamma(\Sigma - l)$ on $V_2(\Sigma) \approx V_2'(\Sigma) \otimes V_1(\Sigma)$ by setting $\varphi = \varphi' \otimes id$.

(ii) One can show that the action of $\Gamma(\Sigma - l)$ on $V_2'(\Sigma)$ is irreducible.

The case $p \equiv 2 \bmod 4$. Then $\varphi(u) = 1$, and the action of $\Gamma(\Sigma - l)$ factors through an action τ of $H_1(\Sigma - l; \mathbf{Z}/2)$. The characters of this group are linear forms $h: H_1(\Sigma - l; \mathbf{Z}/2) \to \mathbf{Z}/2$, with associated isotypic component given by

$$V_{2p}(\Sigma, h) = \{v \in V_{2p}(\Sigma): \tau_a(v) = (-1)^{h(a)} v \text{ for all } a \in H_1(\Sigma - l; \mathbf{Z}/2)\}.$$

Setting $p = 4k - 2$, we thus have the following theorem.

7.9. Theorem. *Let Σ be a closed surface with (colored) structure. Let $k \geq 1$. Then there is a canonical decomposition*

$$V_{8k-4}(\Sigma) = \bigoplus_h V_{8k-4}(\Sigma, h)$$

where the sum is over all cohomology classes $h \in H^1(\Sigma - l; \mathbf{Z}/2)$ (viewed as linear forms on $H_1(\Sigma - l; \mathbf{Z}/2)$).

The case $p \equiv 0 \bmod 4$. Then $\varphi(u) = -1$, and the action of $\Gamma(\Sigma - l)$ factors through an action τ of $\Gamma'(\Sigma - l) = \Gamma(\Sigma - l)/u^2$. Note that this group is the Heisenberg group associated to $H_1(\Sigma - l; \mathbf{Z}/2)$ equipped with the mod 2 intersection form. In particular, for

$a \in H_1(\Sigma - l; \mathbf{Z}/2)$, there is a well defined $[a] \in \Gamma'(\Sigma - l)$, and we have $\tau_{[a+b]} = (-1)^{a \cdot b}\tau_{[a]}\tau_{[b]}$. Hence, the relevant characters of $\Gamma'(\Sigma - l)$ are given by functions $q: H_1(\Sigma - l; \mathbf{Z}/2) \to \mathbf{Z}/2$ such that $q(a + b) = q(a) + q(b) + a \cdot b$, with associated isotypic components given by

$$V_{2p}(\Sigma, q) = \{v \in V_{2p}(\Sigma) \colon \tau_{[a]}(v) = (-1)^{q(a)}v \text{ for all } a \in H_1(\Sigma - l; \mathbf{Z}/2)\}.$$

Setting $p = 4k$, we thus have the following theorem.

7.10. THEOREM. *Let Σ be a closed surface with (colored) structure. Let $k \geq 1$. Then there is a canonical decomposition*

$$V_{8k}(\Sigma) = \bigoplus_q V_{8k}(\Sigma, q)$$

where the sum is over all $\mathbf{Z}/2$-valued quadratic forms q on $H_1(\Sigma - l; \mathbf{Z}/2)$ inducing the intersection form (i.e. such that $a \cdot b = q(a + b) - q(a) - q(b)$).

7.11. *Remark.* Let $a \in H_1(\Sigma - l; \mathbf{Z})$ be represented by an oriented simple closed curve γ around a component of the link $l \subset \Sigma$. Assume this component is colored by i (with $i = 1$ if (Σ, l) is a surface with structure). Then $[a]$ acts by $(-1)^i$ (cf. 7.6), hence $V_{8k}(\Sigma, q)$ (resp. $V_{8k-4}(\Sigma, h)$) is zero except if $q(a) \equiv i \bmod 2$ (resp. $h(a) \equiv i \bmod 2$).

7.12. *Comment.* There is a canonical bijection between the set $Spin(\Sigma - l)$ of spin structures on $\Sigma - l$ and the set of $\mathbf{Z}/2$-valued quadratic forms q on $H_1(\Sigma - l; \mathbf{Z}/2)$ inducing the intersection form (see [13]). Hence, Theorem 7.10 can be viewed as a canonical decomposition of $V_{8k}(\Sigma)$ into submodules associated to spin structures.

7.13. *Computation of the ranks of $V_{8k-4}(\Sigma, h)$ and $V_{8k}(\Sigma, q)$.* We now compute the dimensions of these submodules in the case where $\Sigma = \Sigma_g$, i.e. a closed surface of genus $g \geq 1$ equipped with the empty link.

Recall that $\operatorname{rank}(V_{2p}(\Sigma_g)) = d_g(2p) = \operatorname{trace}_{V_{2p}}(K^{g-1})$, where $K = \sum_{j=0}^{p-2} e_j^2 \in V_{2p}$. (Here, as in Section 5, V_{2p} stands for $V_{2p}(S^1 \times S^1)$.) If we identify Σ_g with the boundary of a regular neighborhood of the graph G in Fig. 12, a basis is given by all $2p$-admissible colorings $(l_0, l_1, \ldots, l_{g-2}, j_1, \ldots, j_{g-1}, j'_1, \ldots, j'_{g-1})$ of G (with colors $< p - 1$).

For $\varepsilon \in \{0, 1\}$, let $\delta_g^{(\varepsilon)}(2p)$ be the number of $2p$-admissible colorings (with colors $< p - 1$) of G, for which the colors j_i are even, and the color l_0 has parity ε. These colorings give a basis of a submodule $V_{2p}^{(\varepsilon)}(\Sigma_g)$ of $V_{2p}(\Sigma_g)$. If $g = 1$, we abbreviate $V_{2p}^{(\varepsilon)}(\Sigma_1)$ by $V_{2p}^{(\varepsilon)}$. This is the submodule of V_{2p} generated by the e_j, where j has parity ε, or equivalently, the submodule

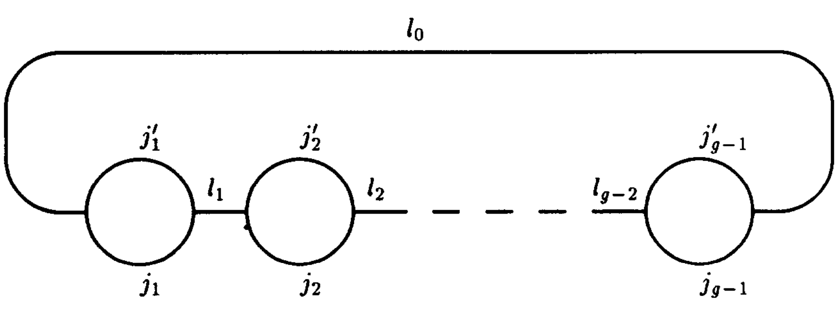


Fig. 12.

generated by polynomials in z whose degree in z has parity ε. Set

$$K_0 = \sum_{\substack{j=0 \\ j \text{ even}}}^{p-2} e_j^2 .$$

Note that multiplication by K_0 preserves both $V_{2p}^{(0)}$ and $V_{2p}^{(1)}$.

7.14. LEMMA. *For $\varepsilon \in \{0, 1\}$, we have $\delta_g^{(\varepsilon)}(2p) = \operatorname{trace}_{V_{2p}^{(\varepsilon)}}(K_0^{g-1})$. Moreover, if $p = 2s$ is even, then*

$$d_g(2p) = 2^g \delta_g^{(1)}(2p) + s^{g-1} = 2^g \delta_g^{(0)}(2p) + (1 - 2^g)s^{g-1}.$$

Proof. The first statement can be proven by an elementary counting argument, as in [1, formula (5.8)] (cf. 5.8).

Next, assume $p = 2s$. Recall that the module V_{2p} has a basis $v_0, \dots, v_{p-2}$ with $zv_j = \lambda_j v_j$ (see 5.8). For $\varepsilon \in \{0, 1\}$ we set $v_j^{(\varepsilon)} = v_j + (-1)^\varepsilon v_{p-2-j}$. One verifies that $v_0^{(0)}, \dots, v_{s-1}^{(0)}$ form a basis of $V_{2p}^{(0)}$, and $v_0^{(1)}, \dots, v_{s-2}^{(1)}$ form a basis of $V_{2p}^{(1)}$. Since K_0 is a polynomial in z^2, and $z^2 v_j^{(\varepsilon)} = \lambda_j^2 v_j^{(\varepsilon)}$ and $\lambda_{p-2-j} = -\lambda_j$, we have $K_0 v_j^{(\varepsilon)} = K_0(\lambda_j) v_j^{(\varepsilon)}$, whence

$$\delta_g^{(0)}(2p) = \sum_{j=1}^{s} K_0(\lambda_{j-1})^{g-1} = \delta_g^{(1)}(2p) + K_0(\lambda_{s-1})^{g-1}.$$

Proceeding as in 5.10, we find that

$$K_0(\lambda_{j-1}) = \sum_{i=0}^{s-1} (e_{2i}(\lambda_{j-1}))^2 = \sum_{i=0}^{s-1} \left(\frac{A^{2j(2i+1)} - A^{-2j(2i+1)}}{A^{2j} - A^{-2j}} \right)^2 = \begin{cases} \dfrac{-p}{(A^{2j} - A^{-2j})^2} & \text{if } j < s \\ s & \text{if } j = s. \end{cases}$$

It follows that $K_0(\lambda_{j-1}) = \frac{1}{2} K(\lambda_{j-1})$, for $1 \le j \le s - 1$, and $K_0(\lambda_{s-1}) = s = K(\lambda_{s-1})$. Since

$$d_g(2p) = \sum_{j=1}^{2s-1} K(\lambda_{j-1})^{g-1} = K(\lambda_{s-1})^{g-1} + 2 \sum_{j=1}^{s-1} K(\lambda_{j-1})^{g-1}$$

the result follows by an easy computation. □

7.15. *Notation.* It is well known [13] that two quadratic forms with the same Arf invariant are in the same orbit under the action of the diffeomorphism group of Σ_g. Hence, the rank of $V_{8k}(\Sigma_g, q)$ depends on q only through its Arf invariant. We set $d_g^{(\varepsilon)}(8k) = \operatorname{rank}(V_{8k}(\Sigma_g, q_\varepsilon))$, where q_ε has Arf invariant $\varepsilon \in \{0, 1\}$.

Similarly, any two nonzero mod 2 cohomology classes are in the same orbit under the action of the diffeomorphism group of Σ_g. Hence, the rank of $V_{8k-4}(\Sigma_g, h)$ is the same for all nonzero cohomology classes $h \in H^1(\Sigma_g; \mathbf{Z}/2)$. We denote this rank by $d_g^{(1)}(8k - 4)$, and let $d_g^{(0)}(8k - 4)$ be the rank of $V_{8k-4}(\Sigma_g, 0)$.

7.16. THEOREM. *For $k \ge 1$ and $\varepsilon \in \{0, 1\}$, one has*

$$d_g^{(\varepsilon)}(8k) = 2^{-2g}(d_g(8k) + (2k)^{g-1}((-1)^\varepsilon 2^g - 1)).$$

Proof. Choose simple closed curves $\alpha_0, a_1, \dots, \alpha_{g-1}$ on Σ_g such that α_0 is a meridian around the arc of the graph G colored by l_0, and, for $i \ge 1$, α_i is a meridian around the arc colored by j_i. Let $a_i \in H_1(\Sigma_g; \mathbf{Z})$ be the class of α_i. It follows from 7.6(i) that $V_{8k}^{(0)}(\Sigma_g)$ is precisely the submodule of $V_{8k}(\Sigma_g)$ fixed by $[a_0], \dots, [a_{g-1}]$. But this is the orthogonal sum of 2^g submodules of the form $V_{8k}(\Sigma_g, q)$, where the quadratic forms q all have Arf invariant zero. Hence,

$$d_g^{(0)}(8k) = \frac{1}{2^g}\text{rank } V_{8k}^{(0)}(\Sigma_g) = \frac{1}{2^g}\delta_g^{(0)}(8k).$$

Thus, the result for Arf invariant zero follows from Lemma 7.14. Now there are $2^{g-1}(2^g+1)$ quadratic forms with Arf invariant 0, and $2^{g-1}(2^g-1)$ quadratic forms with Arf invariant 1, whence

$$d_g(8k) = 2^{g-1}(2^g+1)d_g^{(0)}(8k) + 2^{g-1}(2^g-1)d_g^{(1)}(8k).$$

This implies the formula for Arf invariant one.

Example. For $k = 1$, one has $d_g^{(0)}(8) = 1$ and $d_g^{(1)}(8) = 0$, and

$$\text{rank } V_8(\Sigma_g) = d_g(8) = 2^{g-1}(2^g+1)$$

is the number of spin structures on Σ_g with Arf invariant zero.

7.17. THEOREM. *For $k \geq 1$, one has*

$$d_g^{(1)}(8k-4) = 2^{-2g}(d_g(8k-4) - (2k-1)^{g-1})$$
$$d_g^{(0)}(8k-4) = d_g^{(1)}(8k-4) + (2k-1)^{g-1}.$$

Proof. One proceeds as above to show that $V_{2p}^{(1)}(\Sigma_g)$ is the orthogonal sum of 2^g submodules of the form $V_{8k}(\Sigma_g, h)$, where the linear forms h are all nonzero. Hence, $d_g^{(1)}(8k-4) = (1/2^g)\delta_g^{(1)}(8k-4)$ and the result follows from Lemma 7.14 by a computation. □

APPENDIX A: ALGEBROIDS AND MORITA EQUIVALENCE

Definition (see Mitchell [27]). Let k be a commutative ring. A *k-algebroid* (or a k-category or a k-linear category) is a category Λ (which is supposed to be small, or, at least, to have a small skeleton) such that each morphism set is endowed with the structure of a k-module in such a way that the composition law is k-bilinear.

Remark. A k-algebroid with only one object is a k-algebra. For this reason, the morphisms of a k-algebroid Λ will be called elements of Λ.

Definition. Let Λ be a k-algebroid. A *left Λ-module* is a functor from Λ to the category of k-modules, such that the induced maps between the morphism sets are k-linear. A *right Λ-module* is a left Λ^{op}-module. If Λ' is another k-algebroid, a *$\Lambda \times \Lambda'$-bimodule* is a functor from the category $\Lambda \times (\Lambda')^{\text{op}}$ to the category of k-modules.

Notation. For convenience, for objects a and b in a k-algebroid Λ, we will denote by ${}_a\Lambda_b$ the k-module $\text{Hom}_\Lambda(a, b)$, and the composite $\beta \circ \alpha$ of two morphisms $\alpha \in {}_a\Lambda_b$ and $\beta \in {}_b\Lambda_c$ will be written $\alpha\beta$. Then the composition law is a map from ${}_a\Lambda_b \otimes_k {}_b\Lambda_c$ to ${}_a\Lambda_c$.

Similarly, if M is a left or right Λ-module and a is an object in Λ, the k-module $M(a)$ will be denoted by ${}_aM$ or M_a. If M is a $\Lambda \times \Lambda'$-bimodule, a is an object in Λ and b an object in Λ', the k-module $M(a, b)$ will be written ${}_aM_b$. If α is a morphism in ${}_a\Lambda_b$ and u is an element in ${}_bM$, the image of u by the map $M(\alpha)$ will be written αu. Similar notation will be used for right modules and bi-modules.

Remark. If Λ is a k-algebroid, the category of left or right Λ-modules is an abelian category. It has almost all the properties of the category of modules over a ring or an algebra. The same holds for bimodules.

Definition. Let Λ be a k-algebroid. Let M be a right Λ-module and let N be a left Λ-module. The *tensor product of* M with N is the k-module, denoted by $M \otimes_\Lambda N$, which is the quotient of the k-module

$$\bigoplus_a M_a \otimes_k {}_aN$$

where the sum runs over all objects in Λ (or, if Λ is not small, in a small skeleton of Λ), by the submodule generated by the relations

$$u\alpha \otimes v \equiv u \otimes \alpha v$$

where a and b are objects of Λ, $u \in M_a$, $v \in {}_bN$, and $\alpha \in {}_a\Lambda_b$.

Remark. If M and N are bimodules over k-algebroids, the left action on M and the right action on N induce a bimodule structure on $M \otimes_\Lambda N$. If Λ is a k-algebroid, tensorization on the right (or on the left), by Λ over Λ, is naturally equivalent to the identity functor.

Definition. Two k-algebroids Λ and Δ are said to be *Morita equivalent*, if there is a functor F, from the category of left Λ-modules to the category of left Δ-modules, which is a k-linear equivalence of categories.

If Λ is a k-algebroid, and $\{u_i\}$ are elements in Λ, we can define the two-sided ideal generated by these elements, i.e. the sub-bimodule of Λ generated by $\{u_i\}$.

The following result is a key technical ingredient of this paper. Let Λ be a k-algebroid. Let $\{a_i\}_{i\in I}$ be a family of objects in Λ, and, for each $i \in I$, let ε_i be an idempotent in the algebra ${}_{a_i}\Lambda_{a_i}$. Denote by Δ the following k-algebroid.

The objects of Δ are the elements of the index set I, and the morphisms are defined by

$${}_i\Delta_j = \varepsilon_i \, {}_{a_i}\Lambda_{a_j} \varepsilon_j.$$

Let $i \in I$, and let a be an object in Λ. Let E be the $\Delta \times \Lambda$-bimodule defined by ${}_iE_a = \varepsilon_i \, {}_{a_i}\Lambda_a$ and let E' be the $\Lambda \times \Delta$-bimodule defined by ${}_aE'_i = {}_a\Lambda_{a_i}\varepsilon_i$.

THEOREM. *Suppose that the idempotents ε_i generate Λ as a two-sided ideal. Then the bimodule $E \otimes_\Lambda E'$ is isomorphic to Δ and $E' \otimes_\Delta E$ is isomorphic to Λ.*

Consequently, tensoring (on the left or right), by the modules E and E', yields inverse Morita equivalences of the algebroids Δ and Λ. Moreover, these equivalences are compatible with tensor product.

Proof. The correspondence $\varepsilon_i u \otimes v\varepsilon_j \mapsto \varepsilon_i uv\varepsilon_j$ defines an isomorphism $E \otimes_\Lambda E' \xrightarrow{\approx} \Delta$.

The correspondence $\alpha\varepsilon_i \otimes \varepsilon_i\beta \mapsto \alpha\varepsilon_i\beta$ gives rise to a morphism $\varphi: E' \otimes_\Delta E \to \Lambda$. By assumption, for all objects a of Λ there exist finitely many elements $\alpha_i \in {}_a\Lambda_{a_i}$ and elements $\beta_i \in {}_{a_i}\Lambda_a$, such that

$$1_a = \sum_i \alpha_i\varepsilon_i\beta_i.$$

Let a and b be objects in Λ. Let f be the morphism from ${}_a\Lambda_b$ to ${}_aE' \otimes_\Delta E_b$ defined by $f(\alpha) = \sum_i \alpha_i\varepsilon_i \otimes \varepsilon_i\beta_i\alpha$. One checks that f and φ are mutually inverse. Hence φ is an isomorphism.

Definition. If Δ is a k-algebroid, and M a Δ-bimodule, the Hochschild homology module $H_0(\Delta, M)$ is the quotient of the module $\oplus_a {}_aM_a$ by the relations

$$uv \equiv vu$$

for all $u \in {}_a\Delta_b$ and all $v \in {}_bM_a$. A Morita equivalence induces an isomorphism between Hochschild homology modules.

The module $H_0(\Delta, \Delta)$ is simply denoted by $H_0(\Delta)$.

APPENDIX B: p_1-STRUCTURES

For the relevant classical algebraic topology, see for instance [31].

A p_1-structure up to homotopy is the analogue of a spin structure, where the second Stiefel–Whitney class w_2 is replaced by the first Pontryagin class p_1. For 3-manifolds, it is equivalent to Atiyah's "2-framings" [4].

Definition. Let X be the homotopy fiber of the map $p_1 : BO \to K(\mathbf{Z}, 4)$ corresponding to the first Pontryagin class of the universal stable bundle γ over BO. Let γ_X be the *pull-back* of γ over X. A *p_1-structure* on a manifold M is a fiber map from the stable tangent bundle of M, τ_M, to γ_X.

Remark. (i) Such a structure induces a lifting of a classifying map of τ_M. Note that we did not say "homotopy class of map". This allows manifolds with p_1-structure to be glued along parts of their boundary.

(ii) Notice that as we have just defined it, a p_1-structure on M does not include an orientation of M. Hence, a p_1-structure on an oriented manifold M canonically induces one on $-M$ (the same manifold with opposite orientation).

(iii) There is an obvious notion of *p_1-surgery*, that is, we demand that the trace of the surgery has a p_1-structure. Notice, however, that if M_2 is obtained from M_1 by (ordinary) surgery of index one or two, then every p_1-structure on M_1 extends over the trace of the surgery (uniquely up to homotopy), and hence determines a p_1-structure on M_2 (uniquely defined up to homotopy).

Notation. If M^3 is an oriented closed 3-manifold, then there is a compact oriented 4-manifold W, with $\partial W = M$. If α is a p_1-structure on M, let $p_1(W, \alpha) \in H^4(W, M; \mathbf{Z})$ denote the obstruction to extending it to W. Define

$$\sigma(\alpha) = 3 \text{ signature}(W) - \langle p_1(W, \alpha), [W] \rangle \in \mathbf{Z}.$$

(Here $[W]$ denotes the fundamental class and $\langle\, , \rangle$ denotes evaluation of cohomology on homology.) By Hirzebruch's signature theorem, this number is independent of W. (This is equal to 3 times Aityah's σ [4].)

The following facts are easily proven using obstruction theory.

PROPOSITION. (i) *The set of homotopy classes (rel. boundary) of p_1-structures, on an oriented, compact, connected 3-manifold, is affinely isomorphic to $\mathbf{Z}$. Moreover, if the manifold is closed, then the map σ is such an affine isomorphism.*

(ii) *On manifolds of dimension less than or equal to two, p_1-structures are unique up to homotopy.*

Remark. The cobordism group $\Omega_3^{p_1}$, of oriented 3-manifolds with p_1-structure, is isomorphic to $\mathbf{Z}/3\mathbf{Z}$, the isomorphism being induced by the invariant σ.

The extended mapping class group. (Cf. [4].) Let Σ be a closed oriented surface with p_1-structure and banded link. We define $\tilde{\mathcal{M}}(\Sigma)$ to be the set of equivalence classes represented by the mapping cylinders, M_f, of orientation preserving diffeomorphisms, f, from the surface to itself, which send the given banded link to itself, preserving its orientation, together with a p_1-structure on the mapping cylinder, extending the given p_1-structure on the two copies of Σ in M_f. (Here equivalence is as in the definition of $C_2^{p_1}$.) If Σ is connected, the forgetful map is an epimorphism from $\tilde{\mathcal{M}}(\Sigma)$ to the classical mapping class group $\mathcal{M}(\Sigma)$ (without p_1-structure), with kernel $\mathbf{Z}$.

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C. B., N. H. – URA 758 du CNRS
Université de Nantes
Département de Mathématiques
2 rue de la Houssinière
44072 NANTES Cedex 03, France

G. M., P. V. – URA 212 du CNRS 'Théories Géométriques'
Université Paris VII
U.F.R. de Mathématiques
Tour 45-55, 5ème étage, 2 place Jussieu
Case 7012
75251 Paris Cedex 05, France