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Using Unavoidable Set of Trees to Generalize Kruskal's Theorem

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INTRODUCTION

Termination is an important property for term rewriting systems. To prove termination, N.Dershowitz [1] introduces quasi-simplification orderings that are monotonic extensions of the embedding relation. He proves that they are well quasi-ordered and a fortiori well-founded by using a theorem from Kruskal [5], which shows that the simple tree insertion order TIO (defined below) is a well quasi-ordering over a certain set of trees. (Well-founded means that every nonempty set contains at least one minimal element; well quasi-ordered means that every nonempty set contains at least one and at most a finite number of noncomparable minimal elements.) Dershowitz's method is powerful, but cannot be used when the rewriting system contains a rule whose right hand side is embedded in the left hand side. The purpose of this paper is to overcome this constraint, when the rewriting system uses a finite ranked alphabet, by generalizing Kruskal's theorem to obtain a family of quasi-orders TIO(S, ω) that are strictly included in TIO but are still well quasi-orders. This generalization is parallel to the generalization described in the next paragraph.

G. Higman [3] includes a well-known subsidiary result, Theorem 4.3, which has the following result as a special case: The set of all words Σ^* over the finite alphabet Σ is well quasi-ordered by the simple word insertion order WIO. The relation t WIO t' means that word t' can be obtained from word t by inserting arbitrary words anywhere in t, including at the very

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beginning and the very end. A. Ehrenfeucht, D. Haussler, and G. Rozenberg [2] have generalized this result, and obtain a well quasi-order by permitting only words from a specified set to be inserted, if that set satisfies a certain property, and if later insertions may be made inside earlier ones. Specifically, given a finite alphabet Σ and a set S of words over Σ, they define two concepts, unavoidability of S and the word insertion order WIO(S), and they prove that WIO(S) is a well quasi-order if and only if S is unavoidable.

S is defined to be unavoidable with avoidance bound k if every word t of length $\geq$ k contains some word S from S as a factor or subword, i.e., t contains S as a consecutive block of letters. As illustrations, if $\Sigma = \{a, b\}$, then it is easy to see that S = {aaa, ab, ba, bbbb} is unavoidable with bound 4 and S' = {aaa, ab, bbbb} is unavoidable with bound 6. The word insertion order WIO(S) is defined to be the transitive closure of the word insertion relation I_S. t I_St' if and only if either (i) t = t'or (ii) t' can be obtained from t by inserting some word from S in t. Thus t WIO(S) t' if and only if t' can be obtained by starting with t and performing several insertion operations using words from S. Because later insertions may be made inside previous ones, the sequence ultimately found in t' between two positions that were adjacent in t may be a complicated combination of words from S. Also note that if several words from S are inserted in t to make t', then t' contains the last word inserted as a factor. This is the central connection on which the theorem of Ehrenfeucht et al. depends.

Now we describe the simple tree insertion order TIO and a special case of Kruskal's theorem. In this paper, trees are always taken to be ordered rooted trees, so the children vertices of each vertex are linearly ordered. Let Σ be a finite alphabet, and let $T(\Sigma)$ be the set of all trees with vertices labeled by elements from Σ. A special case of the main theorem from Kruskal [5] states that $T(\Sigma)$ is well quasi-ordered under TIO. Informally, the relation t TIO t'means that tree t' can be obtained from tree t by inserting arbitrary trees from $T(\Sigma)$ anywhere in t. This means that an arbitrary tree can be inserted between a vertex and a child vertex. Also, an arbitrary tree can be inserted before the root, and another following any terminal vertex. In addition, a tree can be inserted following a nonterminal vertex, between any adjacent pair of its children vertices.

In this paper, our goal is to generalize Kruskal's theorem on trees parallel to the way in which Ehrenfeucht et al. [2] generalize Higman's theorem on words. To make our theorem work, however, we need to make two changes. First, we need to assume that the finite alphabet Σ is ranked. Second, we need to introduce a special new element ω that is not in Σ. (The meaning of ranked and the use of ω are explained below.) Given a finite ranked alphabet Σ and a set S of trees over Σ, we define two concepts, unavoidability of S and the tree insertion order TIO(S, ω), and we prove that TIO(S, ω) is a well quasi-order if and only if S is unavoidable. (Later in the paper, TIO(S, ω) is written $\leq_{S\omega}$ for brevity.)

To explain informally what unavoidability and TIO(S, ω) mean, we start with some elementary definitions, in which we emphasize the parallelism between words and trees. A word over Σ has positions that are labeled by elements from Σ. A tree over Σ has vertices that are labeled by elements from Σ. Informally, we may refer to a position in a word or to a vertex in a tree by its label, when this is not ambiguous. The length of a word is the number of positions it has. The depth of a tree is the maximum number of vertices in any path from the root to a leaf. Σ is said to be a ranked alphabet if each element of Σ has an associated nonnegative integer called its arity. The special element ω has arity 0. A tree is said to respect the arity values if for every vertex, the arity value of its label is equal to the number of its children vertices. If Σ is ranked, then T(Σ) means the set of all trees over Σ that respect the arity values, and $T(\Sigma,\omega) = T(\Sigma \cup \omega)$ is defined similarly.

The set S is defined to be unavoidable with avoidance bound k if every tree twith depth ≥ k contains some tree S from S as a factor. A factor of t, see Figure 1, means any tree that can be obtained from t in the following way.

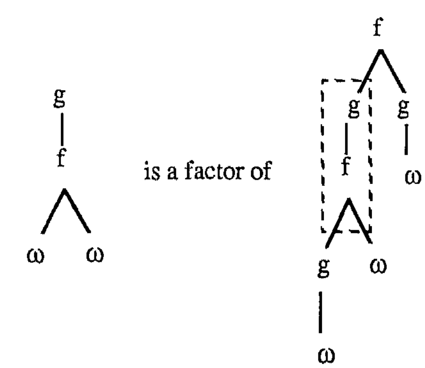


Figure 1

First, for any vertex u in t, take the suffix tree t/u of t, i.e., the portion of t that is suspended from u. (If u is the root of t, then t/u = t.) Then choose any set of incomparable vertices v_i in t/u,delete all vertices below any v_i , and relabel every v_i with ω. The result is a factor of t. We say the factor is located at vertex u in t. Notice that there is a natural embedding of the vertices of the factor into the vertices of tree that contains it. This is called the factor embedding.

To insert a word S at position i of another word t means to insert S just before i, i.e., to start S at position i, and to put the suffix word t/i just after S. To insert a tree S at vertex u of another tree t means to insert S just before u, i.e., to start S at vertex u, and to put the suffix tree t/u just after S -- but where after S? Unlike a word, tree S has many terminal vertices, and t/u could be inserted following any one of them. Furthermore, t/u could also be inserted following a nonterminal vertex, in between any adjacent pair of its children vertices.

To specify in what location the suffix tree will go, we can add a new terminal vertex to S at that location. Then putting t/u at that location is substitution of t/u for the new vertex. To indicate the special meaning of the new vertex, we label it with the special symbol ω of arity 0 that does not belong to Σ. To make the theorem work, however, we are forced to extend this idea and permit each of the trees in S to have one or more terminal vertices that are labeled by ω. When S is inserted at vertex u in t, the suffix tree t/u may be substituted for any one of the ω labeled vertices in S. Parallel to the definition of WIO(S) above, a tree insertion order TIO(S) related to TIO(S, ω) is defined to be the transitive closure of the tree insertion relation I_S. (Later in the paper, TIO(S) is written $\leq_S$.) t I_S t' if and only if either t = t'or t'can be obtained from t by inserting some tree from S in t. Thus t TIO(S) t' if and only if t' can be obtained by starting with t and performing several insertion operations using trees from S. Note that a later insertion may be made inside a previous one. Also note that if several trees from S are inserted in tree t to make tree t' then t'contains the last tree inserted as a factor. This is the central connection on which our theorem depends.

Because of the excess ω vertices, we are forced to deal in a general way with trees containing arbitrarily many ω vertices, which is why we introduced T(Σ, ω) above. To make our theorem work, we also need to use TIO(S, ω), which is a broader relation than TIO(S). TIO(S, ω) is the transitive closure of $I_S \cup I_\omega$. t I_ω t' if and only if either (i) t = t' or (ii) t' can be obtained from t by substituting an arbitrary tree from T(Σ, ω) for any ω vertex.

An illustration of an unavoidable set of trees may be helpful. Let Σ = {f, g, a, b} with arities 2, 1, 0, 0. Then it can be proved in a few lines that the following set S of trees is unavoidable with avoidance bound 4.

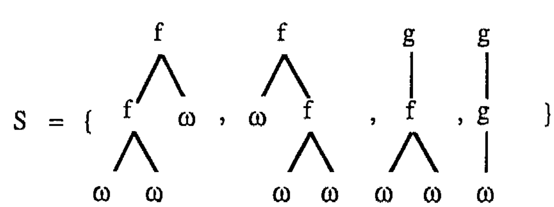


Figure 2

It is not difficult to prove that if TIO(S, ω) is a well quasi-order, then S is unavoidable. This is Proposition 7.2, which is proved in Section 7 in one paragraph, roughly as follows. Assume that TIO(S, ω) is a well quasi-order, and that S is not unavoidable. Then for every k, there is a tree deeper than k which contains no factor from S. Thus there is an infinite sequence of such trees. By the well quasi-ordering assumption, this infinite sequence must contain a pair of trees, t and t', such that t TIO(S, ω) t'. Except for some details which must be taken care of, this means that a tree from S has been inserted into t', and hence that t' contains a factor from S, which is a contradiction.

It is much more difficult to prove that if S is unavoidable, then TIO(S, ω) is a well quasi-order. The proof, which occupies almost the whole paper, consists of three parts. The first part, in Section 3, is to prove Theorem 3.3, a compactness result: If S is unavoidable with avoidance bound k then there exists a finite subset S' of S that is also unavoidable with avoidance bound k. Since TIO(S', ω) is included in TIO(S, ω), if the former is a well quasi-ordering, then the latter must be also. Thus it is legitimate to assume from the beginning that S is finite.

The second part, in Section 4, is to prove two structure theorems, Theorems 4.14 and 4.22. They use both the unavoidability and the finiteness of S. To state them, we introduce some definitions informally. Suppose E is a set of trees. A concatenation of trees from E can be described in terms of the diagram as a set of trees $e_1,..., e_n$ from E connected in a chain: $e_1$is at the top, e_2 is substituted for an ω vertex of e_1, e_3 is substituted for an ω vertex of e_2, and so on to any length. The set of all concatenations is called E*. A dendrite of trees from E can be described as a set of trees from E connected in a tree arrangement: one tree at the top, some trees substituted for ω vertices of the top tree, some trees substituted for ω vertices of trees at the second level, and so on. The set of all dendrites is called E*. A nesting of trees from a set F into a set E means a tree obtained by inserting some trees from F into a tree from E at any vertices. E[F] means the set of all such nestings. An internal nesting of trees from a set F into a set E means a tree obtained by inserting trees from F into a tree from E at any internal vertices (i.e., all vertices except the root). E[F; internal] means the set of all internal nestings. Now make the following recursive definition:

$T_0 = S$,

$S_n = T_n{}^*$ for $n\geq 0$,

$T_{n+1} = S[\, S_n \,;\text{internal}\,]$ for $n\geq 0$.

Intuitively, and ignoring the concatenation steps for the moment, T_n is the set of trees we can get by inserting trees from S into trees from S ... into trees from S, where the insertion depth is limited to n levels. More precisely, concatenations of trees are inserted at each step. Finally, define R_k to be all trees of depth ≤ k.

Theorem 4.1: Suppose Σ is a finite ranked alphabet. Suppose S is a finite subset of T(Σ, ω) not containing ω that is unavoidable with avoidance bound k. Then T(Σ) is contained in $R_k{}^*[S_k]$.

The second structure theorem states that under the same hypothesis, T(Σ) is contained in a subset of $R_k*[S_k]$. To describe the subset, some more definitions are needed. Trees in R_k are called residual trees. Consider a tree t in $R_k*[S_k]$. Because t is in this set, it can be decomposed as a dendrite of trees in $R_k[S_k]$, where each of the trees in $R_k[S_k]$ is in turn decomposed into a nesting of trees from S_k into R_k, and so on. Of course, many different decompositions may be possible. Relative to a particular decomposition, every vertex in t can be identified as belonging ultimately either to a residual tree or to a tree in S. Vertices belonging to residual trees are called residual vertices. Any path from the root of t to a leaf of t contains a certain number of residual vertices. The residual branch height of this leaf, RBH for short, is the number of such residual vertices. The RBH of the decomposition is the maximum RBH over all leaves. The RBH of t is the minimum RBH for any decomposition.

Theorem 4.2: Suppose Σ is a finite ranked alphabet. Suppose S is a finite subset of T(Σ, ω) not containing ω that is unavoidable with avoidance bound k. Then T(Σ) is contained in the set of trees in $R_k*[S_k]$ that have RBH≤ k.

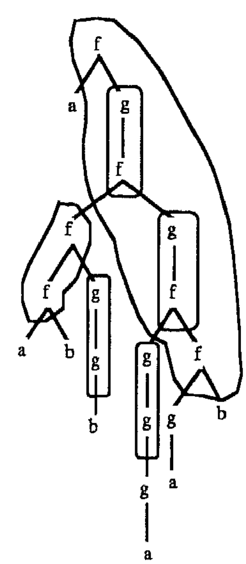


Figure 3

The third part of the proof, in Sections 5 and 6, is to prove a well quasi-ordering result:

Theorem 6.1: Suppose Σ is a finite ranked alphabet. Suppose S is a finite subset of T(Σ, ω) not containing ω, and Q is a finite subset of T(Σ) . Then any subset of $Q*[S_n]$ having bounded RBH is well quasi-ordered under TIO(S, ω).

When this result is used, Q is R_k. This part of the proof uses only the finiteness of S and not its unavoidability. Ignoring certain technical difficulties, the proof goes like this. First we show that S_n is well quasi-ordered for every n. Then we show that $Q[S_n]$ is well quasi-ordered for every n. Last we show that for every n, any subset of $Q^*[S_n]$ having bounded RBH is well quasi-ordered.

In Section 7, the three parts of the proof are quickly put together to yield our main theorem: Theorem 7.3: Suppose Σ is a finite ranked alphabet. Suppose S is a finite subset of $T(\Sigma, \omega)$ not containing ω. Then S is unavoidable if and only if $T(\Sigma)$ is is well quasi-ordered under TIO(S, ω).

To illustrate the purpose for which the theorem was developed, we then present a simple application, by proving termination of a particular term-rewriting system consisting of a single rule. Previous methods for proving termination cannot deal with systems like this one, because the left-hand side of the rule is embedded in the right-hand side. In Section 8 we discuss an extension of our main theorem to be published in the future, in which the alphabet Σ is not finite but is an infinite well quasi-ordered set.

1 PRELIMINARIES

We use mostly standard language theoretic terminology and notation. We use **N** (resp. $\mathbf{N}_+$) to denote the set of non negative (resp. strictly positive) integers. For a finite set S, |S| denotes its cardinality. For sets S_1 and S_2, $S_1 - S_2$ denotes the set theoretic difference between S_1 and S_2. A partition of a set E is a set of subsets $\{ E_i \mid 1 \leq i \leq n \;\; E_i \subset E\}$ such that $\cup_{1 \leq i \leq n} E_i = E$ and $E_i \cap E_j = \emptyset$ for $i \neq j$.

1.1 Relations and Orderings

We define a binary relation R on a a set A as a subset of A^2. We say that a and b are R-related, written aRb, if and only if $(a,b) \in R$. We recall some properties of a binary relation R:

R is reflexive if and only if for all $a \in A$, aRa;

R is irreflexive if and only if there is no $a \in A$ such that aRa;

R is transitive if and only if for all a, b, $c \in A$, aRb and bRc imply aRc;

R is symmetric if and only if for all a, $b \in A$, aRb implies bRa;

R is antisymmetric if and only if for all a, $b \in A$, aRb and bRa imply a=b;

We define a quasi-ordering on a set A as a reflexive transitive binary relation, a partial ordering as a reflexive, antisymmetric transitive binary relation, and a total ordering R as a partial ordering satisfying the additional condition: for all $a,b \in A$, a R b or b R a. For any quasi-ordering $\leq$, $a<b$ will stand for $a\leq b$ and $a\neq b$, and $a\geq b$ for $b\leq a$ ($<$ is the associated strict ordering). We also define the associated equivalence relation, $\approx$, by $a\approx b$ if and only if $a\leq b$ and $a\geq b$.

The cartesian product of two binary relations R_1 and R_2 on sets A_1 and A_2 respectively, written $R_1{\times}R_2$, is a binary relation on $A_1{\times}A_2$ defined by $(a_1, a_2)\ R_1{\times}R_2\ (b_1, b_2)$ if and only if $a_1R_1b_1$ and $a_2R_2b_2$. Since we define a relation as a subset, the inclusion, denoted $\subset$, of relations is only the inclusion of sets, and a sequence of relations (R_i) is increasing if $R_i\subset R_{i+1}$ for every integer i.

We also specify in this section some notation for infinite sequences used through the paper. Let (θ_i) be an infinite sequence. An infinite subsequence (θ'_i) denotes $(\theta_{\gamma(i)})$ where γ is some ascending map from $\mathbf{N}_+$ to $\mathbf{N}_+$ such that for every i, $\theta'_i = \theta_{\gamma(i)}$. Let E be a set of sequences, $(t_i)_{i\in N}$. A minimal sequence in E relative to a total ordering $\leq$ on the elements t_i is a sequence (θ_i) belonging to E such that:

$\theta_0 = \inf \{ t_0 \mid \text{for all sequences } (t_i) \text{ such that } (t_i) \in E\}$

$\theta_{i+1} = \inf \{ t_{i+1} \mid \text{for all sequences } (t_i) \text{ such that } (t_i) \in E \text{ and } \forall j \ (1\leq j\leq i)\ t_j = \theta_j \}$

1.2 Words and Trees

Let Σ be an alphabet. A word on Σ is a mapping $w:\{1,2,\ldots,n\}\rightarrow\Sigma$ for some $n\in\mathbf{N}$. We denote by w_i the element w(i) for i belonging to dom(w)=$\{1,2,\ldots,n\}$ so $w=w_1\ldots w_n$. We use ε for the empty word (where n=0) and dom(w)= $\emptyset$. The length of a word w, denoted by $|w|$, is the number of elements of dom(w). Let $u=u_1\ldots u_n$ and $v=v_1\ldots v_p$ be two words. The word built by concatenation of u and v and denoted u.v or uv is defined by dom(uv)=$\{1,\ldots,n+p\}$, uv(i)=u(i) for $1\leq i\leq n$ and uv(i)=v(i-n) for $n+1\leq i\leq n+p$. The set of words on Σ is denoted by Σ^*. Let the word u/i be the suffix of u at i defined by dom(u/i)=$\{j\in \mathbf{N}_+ \mid i+j-1\in \text{dom}(u)\}$ and u/i(j)=u(i+j-1).

Let us define now some binary relations on words in Σ^*.

–Prefix ordering: $\leq_{prefix}$

$u \leq_{prefix} v$ if and only if dom(u) $\subset$ dom(v) and u(i)=v(i) for all $i \in$ dom(u).

–Factor ordering: $\leq_{factor}$

$u \leq_{factor} v$ if and only if $u \leq_{prefix} v/i$ for some i. With this definition $u \leq_{factor} v$ if and only if there exist two words w_1 and w_2 such that w_1uw_2=v. We say that u is a factor of v.

–Head symbol ordering: HSO(≤)

For any quasi-ordering ≤ on Σ, u HSO(≤)v if and only if u(1)≤v(1).

–Word insertion ordering:WIO(≤)

For any quasi-ordering ≤ on Σ^*, uWIO(≤)v if and only if either (i) u=v or (ii) uWIO(≤)v/2 or (iii) u≤v and u/2WIO(≤)v/2.

–Word insertion ordering modulo a subset: WIO(S)

For any subset S of Σ^*, u I_S v if and only if there exist u_1, u_2 in Σ^* and s in S such that $u = u_1u_2$ and $v = u_1su_2$. WIO(S) is the transitive closure of I_S.

These relations are quasi-orderings on words.

If $u \leq_{prefix} v$, then there is a unique word w such that u.w=v. This word is denoted u\v and spoken as "u under v". Thus u\v is the unique word such that u.u\v=v. Let S be a subset of Σ^*. S is said to be unavoidable if there exists an integer k such that for every word u of length greater than k there exists s in S with $s \leq_{factor} u$.

We also use the classical notation on trees.

Let Σ be an alphabet. A tree t on Σ is defined by a subset, Vertex(t), of N_+^*, the set of words on N_+, and a map, named also t, from Vertex(t) into Σ such that:

i) Vertex(t) is closed under taking prefixes, i.e., $\forall$ u ∈ Vertex(t) $\forall$ v $\leq_{prefix}$ u v∈Vertex(t)

ii) $\forall$ u ∈ Vertex(t), u.i ∈ Vertex(t) ⇒$\forall$ j < i u.j ∈Vertex(t)

The elements of Vertex(t) are the vertices of the tree t. Let u be a vertex of t. t(u) is the label of the vertex u. The vertex ε is the root of the tree and the associated label is the head-symbol. The order of a vertex is the number of vertices immediately below it. The maximum vertices under the prefix ordering are the leaves. We say that two vertices u and v are incomparable if and only if the words u and v are incomparable under the prefix ordering $\leq_{prefix}$. We define height(u), also written |u|, as the length of the finite sequence u∈N_+^* and depth(t) as the maximum of {height(u) | u ∈Vertex(t)}. When Σ is a ranked alphabet, the unique arity of a symbol f in Σ is denoted by ar(f). We denote by Σ_i the subset of Σ whose elements have an arity equal to i.

A tree t on a ranked alphabet Σ is a tree which satisfies the additional property

iii) $\forall$ u ∈ Vertex(t), ar(t(u)) = n ⇒ u.n ∈Vertex(t) and u.(n+1) ∉Vertex(t).

T(Σ) denotes the set of terms (or trees) over Σ. When necessary, we add to the alphabet Σ a set of variables which have arity 0. These variables will be denoted by ω, ω_1,..., ω_n , ω'.... T(Σ,ω) denotes the set of trees over Σ and {ω}.

Call a vertex of t internal if v≠ε. Let Internal(t) = Vertex(t)-{ε}. Let u ∈Vertex(t) . We denote by t/u the tree such that Vertex(t/u) = {v | uv∈Vertex(t)} and t/u(v) = t(uv). We call it the subtree of t at u. Let u ∈Vertex(t). We denote by t [u ← t'] the substitution of the tree t' in the tree t at the vertex u as in [4] where Vertex $(t[u \leftarrow t'])$ = (Vertex(t) - Vertex(t/u)) $\cup$ u.Vertex(t') and $t[u \leftarrow t'](v) = t(v)$ if $v \in$ Vertex(t) - Vertex(t/u)), $t'(u \backslash v)$ otherwise. The notation $t[u_i \leftarrow t_i \mid 1 \leq i \leq n]$ will be used instead of $t\,[u_1 \leftarrow t_1, \ldots, u_n \leftarrow t_n]$ provided the u_i are incomparable vertices. In addition to that, when a symbol ω occurs only once in the tree t, we denote by $t[\omega \leftarrow t']$ the substitution of t' at the vertex u such that $t(u) = \omega$.

Let us define now some orderings on trees in $T(\Sigma,\omega)$.

–Factor ordering: $\leq_{factor}$

s is a factor of t, or s $\leq_{factor}$ t, if t can be obtained from s in two stages:

(1) Substitute trees from $T(\Sigma,\omega)$ for terminal ω–vertices of s.

(2) Substitute the preceding result for a terminal ω–vertex of a tree from $T(\Sigma,\omega)$.

In other words s $\leq_{factor}$t if there exist an integer $n \geq 0$, trees $t_0, \ldots t_n \in T(\Sigma,\omega)$, vertices $u_0 \in$ Vertex(t) and $u_1, \ldots u_n \in$ Vertex(s) with $t_0(u_0) = \omega$, $s(u_i) = \omega$ for $1 \leq i \leq n$, such that $t = t_0[u_0 \leftarrow s[u_i \leftarrow t_i \mid 1 \leq i \leq n]\,]$. Such a decomposition of t is called a factorization of t with respect to s. The vertex of factorization is u_0. The depth of factorization is $|u_0|$. If S is a subset of $T(\Sigma,\omega) - \{\omega\}$, a factorization of t with respect to S means a factorization of t with respect to any $s \in S$.

–Vertex ordering: $\leq_{vertex}$

s $\leq_{vertex}$ t if and only if the number of vertices of s is less than the number of vertices of t.

–Head symbol ordering: HSO($\leq$)

For any quasi-ordering $\leq$ on Σ, s HSO($\leq$) t if and only if $s(\varepsilon) \leq t(\varepsilon)$.

–Tree insertion ordering: TIO($\leq$)

For any quasi-ordering $\leq$ on $T(\Sigma)$, define the relation TIO($\leq$) on $T(\Sigma)$ recursively by s TIO($\leq$) t if and only if either

s = t, or

there exists $i \in$ Vertex(t) such that s TIO($\leq$) t/i, or

$s \leq t$ and s/1s/2...s/m WIO(TIO($\leq$)) t/1t/2... t/n where m (resp.n) is the order of the root in s (resp. t).

TIO(HSO($\leq$)) is the tree ordering used in Kruskal [5].

It can be easily proved that these relations are quasi-orderings on trees.

1.3 Well-Foundedness and Well Quasi-Ordering

Definition 1.1

Given a set A and a quasi-ordering $\leq$ on A, $\leq$ and $<$ are both called well-founded (or nœtherian) if and only if each strictly descending sequence is finite.

Definition 1.2

Given a set A and a quasi-ordering $\leq$ on A, $\leq$ is a well quasi-ordering on A if and only if $\leq$ is well founded and each set of pairwise incomparable elements is finite.

We recall some important properties of these quasi-orderings.

Proposition 1.3

Let $\leq_1$ and $\leq_2$ be two quasi-orderings on a set A. If $\leq_1$ is included in $\leq_2$, then $\leq_2$ nœtherian implies $\leq_1$ nœtherian, and $\leq_1$ a well quasi-ordering implies $\leq_2$ a well quasi-ordering.

Note that the two implications operate in reverse directions. This is why we use well quasi-ordering instead of nœtherian in most what follows.

Proposition 1.4

Let $\leq_1$ (resp. $\leq_2$) be a well quasi-ordering on a set A_1 (resp.A_2). The cartesian product $\leq_1 \times \leq_2$ is a well quasi-ordering on $A_1 \times A_2$.

Proposition 1.5

For any quasi-ordering $\leq$ on a set A, the following conditions are equivalent (Higman[3]):

(i) $\leq$ is a well quasi-ordering on A

(ii) there is no infinite nowhere ascending sequence (i.e.for each infinite sequence (x_i) of elements in A, there exist $i < j$ such that $x_i \leq x_j$).

(iii) each infinite sequence of elements in A contains an infinite ascending subsequence.

The insertion ordering on trees and words defined above are well quasi-ordering when the quasi-ordering used to build them satisfies this property.

Theorem 1.6 Higman [3]

Let $\leq$ be a well quasi-ordering on Σ. Thus, WIO($\leq$) is a well quasi-ordering on Σ^*.

Theorem 1.7 Kruskal [5]

Let $\leq$ be a well quasi-ordering on Σ. Thus, TIO(HSO($\leq$)) is a well quasi-ordering on $T(\Sigma)$.

Theorem 1.8 Kamin-Lévy[7]

Let $\leq$ be a well quasi-ordering on T(Σ). Thus, TIO($\leq$) is a well quasi-ordering on T(Σ).

Theorem 1.9 Erhenfeucht-Haussler-Rozenberg[2]

Let S be a subset of Σ^*. WIO(S) is a well quasi-ordering on Σ^*if and only if S is unavoidable.

We wish to generalize this result to trees as an extension of Kruskal's theorem. In order to do that, we define new operations on trees and give some properties of these operations used later on. In section 5 we define $\leq_S$ (same as TIO(S)) and $\leq_{S\omega}$ (same as TIO(S, ω)). These are both analogous to WIO(S).

2 INSERTIONS IN A TREE

Definition 2.1 Insertion of a tree s $\in$ T (Σ,ω) in a tree t at the vertex u.

(i) If u is a vertex of t and v is an ω-vertex of s, then t [(u,v)] s is defined to be $t[u \leftarrow s[v \leftarrow t/u]]$.

(ii) t [u] s = {t [(u,v)] s | v $\in$ Vertex (s) and s(v) = ω}.

(iii) We use the abbreviations t ♦ s for t [ε] s where ε is the root of the tree t and t[]s when it is not necessary to specify the vertex of insertion.

Note that the insertion of a tree s in a tree t at the vertex u defines a set of trees rather than a tree, and that a tree in T (Σ) cannot be inserted because it has no ω-vertices. Note also that s is above t in the diagram of t ♦ s.

Example 2.2

Let Σ be {y , g , f } with arities 0, 1, 2 respectively.

Insertion of s in t at vertex 2.

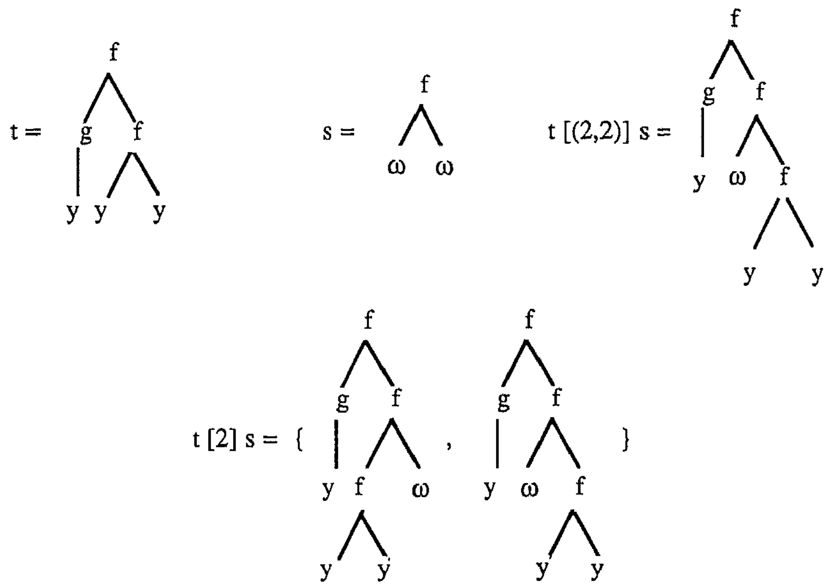


Figure 2.1

Proposition 2.3

Let t_1, t_2, t_3 be trees in $T(\Sigma,\omega)$. The set $(t_1 \blacklozenge t_2) \blacklozenge t_3$ is strictly included in $t_1 \blacklozenge (t_2 \blacklozenge t_3)$ but the converse is false (see Figures 2.2 and 2.3).

Proof. Let t be an element of $(t_1 \blacklozenge t_2) \blacklozenge t_3$. There exist v_2 in Vertex(t_2) and v_3 in Vertex(t_3) such that $t = (t_1 [(\varepsilon, v_2)] t_2) [(\varepsilon, v_3)] t_3$. Thus $t = t_1 [(\varepsilon, v_3 v_2)] (t_2 [(\varepsilon, v_3)] t_3)$ belongs to $t_1 \blacklozenge (t_2 \blacklozenge t_3)$. Another tree $t' = t_1 [(\varepsilon, v_2)] (t_2 [(\varepsilon, v_3)] t_3)$ always belongs to $t_1 \blacklozenge (t_2 \blacklozenge t_3)$ but belongs to $(t_1 \blacklozenge t_2) \blacklozenge t_3$ if and only if v_3 is a prefix of v_2.

Without changing the notation we extend the operation $\blacklozenge$ to sets of trees and define two iterated versions of it, $E^{[k]}$ and E^k.

Definition 2.4

Let E and F be subsets of $T(\Sigma,\omega)$ and t an element of $T(\Sigma,\omega)$.

$t \blacklozenge E = \cup_{\tau \in E}\, t \blacklozenge \tau$ and $E \blacklozenge F = \cup_{t \in E}\, t \blacklozenge F$

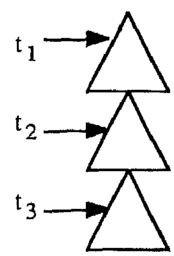
$E^{[0]} = \{\omega\}$ and $E^{[k]} = E^{[k-1]} \blacklozenge E$ for an integer $k > 0$

$E^{[*]} = \cup_{k \in N} E^{[k]}$

$E^0 = \{\omega\}$ and $E^k = E \blacklozenge E^{k-1}$ for an integer $k>0$

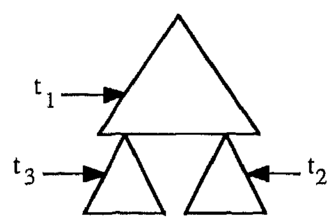
$E^* = \cup_{k \in N} E^k$

As a consequence of the definition, we remark $E^1 = E^{[1]} = E$. It is important to notice the difference between $E^{[k]}$ and E^k. As a consequence of Proposition 2.3, $E^k \supseteq E^{[k]}$. A typical element of $E^{[2]}$ is shown on Figure 2.2, while Figures 2.2 and 2.3 both show elements in E^2.



$\in (t_3 \blacklozenge t_2) \blacklozenge t_1 \subset t_3 \blacklozenge (t_2 \blacklozenge t_1)$

Figure 2.2



$\in t_3 \blacklozenge (t_2 \blacklozenge t_1)$

Figure 2.3

Elements in $E^{[*]}$ are called concatenations of elements in E. Elements in E^* are called dendrites.

Lemma 2.5

$E^{[*]} \blacklozenge E = E^{[*]}$.

Proof. This property is a direct consequence of the definition above.

We generalize the previous definition to the insertion of several trees. The method of combining trees given here differs from the method used in a factorization in several ways. Here, trees are inserted anywhere; there they are inserted only at ω-vertices. (At an ω-vertex, insertion is equivalent to substitution.) As a result, here the insertion vertices can be comparable; there they cannot. We give a recursive definition. When the set U of vertices of insertion is not a set of pairwise incomparable vertices, we split U into two parts: the maximal vertices under the prefix ordering and the remainder.

Definition 2.6 Insertion of n trees $s_1,...,s_n \in T(\Sigma,\omega)$ in a tree t at the vertices $u_1,...,u_n \in$ Vertex(t).

(i) Let $\{u_i \in \text{Vertex}(t) \mid 1 \leq i \leq n\}$ be a set of incomparable vertices and $\{s_i \in T(\Sigma,\omega) \mid 1 \leq i \leq n\}$ be n trees to be inserted. For any integer i ($1 \leq i \leq n$), let $u_i \in \text{Vertex}(t)$, $v_i \in \text{Vertex}(s_i)$ and $s_i(v_i) = \omega$. We write $t = t\,[\ldots(u_i, v_i)\ldots](\ldots s_i \ldots)$ if and only if $t' = t\,[u_i \leftarrow s_i[v_i \leftarrow t/u_i] \mid 1 \leq i \leq n]$.

(ii) $t\,[\ldots u_i \ldots](\ldots s_i \ldots)$ denotes the following set:
$\{t\,[\ldots(u_i, v_i)\ldots](\ldots s_i \ldots) \mid \text{for every } i,\ v_i \in \text{Vertex}\,(s_i) \text{ and } s_i(v_i) = \omega\}$.
We use the abbreviation $t\,[\,](\ldots s_i \ldots)$ when we do not want to specify the vertices of insertion.

(iii) Let $U \subset \text{Vertex}(t)$ be the set of insertion vertices, ρ a map from U to $T(\Sigma,\omega)$ and $\text{Max}\,(U) = \{\, u \mid u \in U,\ u \text{ maximum for the prefix ordering}\,\}$. The insertion is recursively defined by

$t\,[U]\,\rho\,(U) = \{t\}$ if U is void, and

$t\,[U]\,\rho\,(U) = \bigcup_{\tau \in t[\text{Max}\,(U)]\,\rho\,(\text{Max}\,(U))} \tau\,[U - \text{Max}\,(U)]\,\rho\,(U - \text{Max}\,(U))$ if not.

We also use the abbreviation $t\,[\,](\ldots s_i \ldots)$ when we do not need to specify the vertices of insertion.

We draw a figure to show how these insertions are performed.

case (i)

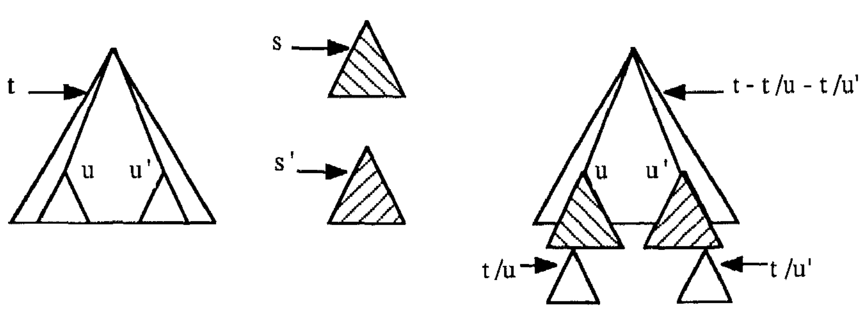


Figure 2.4

case (ii)

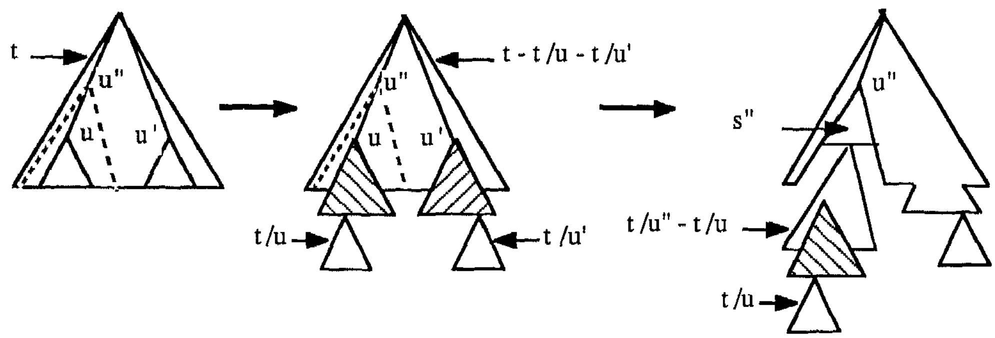


Figure 2.5

We state some tree insertion identities. Let t_1, t_2, t_3 be three trees in $T(\Sigma,\omega)$. We compute $(t_1[(u_1,\mu_2)]t_2)[(v_1,\mu_3)]t_3$ as a function of the relative positions of the insertion vertices.

Proposition 2.7

Let t_1, t_2, t_3 be three trees in $T(\Sigma,\omega)$. Let u_1 be a vertex of t_1, μ_2 (resp. μ_3) a terminal vertex of t_2 (resp. t_3) such that $t_2(\mu_2)=\omega$ (resp. $t_3(\mu_3)=\omega$).

(1) If u_1 and v_1 are incomparable in the prefix ordering, then v_1 is in t_1, and

$$(t_1[(u_1,\mu_2)]t_2)[(v_1,\mu_3)]t_3 = t_1[(v_1,\mu_3)(u_1,\mu_2)](t_3,t_2)$$

(2) If v_1 is a prefix of u_1, then v_1 is in t_1, and

$$(t_1[(u_1,\mu_2)]t_2)[(v_1,\mu_3)]t_3 = t_1[(u_1,\mu_2)(v_1,\mu_3)](t_2,t_3)$$

(3) If u_1 is a prefix of v_1 and $u_1\mu_2$ is a prefix of v_1 (Figure a), then

$v_1 = u_1\mu_2 w'$ and $u_1 w'$ is in t_1, so

$$(t_1[(u_1,\mu_2)]t_2)[(v_1,\mu_3)]t_3 = t_1[(u_1,\mu_2)(u_1w',\mu_3)](t_2,t_3)$$

(4) If u_1 is a prefix of v_1 and v_1 a prefix of $u_1\mu_2$ (Figure b. Insertion inside t_2), then

$v_1 = u_1 w$ and $u_1\mu_2 = v_1 w'$ so $u_1\mu_2 = u_1 ww'$ so $\mu_2 = ww'$, and

$$(t_1[(u_1,\mu_2)]t_2)[(v_1,\mu_3)]t_3 = t_1[(u_1, w\mu_3 w')](t_2[(w,\mu_3)]t_3)$$

(5) If u_1 is a prefix of v_1, and v_1 and $u_1\mu_2$ incomparable (Figure c. Insertion inside t_2)

$v_1 = u_1 w$

$$(t_1[(u_1,\mu_2)]t_2)[(v_1,\mu_3)]t_3 = t_1[(u_1,\mu_2)](t_2[(w,\mu_3)]t_3)$$

Proof. The only necessary case to prove is the first one. The others follow from the definition. Let $\tau = t_1[u_1 \leftarrow t_2[\mu_2 \leftarrow t_1/u_1]]$. Then the subtree of τ at vertex v_1, τ/v_1, is equal to the subtree of t_1 at v_1, t_1 / v_1. Then the following equalities are satisfied.

Let θ be equal to $(t_1[(u_1, \mu_2)]t_2)\,[(v_1, \mu_3)]\,t_3$.

$$\begin{aligned}
\theta &= \tau[v_1 \leftarrow t_3\,[\mu_3 \leftarrow \tau/v_1]] \\
&= \tau\,[v_1 \leftarrow t_3\,[\mu_3 \leftarrow t_1/v_1\,]] \\
&= t_1[u_1 \leftarrow t_2\,[\mu_2 \leftarrow t_1/u_1\,]][v_1 \leftarrow t_3\,[\mu_3 \leftarrow t_1/v_1\,]] \\
&= t_1[v_1 \leftarrow t_3\,[\mu_3 \leftarrow t_1/v_1\,]]\,[u_1 \leftarrow t_2\,\,[\mu_2 \leftarrow t_1/u_1\,]] \\
&= (t_1\,[(v_1\,,\mu_3\,)\,]\,t_3\,)\,[\,(u_1,\mu_2\,)]\,t_2 \\
&= t_1\,[(v_1\,,\mu_3\,)\,(u_1, \mu_2\,)]\,(t_3\,, t_2\,) \quad \text{by definition}
\end{aligned}$$

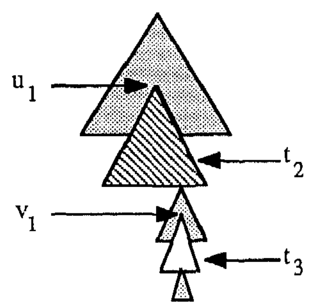


Figure a

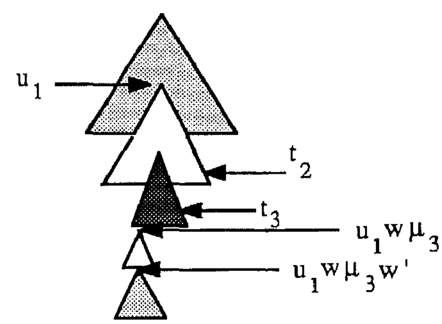


Figure b

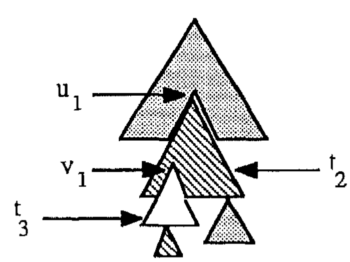


Figure c

Figure 2.6

Proposition 2.8

Let t and s be trees in $T(\Sigma,\omega)$. Let t' belong to t[ ]s. Then $s \leq_{factor} t'$.

Proof. If t' belongs to t[ ]s there are u in Vertex (t) and v in Vertex (s) with $s(v) = \omega$ such that $t' = t[u \leftarrow s[v \leftarrow t/u]]$. Let $t_1 = t[u \leftarrow \omega]$ and $t_2 = t/u$. Then $t' = t_1[u \leftarrow s[v \leftarrow t_2]]$ and thus $s \leq_{factor} t'$.

Proposition 2.9

Let t, t', s and θ be trees in T(Σ,ω). Let $s \leq_{factor} t$ and $\theta \in t' \blacklozenge t$. Then $s \leq_{factor} \theta$.

Proof. As $\theta \in t' \blacklozenge t$, by Proposition 2.8, $t \leq_{factor} \theta$. Now $s \leq_{factor} t$ and transitivity complete the proof.

As a consequence of these properties we get $t \in t_n \blacklozenge (t_{n-1} \blacklozenge \ldots (t_2 \blacklozenge t_1[\,]s))\ldots)$ implies $s \leq_{factor} t$. The converse holds also:

Proposition 2.10

Let t and s be two trees in T(Σ, ω). Then $s \leq_{factor} t$ if and only if there exist an integer n and n trees $\theta_1, \ldots, \theta_n$ in T(Σ, ω) such that $t \in \theta_n \blacklozenge (\theta_{n-1} \blacklozenge \ldots (\theta_2 \blacklozenge (\theta_1[\,]\, s))\ldots)$.

Proof. Let t and s be such that $s \leq_{factor} t$. There are, by definition, $t_0, t_1, \ldots, t_n$ in T(Σ, ω), u a terminal ω -vertex of t_0 and u_i $(1 \leq i \leq n)$ n terminal ω -vertices of s such that the following equalities hold.

$$t = t_0[u \leftarrow s[u_i \leftarrow t_i \mid 1 \leq i \leq n]]$$
$$= (t_n[(\varepsilon,u_n)]\,(t_{n-1}[(\varepsilon,u_{n-1})]\,(\ldots(t_2[(\varepsilon,u_2)](t_1[(\varepsilon,u_1)]s))\ldots)))[(\varepsilon,u)]t_0$$
$$= t_n[(\varepsilon,uu_n)]\,(t_{n-1}[(\varepsilon,uu_{n-1})]\,(\ldots(t_2[(\varepsilon,uu_2)]\,(\tau[(u,u_1)]s))\ldots)) \text{ where } \tau = t_0[u \leftarrow t_1].$$

Thus $t \in \theta_n \blacklozenge (\theta_{n-1} \blacklozenge \ldots (\theta_2 \blacklozenge \theta_1[\,]\, s)\ldots)$ is a characterization of the relation $s \leq_{factor} t$.

3 UNAVOIDABLE SETS

Informally, a subset S of T(Σ, ω) is unavoidable if every tree which is large enough contains a factor belonging to S. Let us define this formally in terms of the concept above.

Definition 3.1 Factor - unavoidable

A subset S of T(Σ, ω) is said to be factor - unavoidable if it does not contain the tree ω and if there exists an integer k such that for every tree t in T(Σ, ω) with depth (t) > k there exists s in S such that $s \leq_{factor} t$. We call k the avoidance bound.

Remark. By definition an unavoidable set does not contain the tree ω. By Proposition 2.10, another way to describe the property that a subset S is factor - unavoidable with avoidance bound k is the following:

$\forall t \in T(\Sigma, \omega)$, depth (t) > k $\Rightarrow$

$\exists s \in S, \exists t_1,\ldots,t_k \in T(\Sigma, \omega), t \in t_k \blacklozenge \ldots (t_2 \blacklozenge (t_1[\,]\, s)$.

Example 3.2

Let $\Sigma = \{y, g, h\}$ with arities 0, 1, 2 respectively.

Let $s_1 = f(f(\omega,\omega),\omega)$, $s_2 = f(\omega, f(\omega,\omega))$, $s_3 = g(g(\omega))$ and $s_4 = g(f(\omega,\omega))$, as in the introduction.

Let $S = \{s_1, s_2, s_3, s_4\}$. It is easy to prove it unavoidable with avoidance bound 2.

Let $t = f(g(f(f(\omega,\omega), g(\omega))), g(f(g(\omega), \omega)))$, $t_1 = f(f(\omega,\omega), g(f(g(\omega),\omega)))$ and $t_2 = g(\omega)$.

Then $t \in t_2 \blacklozenge t_1[\,]\, s_4$ (Figure 3.1).

But $S_1 = \{s_1, s_2, s_4\}$ is not unavoidable. For any integer l greater than 2, the tree $g^l(\omega)$, defined by $g^l(\omega) = g(g^{l-1}(\omega))$, has no factor in S_1.

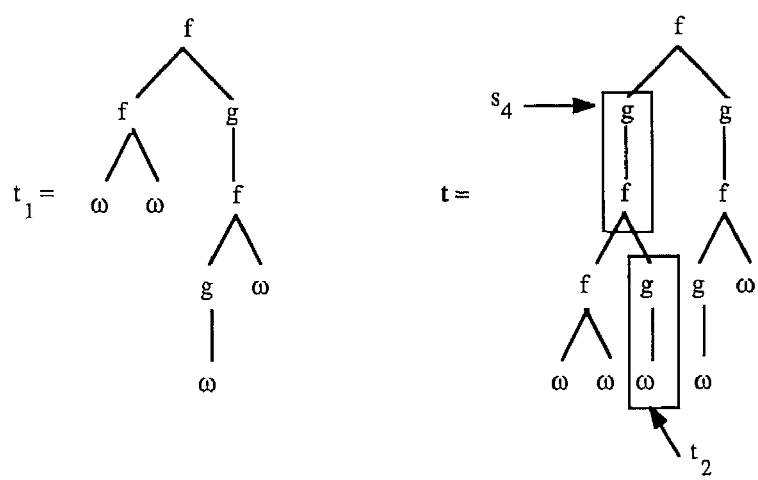


Figure 3.1

Now, we show that, when Σ is a finite ranked alphabet, every unavoidable set includes a finite set which is unavoidable with the same bound.

Theorem 3.3 (Compactness)

Let Σ be a finite ranked alphabet. When $S \subset T(\Sigma, \omega)$ is factor - unavoidable in $T(\Sigma, \omega)$ with avoidance bound k, there exists a finite set $F \subset S \cap (T(\Sigma, \omega) - T(\Sigma))$ that is factor - unavoidable in $T(\Sigma, \omega)$ with the same bound.

Proof. Let $F = S \cap \{t \mid depth(t) \leq k\}$. Clearly F is finite. To prove F unavoidable, let t be a tree in $T(\Sigma,\omega)$ with $depth(t) \geq k+1$. Let us substitute ω in t at each vertex u whose length is k+1 and for which $ar(t(u)) \neq 0$. The new tree is denoted by $t_0 = t[u \leftarrow \omega \mid height(u)=k+1, ar(t(u)) \neq 0]$. Because $depth(t_0) = k+1$ there is a tree s in S such that $s \leq_{factor} t_0$. Thus $depth(s) \leq k+1$. Since $t_0 \leq_{factor} t$, $s \leq_{factor} t$. This shows F is factor-unavoidable. It is easy to prove that the avoidance bound remains unchanged.

Lemma 3.4

Let Σ be a finite ranked alphabet. When $S \subset T(\Sigma, \omega)$ is factor - unavoidable in $T(\Sigma, \omega)$ with avoidance bound k, for every term t such that $depth(t) \geq k+1$ there exist a tree s in S, a vertex u of t and n trees $t_1, \ldots, t_n$ such that $t = t_1[u \leftarrow s[u_i \leftarrow t_i \mid 2 \leq i \leq l][u_1 \leftarrow t_1/u]]$ where u_i $(1 \leq i \leq l)$ are terminal vertices of s and $height(u) \leq k$.

Proof. It is an immediate consequence of the previous theorem, taking in account the fact that s is different from ω.

4 STRUCTURE Theorem

Now we are going to show how it is possible to build every tree from trees whose depth is less than the avoidance bound by insertion of unavoidable trees. From an unavoidable set S we build by induction sets of trees T_n and S_n for $n \geq 0$. We use two operations, insertion at any internal vertex, and concatenation, which is insertion at the root.

T_0 is S. For every integer n, each element of S_n is the concatenation of a sequence of trees belonging to T_n. Each element of T_n is built by insertion, at internal vertices, of trees from S_{n-1} in an element of S (Figure 4.1).

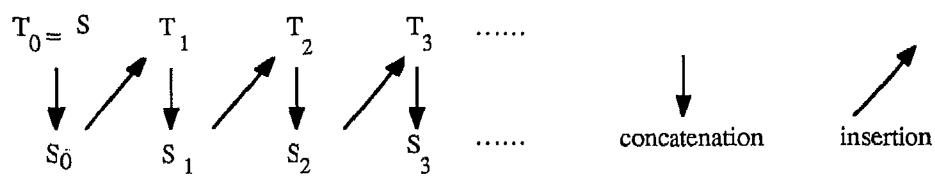


Figure 4.1

Definition 4.1

Let Σ be an alphabet, ω a variable not belonging to Σ and S a subset of $T(\Sigma, \omega) - T(\Sigma)$. Let us define the following sets.

$T_0 = S$

For $n \geq 0$, $S_n = T_n^{[*]}$

For $n \geq 1$, $T_n = \bigcup_{s \in S} \bigcup_{U \subset \text{Internal}(s)} \bigcup_{\rho : U \to S_{n-1}} s[U]\rho(U)$

Let $\theta_1, \theta_2, \theta_3$ belong to S_{n-1} and s to S. Figure 4.2 displays an element of T_n.

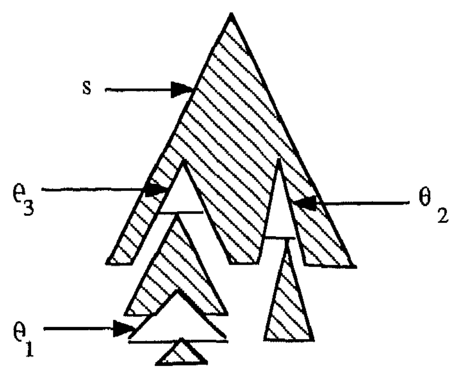


Figure 4.2

Lemma 4.2

For every integer n, the tree ω belongs to S_n and $T_n \subset S_n$.

Proof. For every set E, $E^{[*]} \supset E^{[0]} = \{\omega\}$ and $E^{[*]} \supset E^{[1]} = E$ by definition.

Comment. ω does not belong to T_n for any n.

Lemma 4.3

For every integer n, $S \subset T_n$.

Proof. For any tree s, when U is void, $s\,[\,U]\,\rho(U) = s$.

Lemma 4.4

For every integer n, $T_n \subset T_{n+1}$ and $S_n \subset S_{n+1}$.

Proof. If $T_n \subset T_{n+1}$ then $S_n \subset S_{n+1}$ by definition, and if $S_n \subset S_{n+1}$ then $T_{n+1} \subset T_{n+2}$. As $T_0 = S \subset S^{[*]} = T_1$, the Proposition is true by induction.

Comment. In general, it is not true that $S_{n-1} \subset T_n$. Furthermore we show, on the example below, that, in general, $S_n \blacklozenge S_n$ is not included in S_n. Let $S=\{f(\omega,\omega),g(g(\omega))\}$. Let $t_1=g(g(\omega))$, $t_2=g(g(\omega))$ two trees different from ω in S_n, $t_3=f(\omega,\omega)$ a tree in T_n. Then $t_3[1\leftarrow t_1] \in S_n$, $t_3[1\leftarrow t_1,2\leftarrow t_2]=f(g(g(\omega)),g(g(\omega))) \notin S_n$ but $f(g(g(\omega)),g(g(\omega)))\in S_n \blacklozenge S_n$.

Let t_1, t_2, t_3 belong to T_n. We represent an element which belongs to S_n (Figure 4.3, left) and one element which belongs to $S_n \blacklozenge S_n$ and not to S_n (Figure 4.3, right).

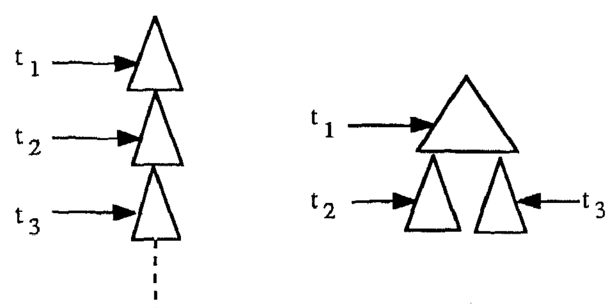


Figure 4.3

Lemma 4.5

Let n, n' be two integers such that n'<n. $S_n \blacklozenge S_{n'} \subset S_n$.

Proof. We prove this result by induction on n. The basic case is n=1 and n'=0. Let $t \in S_1$ and $t' \in S_0$. As t' belongs to S_0, $t' = (((\tau_m \blacklozenge \tau_{m-1}) \blacklozenge \cdots)\, \tau_2) \blacklozenge \tau_1$ with for every i $(1 \leq i \leq m)$ $\tau_i \in S=T_0 \subset T_1$. Let θ be an element of $t \blacklozenge t'$. There exists an index i $(1 \leq i \leq m)$ such that $\theta \in ((\tau_i[u\leftarrow t, v\leftarrow ((\tau_m \blacklozenge \cdots) \blacklozenge \tau_{i+1}\,)] \blacklozenge \tau_{i-1}) \blacklozenge \cdots) \blacklozenge \tau_1$ with u, v terminal ω-vertices of τ_i. In order to get the result, it suffices to prove that $\tau_i[u\leftarrow t, v\leftarrow((\tau_m \blacklozenge \cdots)\blacklozenge \tau_{i+1})]$ belongs to S_1. By definition $\tau_i[v\leftarrow((\tau_m \blacklozenge \cdots)\blacklozenge \tau_{i+1})]$ belongs to T_1. By definition, once again, $\tau_i[u\leftarrow t, v\leftarrow((\tau_m \blacklozenge \cdots)\blacklozenge \tau_{i+1})]$ is a subset of $t \blacklozenge \tau_i[v\leftarrow((\tau_m \blacklozenge \cdots)\blacklozenge \tau_{i+1})]$ which is included in S_1 by Lemma 2.5. We suppose now the property true for every p and p' with $p'<p$ and $p<n$. Let $t\in S_n$ and $t'\in S_{n'}$. As t' belongs to $S_{n'}$, $t' = (((\tau_m \blacklozenge \tau_{m-1}) \blacklozenge \cdots) \tau_2) \blacklozenge \tau_1$ with for every i $(1\leq i\leq m)$ $\tau_i \in T_{n'} \subset T_{n-1}$. Let θ be an element of $t \blacklozenge t'$. Thus $\theta \in ((\tau_i[u\leftarrow t, v\leftarrow((\tau_m \blacklozenge \cdots) \blacklozenge \tau_{i+1})] \blacklozenge \tau_{i-1}) \blacklozenge \cdots) \blacklozenge \tau_1$. Note that $((\tau_m \blacklozenge \cdots)\blacklozenge \tau_{i+1})$ belongs to $S_{n'} \subset S_{n-1}$. $\tau_i[u\leftarrow t, v\leftarrow((\tau_m \blacklozenge \cdots)\blacklozenge \tau_{i+1})]$ belongs to S_n because, either v is a vertex of the element of S from which τ_i is built and, by definition, $\tau_i[v\leftarrow((\tau_m \blacklozenge \cdots)\blacklozenge \tau_{i+1})]$ belongs to T_n, or v is a vertex v_τ of an element τ of $S_{n'-1}$ which is a factor of τ_i and by induction $\tau[v_\tau\leftarrow((\tau_m \blacklozenge \cdots)\blacklozenge \tau_{i-1})]$ belongs to S_{n-1} and thus $\tau_i[v\leftarrow((\tau_m \blacklozenge \cdots)\blacklozenge \tau_{i+1})]$ belongs to T_n. By definition, once again, $\tau_i[u\leftarrow t, v\leftarrow((\tau_m \blacklozenge \cdots)\blacklozenge \tau_{i+1})]$ is a subset of $t \blacklozenge \tau_i[v\leftarrow((\tau_m \blacklozenge \cdots)\blacklozenge \tau_{i+1})]$ which is included in S_n by Lemma 2.5. As a particular case we get $S_n \blacklozenge S \subset S_n$ for every integer n.

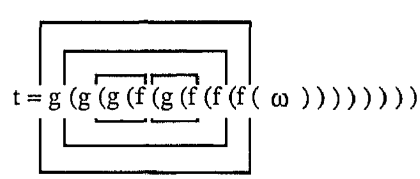
Definition 4.6

We call the nesting level of a tree t the smallest integer n such that t belongs to S_n.

In the Lemma below we prove that inserting a tree $s \in S$ into a tree $t \in S_n$ sufficiently near the root does not increase the nesting level. The following example gives an intuitive idea of this property.

Example 4.7

Let $\Sigma = \{f, g, y\}$ with arities 1,1,0 respectively and ω be a variable.
Let $S = \{s= g(f(\omega))\}$ and $t=g(g(g(f(g(f(f(f(\omega))))))))$.
$t[(11,11)]s = g(g(g(f(g(f(g(f(f(f(\omega))))))))))$.
As we can see in Figure 4.5 , t and t[(11,11)]s both belong to S_2 (the brackets show membership in T_0,T_1 and T_2, working from inside out). The nesting level remains unchanged.



$$t\,[(11,11)]\,s = g\,(g\,(g\,(f\,(g\,(f\,(g\,(f\,(f\,(f\,(\ \omega\ ))))))))))$$

Figure 4.5

Proposition 4.8

Let Σ be an alphabet and ω a variable of arity 0 which does not belong to Σ. Let S be a subset of $T(\Sigma,\omega) - T(\Sigma)$. Let t be in S_n and u in Vertex(t) such that $|u| \leq n$. For each s in S, t[u]s is included in S_n.

Proof.(by induction on n)
1) Case n = 0. The insertion can be performed only at the root. Thus $u = \varepsilon$ and $t[u]s = t \blacklozenge s$ is included in S_0 by Lemma 2.5.
2) Let us suppose the property true for every n'<n. If $u = \varepsilon$, $t[u]s = t \blacklozenge s$ is included in S_n by Lemma 4.5. If $u \neq \varepsilon$ there exist k trees t_j in T_n, an integer j $(1 \leq j \leq k)$ and a vertex u_j of t_j such that $t \in (\ldots(t_k \blacklozenge t_{k-1}) \blacklozenge \ldots) \blacklozenge t_1$ and $t[u]s \subset (\ldots(\ldots(t_k \blacklozenge t_{k-1}) \blacklozenge \ldots(t_j[u_j]s)) \blacklozenge \ldots) \blacklozenge t_1$. By definition, if $t_j[u_j]s$ is included in T_n, then t[u]s is included in S_n.Then it is sufficient to prove that, if $t \in T_n$, $s \in S$, $u \in$ Vertex(t) with $|u| \leq n$ and $v \in$ Vertex(s) with $s(v) = \omega$, t [(u, v)] s belongs to T_n. By definition, there exist a tree τ belonging to S, m trees θ_i $(1 \leq i \leq m)$ belonging to S_{n-1} and for every i $(1 \leq i \leq m)$ a pair (u_i, v_i) from Vertex(τ) $\times$ Vertex(θ_i) such that $t = \tau\,[\ldots(u_i, v_i)\ldots](\ldots\theta_i\ldots)$. In order to show that t[(u,v)]s belongs to T_n we consider two subcases i) and ii).
i) There exists an integer i $(1 \leq i \leq m)$ such that $u_i \leq_{prefix} u$ (i.e. $u = u_i w$) and $w \in$ Internal(θ_i). Intuitively, the insertion is performed inside θ_i. Consider the relationship between u and $u_i v_i$. $u >_{prefix} u_i v_i$ is impossible because u is in θ_i. If $u = u_i$, $\theta_i \blacklozenge s \subset S_{n-1}$ by Lemma 4.5, so t[u]s is included in T_n.

If u and $u_i v_i$ are incomparable, then by Proposition 2.7, case 5,
$t[(u,v)]s = \tau[\ldots(u_{i-1},v_{i-1}),(u_i,v_i),(u_{i+1},v_{i+1})\ldots](\ldots\theta_{i-1},\theta_i[(w,v)]s,\theta_{i+1}\ldots)$

If $u \leq_{prefix} u_i v_i$, $uw_i = u_i w w_i = u_i v_i$ and thus $ww_i = v_i$. By Proposition 2.7, case 4,
$t[(u,v)]s = \tau[\ldots(u_{i-1},v_{i-1}),(u_i,wvw_i),(u_{i+1},v_{i+1})\ldots](\ldots\theta_{i-1},\theta_i[(w,v)]s,\theta_{i+1}\ldots)$
By definition of T_n , $u_i \neq \varepsilon$ thus $n \geq |u| = |u_i|+|w| \geq 1+|w|$ and $|w| \leq n-1$. By induction we get $\theta_i[(w,v)]\,s \in S_{n-1}$ and $t\,[u]\,s \subset T_n$.

ii) $u \in \text{Internal}(\tau)$ but there is no integer i such that $u_i \leq_{prefix} u$ (i.e. $u = u_i w$) and $w \in \text{Internal}(\theta_i)$. Intuitively the insertion is performed inside τ but not inside a θ_i. By Proposition 2.7, cases 1 and 2, there exists $u' \in N_+{}^*$ such that $|u'| \leq |u|$ and $t[(u,v)]\, s = \tau[\ldots(u_i,v_i)\ldots(u',v)]\,(\ldots\theta_i\ldots s)$. As s belongs to S_{n-1}, this tree belongs to T_n by definition.

In conclusion, in both cases t [u] s is included in T_n provided t is in T_n and $|u| \leq n$.

More generally, we might hope to get the same result when an element of $S_{n'}$ is inserted in an element of S_n at a vertex such that $|u| \leq n - n'$, but the result is not true. However, as described in Corollary 4.10, the resulting tree can be split into an element of S_n and an element of $S_{n'}$ (Figure 4.4). If we add the condition that v is internal to the lowest portion of s' (condition (iv) in the following lemma) then the result is true.

Lemma 4.9

Suppose $s \in S_n$, $s' = s'_1[u'_1 \leftarrow s'_2[u'_2 \leftarrow s'_3[\ldots]]] \in S_{n'}$, $u \in \text{Vertex}(s)$, $v \in \text{Vertex}(s')$, $v(s') = \omega$, and that

(i) $n' < n$ and $|u| \leq n - n'$,

(ii) $\forall i\ (1 \leq i \leq l)\ s'_i \in T_{n'}$,

(iii) $\forall i\ (1 \leq i \leq l-1)\ u'_i$ is a terminal ω-vertex of s'_i,

(iv) $v = u'_1 u'_2 \ldots u'_{l-1} v'$ and $v' \neq \varepsilon$.

Then $s[(u, v)]s' \in S_n$.

Proof. We prove the result by induction on n.When n=1, n'=0 and |u|=1 or 0. When $u=\varepsilon$ $s[(u,v)]s' \in S_n$ as seen in Lemma 4.9. When |u|=1, the insertion cannot be performed at an internal vertex of an element of S_0 inserted itself in an element of S. Thus, by definition, the resulting tree belongs to S_1. In the general case, let n be an integer satisfying the induction hypotheses:

For every $m<n$, $n' \leq m$, $s \in S_m$, $|u| \leq m - n'$ and $s' = s'_1[u'_1 \leftarrow s'_2[u'_2 \leftarrow s'_3[\ldots]]] \in S_{n'}$ $s[(u,v)]s' \in S_m$.

Let $s \in S_n$. Then $s = s_1[u_1 \leftarrow s_2[u_2 \leftarrow s_3[\ldots]]]$ with every s_i belonging to T_n and, by definition of T_n, $s_i = \sigma_i[\ldots(u_j{}^i,v_j{}^i)\ldots](\ldots\theta_j{}^i\ldots)$ with $\sigma_i \in S$ and $\theta_j{}^i \in S_{n-1}$. There are three cases:

i) There exists i such that $u \in \text{Internal}(\sigma_i)$. As $S_{n'}$ is included in S_{n-1}, $s_i[(u, v)]s' \in T_n$ and $s[(u, v)]s' \in S_n$ by definition.

ii) There exists i such that u is a vertex u_i. Then by the hypothesis on v, $s[(u, v)]s'$ can be written out using insertions from T_n, then insertions from $T_{n'}$, then insertions from T_n. As $T_{n'}$ is included in T_n, the resulting tree satisfies the definition of S_n.

iii) There exist i and j such that u is an internal vertex u_θ of θ_j^i with $|u_\theta|<|u|$. As $1<|u|\leq n-n'$, and thus $n'<n-1$ and $|u_\theta|\leq n-n'-1$, the induction hypothesis implies that the tree built by insertion of s' in θ_j^i belongs to S_{n-1} and in conclusion $s[(u, v)]s' \in S_n$.

Corollary 4.10

Let $s \in S_n$, $s' = s'_1[u'_1 \leftarrow s'_2[u'_2 \leftarrow s'_3[\ \ldots]]] \in S_{n'}$, $u \in$ Vertex(s) and $v \in$ Vertex(s') with

(i) $n' < n$ and $|u| \leq n - n'$

(ii) $\forall i\ (1 \leq i \leq l)\ s'_i \in T_{n'}$

(iii) $\forall i\ (1 \leq i \leq l-1)\ u'_i$ is a terminal vertex of s'_i

(iv) $v = u'_1 u'_2 \ldots u'_{k-1} v'$ with $k \leq l$ and $v' \neq \varepsilon$.

If v and $u'_1 u'_2 \ldots u'_k$ are incomparable vertices then $s[(u, v)](s' - s'/ u'_1 u'_2 \ldots u'_k) \in S_n$ and $s'/u'_1 u'_2 \ldots u'_k \in S_{n'}$.

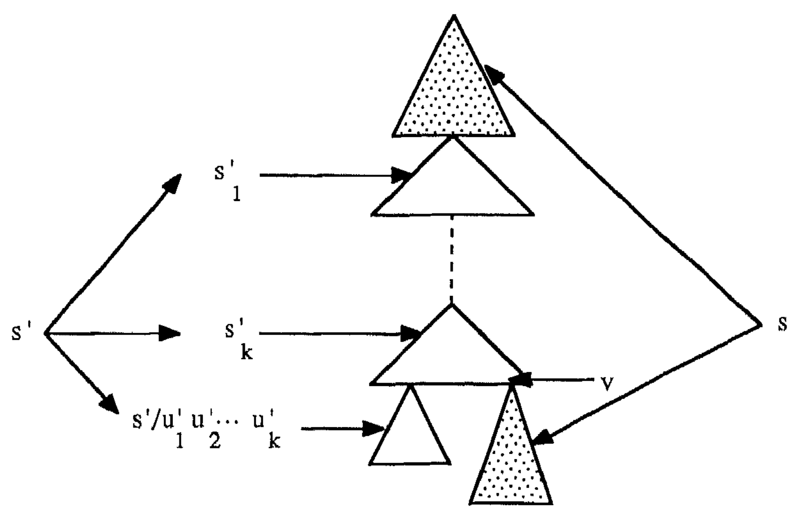


Figure 4.4

We introduce now a subset Q of $T(\Sigma,\omega)$ and we insert inside trees belonging to Q nested trees from S.

Definition 4.11

Let Q be a subset of $T(\Sigma,\omega)$ and S a subset of $T(\Sigma,\omega) - T(\Sigma)$.

$$Q[S_n] = \cup_{t\in Q} \cup_{U\subset Vertex(t)} \cup_{\rho:U\to S_n} t[U]\rho(U) .$$

Remarks and notation.

1) S_n is included in $Q[S_n]$ when ω belongs to Q.

2) For every integer n, Q is included in $Q[S_n]$.

3) From now on, we deal with concatenations of sets like $Q[S_n]$. To simplify notations we write $Q^l[S_n]$ or $Q^*[S_n]$ instead of $(Q[S_n])^l$ or $(Q[S_n])^*$.

Lemma 4.12

Let Q be a subset of $T(\Sigma)$. $Q^*[S_n]$ is stable under the operation $\blacklozenge$, $Q^*[S_n]\blacklozenge Q^*[S_n] = Q^*[S_n]$.

Proof. Notice that Q is a subset of $T(\Sigma)$. This restriction is necessary to avoid the concatenation of trees from Q. Let t and θ be trees in $Q^*[S_n]$. There exist integers l and k such that $t \in Q^l[S_n]$ and $\theta \in Q^k[S_n]$. We show by induction on l+k that there exists an integer m such that $t\blacklozenge\theta\subset Q^m[S_n]$. If $l+k = 2$ (i.e. $l = k = 1$) then, by definition, $t \blacklozenge \theta\subset Q^2[S_n]$. Let us suppose the property true for every pair (l, k) such that $l+k < n$. If $l=1$ then, by definition, $t\blacklozenge\theta\subset Q^{k+1}[S_n]$. Otherwise there are $t_1 \in Q[S_n]$ and $t_2\in Q^{l-1}[S_n]$ such that $t \in t_1 \blacklozenge t_2$. There exists an integer p such that $t_2 \blacklozenge \theta \subset Q^p[S_n]$ (induction hypothesis). From $(t_1 \blacklozenge t_2) \blacklozenge \theta$ included in $t_1 \blacklozenge (t_2 \blacklozenge \theta)$ (Proposition 2.3), we can deduce $t \blacklozenge \theta\subset t_1 \blacklozenge (t_2 \blacklozenge \theta)\subset Q^{p+1}[S_n]$ and this prove the property of stability.

Let S be an unavoidable subset of $T(\Sigma,\omega)$-$T(\Sigma)$ and k its avoidance bound. We are able to prove now that $T(\Sigma)$ is included in $R_k^*[S_k]$ where R_k is the subset of trees from $T(\Sigma)$ whose depth is less than or equal to the avoidance bound k.

Definition 4.13

$R_n = \{t \in T(\Sigma) \mid depth(t)\leq n\}$.

Theorem 4.14 (Structure Theorem)

Let S be an unavoidable subset of $T(\Sigma,\omega) - T(\Sigma)$ with avoidance bound k. Then $T(\Sigma) \subset R_k^*[S_k]$.

Proof. For every integer n, $R_n \subset R_n[S_n] \subset R_n^*[S_n]$, so $T(\Sigma) \subset \cup_{n\in N} R_n \subset \cup_{n\in N} R_n^*[S_n]$. Let θ a minimal tree (with respect to the number of nodes) in $T(\Sigma) - R_k^*[S_k]$. As $R_k \subset R_k^*[S_k]$, depth(θ) ≥ k+1. Furthermore, as S is unavoidable with avoidance bound k, there exists an s in S with $s \leq_{factor} \theta$. Hence, we deduce from Lemma 3.4 the existence of an integer n, n+1 trees $t_0, \ldots, t_n$ belonging to $T(\Sigma,\omega)$ and a vertex u in t_0 with $|u| \leq k$ such that θ belongs to $t_0[u \leftarrow t_1 \blacklozenge (\ldots (t_n \blacklozenge s))]$. Clearly $t_0[u \leftarrow t_n], t_1, \ldots, t_{n-1}$ each has fewer nodes than θ. Since θ is minimal, these trees belong to $R_k^*[S_k]$. By Proposition 4.8, as $|u| \leq k$, $t_0[u \leftarrow (t_n \blacklozenge s)] \subset R_k^*[S_k]$ and then, by Lemma 4.12, $\theta \in R_k^*[S_k]$ and we get a contradiction.

Let S be an unavoidable set with avoidance bound k. The theorem above states that any tree of $T(\Sigma)$ either belongs to R_k whose elements are called remainders, or can be split into trees each of which is built by insertion of nested trees from S into a remainder. From this result we prove furthermore that there exists, for any tree, a decomposition satisfying an additional property: the number of remainder vertices we go through from the root to any leaf is less than the avoidance bound.

Definition 4.15 Residual Branch Height

Informally, RBH(t) is the maximum, over all leaves, of the number of vertices belonging to remainders on each path from the root to the leaf. Formally, let Q and S be two subsets of $T(\Sigma)$ and $T(\Sigma, \omega)$ respectively and k an integer. Let $t \in Q^*[S_k]$ and $t = t_0[\ldots(u_i,v_i)\ldots](\ldots s_i[w_j^i \leftarrow \theta_j^i \mid j \in L_i]\ldots)$ with $t_0 \in Q$, $s_i \in S_k$, $\theta_j^i \in Q^*[S_k]$ be a decomposition of t. Define the residual branch height of this decomposition of t, D(t), with respect to Q and S recursively by

$$RBH(D(t)) = \max(\mathrm{height}(t_0)+1, \max_i (|u_i| + \max_j RBH(\theta_j^i))) \text{ if } t \neq \omega.$$

and RBH(t), the residual branch height of the tree t, by the minimal residual branch height of all such decomposition of t if $t \neq \omega$ and $RBH(\omega) = 0$.

Definition 4.16

Let Q and S be two subsets of $T(\Sigma)$ and $T(\Sigma, \omega)$ respectively and k, m be integers. Define $Q^{||m||}[S_k] = \{t \in Q^*[S_k] \mid RBH(t) \leq m\}$.

We shall need property P.

Definition 4.17 Property P

Let $t \in R_k^*[S_k]$. t satisfies property P if and only if there exists a decomposition D(t) of

$t = t_0[\ldots(u_i,v_i)\ldots](\ldots s_i[w_j^i \leftarrow \theta_j^i \mid j \in L_i]\ldots)$ such that

$height(t_0) \leq k$

$\forall$ i, j, $\theta_j^i \in R_k^*[S_k]$

$\forall$ i, $s_i \in S_k$ (s_i is called a decomposition pattern)

$\forall$ i, u_i minimal under prefix ordering over the vertices of insertion, implies $s_i \in S_{k-|u_i|}$ (a decomposition pattern at a minimal vertex is called a minimal pattern)

$RBH(t) = RBH(D(t)) \leq k+1$

Lemma 4.18

Let $t \in R_k^*[S_k]$ satisfy property (P). Choose a decomposition of t of the kind guaranted by Property P. If s_i is a minimal pattern, then for every j $RBH(\theta_j^i) \leq k+1-|u_i|$.

Proof. It follows from the definition of RBH(t).

Lemma 4.19

Let $t \in R_k^*[S_k]$ satisfy Property P. Choose a decomposition of t of the kind guaranted by Property P. Let w be a vertex of t such that no u_i is a prefix of w. Then $RBH(t/w) \leq RBH(t) - |w|$.

Proof. The decomposition of t/w,D(t/w), induced by that of t satisfiesthe following inequalities:

$$\begin{aligned} RBH(D(t/w)) &\leq \max(height(t_0/w) + 1, \max_{u_i \geq_{prefix} w} (|u_i| - |w| + \max_j (RBH(\theta_j^i)))) \\ &\leq \max(height(t_0) - |w| + 1, \max_{u_i} (|u_i| - |w| + \max_j (RBH(\theta_j^i)))) \\ &\leq RBH(t) - |w| \end{aligned}$$

Thus $RBH(t/w) \leq RBH(D(t/w)) \leq RBH(t) - |w|$.

Lemma 4.20

Let $t \in R_k^*[S_k]$ satisfying Property P. Choose a decomposition of t of the kind guaranted by Property P. Let w be a vertex of t such that no u_i is a prefix of it. Let $\theta = \theta_0[\ldots(v_i,\mu_i)\ldots](\ldots\sigma_i[\ldots]\ldots)$ be a tree satisfying Property P. Suppose that

i) at every minimal vertex v_i, $\sigma_i \in S_{k-|v_i|-|w|}$,

ii) $RBH(\theta) \leq k + 1 - |w|$,

iii) $height(\theta_0) \leq k - |w|$.

Then $t[w \leftarrow \theta]$ satisfies (P).

Proof. We deduce from the given decomposition of θ and the previous one of t a decomposition of $t[w \leftarrow \theta]$ that is $t_0 [w \leftarrow \theta_0] [\ldots(u_i, v_i)\ldots(wv_j,\mu_j)\ldots](\ldots s_i[\ldots] \ldots \sigma_j[\ldots] \ldots)$.

$$\begin{aligned}\text{height}(t_0 [w \leftarrow \theta_0]) &= \max (\text{height} (t_0), |w| + \text{height} (\theta_0)) \\ &\leq \max (\text{height} (t_0), k) \\ &\leq k\end{aligned}$$

$\forall$ i, $s_i \in S_k$. $\forall$ i, $\sigma_i \in S_k$

$\forall$ u_i minimal, $s_i \in S_{k-|u_i|}$

$\forall$ v_i minimal, $\sigma_i \in S_{k-|v_i|-|w|} = S_{k-|wv_i|}$

$$\begin{aligned}\text{RBH} (t[w \leftarrow \theta]) &\leq \max (\text{RBH} (t), |w| + \text{RBH} (\theta)) \\ &\leq \max (\text{RBH} (t), |w| + k + 1 - |w|) \\ &\leq k + 1\end{aligned}$$

Proposition 4.21

Let S be an unavoidable subset of $T(\Sigma,\omega)$-$T(\Sigma)$ with avoidance bound k. Every tree $t \in R_k{}^*[S_k]$ satisfies property (P).

Proof. Let t be a tree in $R_k{}^*[S_k]$ not satisfying (P), minimal with respect to $\leq_{vertex}$. Clearly the elements of R_k satisfy (P), thus height (t) > k. As S is unavoidable with bound k, by Lemma 3.4, there exist $s \in S$, $\theta_0, \theta_1 \ldots \theta_n$ in $T(\Sigma,\omega)$ and u in Vertex(θ_0) such that $|u| \leq k$ and $t=\theta_0[u \leftarrow s[w_i \leftarrow \theta_i \mid 1 \leq i \leq n]]$. Let us choose, among these decompositions, one in which u is minimal under the prefix ordering. For every i ($1 \leq i \leq n$), $t_i = \theta_0[u \leftarrow \theta_i]$ satisfies (P) because $t_i <_{vertex}$ t, and the associated decomposition is $t_i = \tau [\ldots(u_j, v_j)\ldots](\ldots s_j[w_l{}^j \leftarrow \tau_l{}^j \mid l \in L^j]\ldots)$ where v_j is a vertex of s_j. There are two cases.

Case 1. There exist a tree t_i and an index j such that $u_j v_j \leq_{prefix}$ u. In fact, we consider such a u_j minimal for the prefix ordering. In the following, let us use (v, μ) for the pair (u_j, v_j), σ for $s_j \in S_{k-|v|}$, θ_l' for $\tau_l{}^j$, L for L^j and w'_l for $w^j{}_l$ (Figure 4.6).
With this notation, as t_i satisfies (P), we get

$t_i = \tau [(v, \mu)\ldots](\sigma[w'_l \leftarrow \theta_l' \mid l \in L]\ldots)$

$\sigma \in S_{k-|v|}$

$\forall$ l, $\theta_l' \in R_k{}^*[S_k]$, RBH(θ_l') $\leq k + 1 - |v|$ (this is a consequence of Lemma 4.19).

Let K = { $u'j \in$ Vertex(t) | j an integer, $u' \leq_{prefix} v$ and $u'j$ and v incomparable}. For every $w \in K$, the decomposition of the subtree of t at w, t/w, induced by the decomposition of t_i satisfies (P). Furthermore, if v_i is a minimal vertex of this decomposition, the associated minimal pattern belongs to $S_{k-|v_i|-|w|}$ and RBH(t/w) <k + 1 - |w| (Lemma 4.20).

Let $t' = t[v \leftarrow t/v\mu][w \leftarrow \omega \mid w \in K]$ where ω is constant (Figure 4.7). To build t' from t, we remove the decomposition pattern σ and substitute for it the subtree $t/v\mu$, and then substitute the constant ω at all the minimal vertices of t that are not prefixes or suffixes of v.

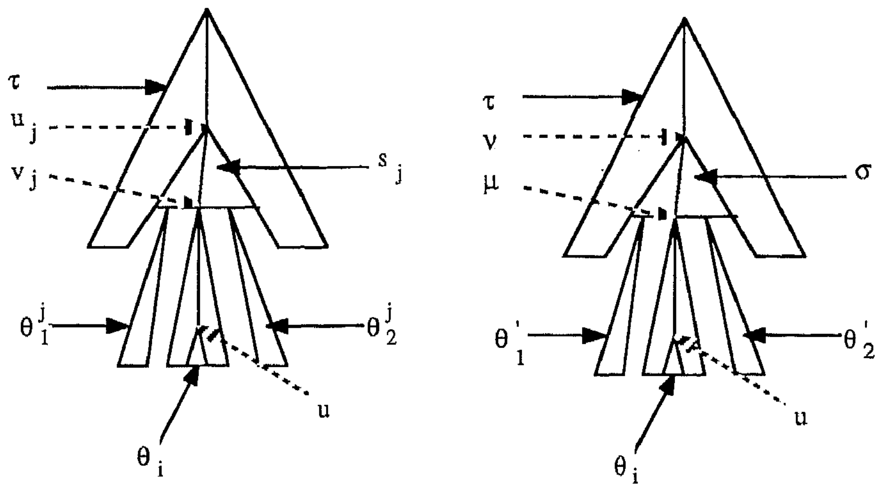


Figure 4.6

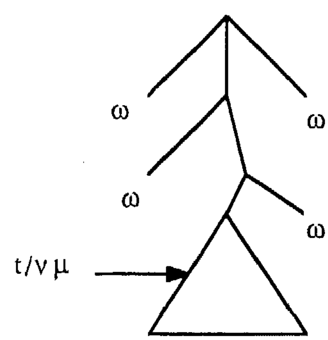


Figure 4.7

As $t' <_{vertex} t$, t' satisfies property (P) and has the decomposition below:

$$t' = \tau'[\ldots(u_i', v_i')\ldots](\ldots s_i'[\, w_j^i \leftarrow \theta_j^i \mid j \in L_i]\ldots)$$

We prove that (P) is preserved when t is rebuilt from t'.

Notice that there is no $u_i'v_i'$ that is a prefix of v because of the shape of the tree above the vertex v and because, in the decomposition of t, the vertex u where there exists an element of S has been chosen minimal for the prefix ordering. Thus we have to consider the two cases below.

–There is no $u_i' <_{prefix} v$. The decomposition of t'/v deduced from the decomposition of t' satisfies (P), RBH(t'/v) <k + 1 - |v| (Lemma 4.19) and height(t'/v)≤height(t')-|v|. Furthermore $\sigma[\mu \leftarrow t'/v][w'_l \leftarrow \theta'_l | l \in L]$ satisfies (P) because $RBH(\sigma[\mu \leftarrow t'/v][w'_l \leftarrow \theta'_l | l \in L])=\max(RBH(t'/v), \max_{l \in L}(RBH(\theta'_l))) \leq k+1-|v|$ and the other properties are trivially satisfied. Now, Lemma 4.20 allows us to conclude that $t=t'[v \leftarrow \sigma[\mu \leftarrow t'/\mu][w_l \leftarrow \theta'_l | l \in L]][w \leftarrow t/w | w \in K]$ satisfies (P) and thus that RBH(t)≤k+1.

–In the other case, some u'_i, u'_0 for example, satisfies $u'_0 <_{prefix} v$. Thus $v=u'_0v'$. Since $u'_0v'_0$ is not a prefix of v, as mentioned above, $v' \leq_{prefix} v'_0$ because of the shape of t' above the vertex v ($v'_0=v'v''$). The insertion of σ is performed inside an element $s'_0 \in S_{k-|u'_0|}$. This insertion of an element of $S_{k-|v|}$ in an element of $S_{k-|u'_0|}$ at a vertex of depth $|v|-|u'_0|$ generates an element σ' of $S_{k-|u'_0|}$ and an element σ'' of $S_{k-|v|}$ as seen in corollary 4.10 (Figure 4.8). Let $v'\mu''$ be the vertex of σ' from which σ'' is hanging. The set $\{\theta_l \mid l \in L\}$ of trees hanging from terminal vertices of σ is split in two subsets, $\{\theta_l \mid l \in L'\}$ hanging from σ' and $\{\theta_l \mid l \in L''\}$ hanging from σ''.

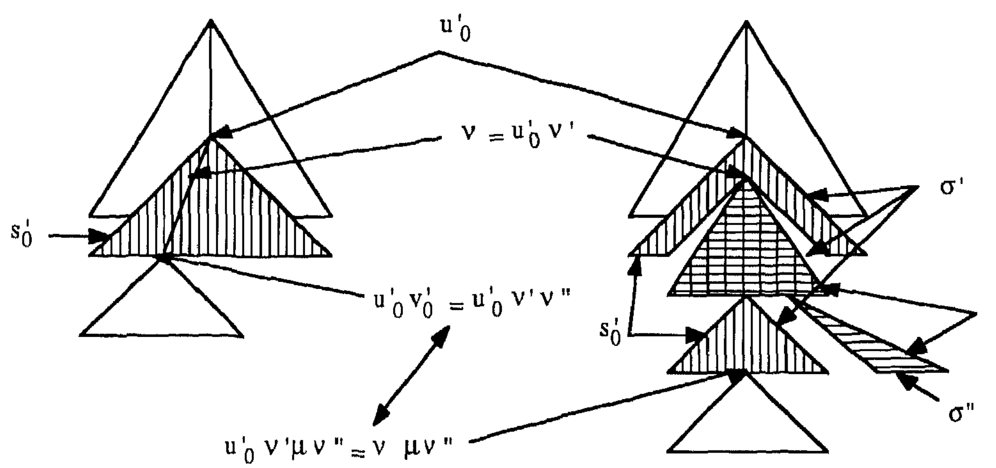


Figure 4.8

With the notation above, t can be seen as the following tree:
$t=t'[u'_0 \leftarrow \sigma'[v'w_l \leftarrow \theta_l | l \in L'][v'\mu \leftarrow t'/v][v'\mu'' \leftarrow \sigma''[w_l \leftarrow \theta_l | l \in L'']]][w \leftarrow t/w | w \in K]$.
Now using the decomposition of t' we get the following decomposition for t and we prove that it satisfies (P).

$t = (\ \tau'[(u'_0, v'\mu v'') \ldots (u'_i, v'_i) \ldots]$

$(\ (\sigma'[v'w_l \leftarrow \theta_l | l \in L'][w_l{}^0 \leftarrow \theta_l{}^0 | l \in L_0]\ [\ v'\mu'' \leftarrow \sigma''[w_l \leftarrow \theta_l | l \in L'']]) \ldots s'_i[w_l{}^i \leftarrow \theta_l{}^i | l \in L_i] \ldots\ ))$

$[w \leftarrow t/w | w \in K]$ with $i \neq 0$.

The head residual tree τ'' of the decomposition of t induced by the decomposition above is equal to $\tau'[\ w \leftarrow \tau/w | w \in K]$. Then the following properties hold.

- $\text{height}(\tau'') \leq \max(\text{height}(\tau), \text{height}(\tau')) \leq k$.
- For $i \neq 0$ the properties of decomposition patterns s'_i are preserved and $\sigma' \in S_{k-|u'_0|}$.
- For every $l \in L'$, $\theta_l \in R_k{}^*[S_k]$

  For every $l \in L_0$, $\theta_l{}^0 \in R_k{}^*[S_k]$

  $\sigma''[w_l \leftarrow \theta_l | l \in L''] \in R_k{}^*[S_k]$ because σ'' and its suspended trees belong to $R_k{}^*[S_k]$.
- $\text{RBH}(t)$ $\leq \max(\text{RBH}(t'), |u'_0| + \max_{l \in L',L''}(\text{RBH}(\theta_l)), \max_{w \in K}(\text{RBH}(t/w) + |w|))$

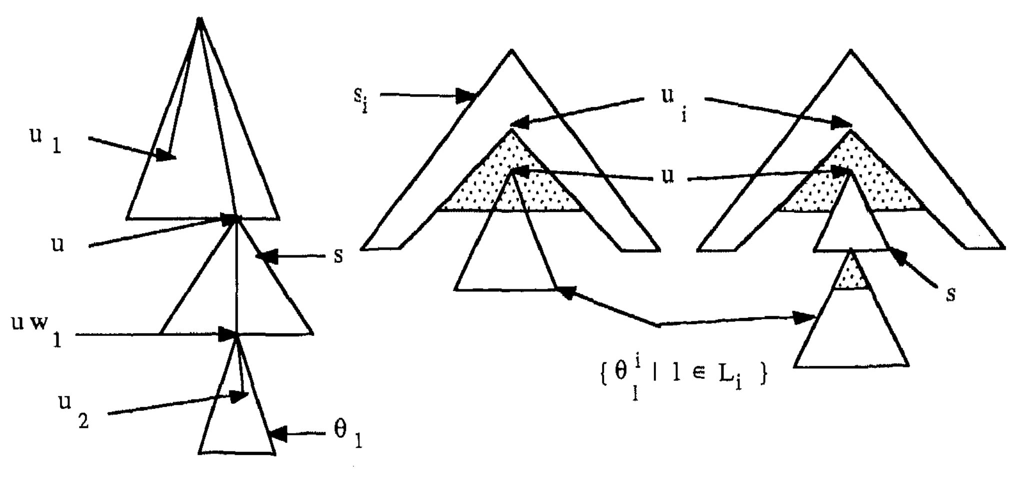
  $\leq \max\ (\text{RBH}(t'), |u'_0| + k - |v| + 1, |w| + k + 1 - |w|)$

  $\leq k+1$

In that case also, we conclude that t satisfies (P).

Case 2. There is no t_i such that there exists an index j with $u_j v_j \leq_{prefix} u$.

We consider then, $t_1 = \theta_0[u \leftarrow \theta_1]$ that satisfies (P). Thus, there exists a decomposition $t_1 = \tau[\ldots(u_j, v_j)\ldots](\ldots s_j[w_l{}^j \leftarrow \theta_l{}^j | l \in L_j]\ldots)$. There are two subcases.



case 2.1 case 2.2

Figure 4.9

Subcase 2.1 There is no j such that $u_j <_{prefix} u$, and thus the decomposition of $\theta_1 = t_1/u$ deduced from the decomposition of t_1 satisfies (P). Furthermore, if v_i is a minimal vertex of the decomposition, the corresponding minimal pattern belongs to $S_{k-|v_i|-|u|}$ and $RBH(\theta_1) \leq k+1-|u|$ by Lemma 4.19.

Subcase 2.2 There is i such that $u_i <_{prefix} u$ ($u = u_i u'$). The insertion of s is performed inside the pattern s_i. We deduce from Lemma 4.9 that $s_i[(u',w_1)]s$ belongs to $S_{k-|u_i|}$ because $s \in S$, $s_i \in S_{k-|u_i|}$ and $|u'| = |u| - |u_i| \leq k - |u_i|$.

For these two subcases we can use the same notation: $t_1[(u,w_1)]s = \tau[v \leftarrow \sigma[w'''_l \leftarrow \theta'''_l | l \in L''']]$ where v is either u or u_i, σ is either s or $s_i[(u',w_1)]s$ and $\{\theta'''_l\}$ is either $\{\theta_1\}$ or $\{\theta^i_l | l \in L_i\}$. In each of these cases, $\sigma \in S_{k-|v|}$, $\theta'''_l \in R_k^*[S_k]$ and $RBH(\theta'''_l) \leq k+1-|v|$. Thus, by Lemma 4.20, $t_1[(u,w_1)]s$ satisfies Property P. Let $K = \{u'j \in Vertex(t_1) | u' <_{prefix} v$ and $u'j$ and v incomparable$\}$. For every w in K, the decomposition of t_1/w deduced from the decomposition of t_1 satisfies (P). Furthermore, if v_i is a minimal vertex of this decomposition, the corresponding minimal pattern belongs to $S_{k-|v_i|-|w|}$ and $RBH(t_1/w) \leq k+1-|w|$ by Lemma 4.18. Let $t_2 = t_1[v \leftarrow \theta_2][w \leftarrow \omega | w \in K]$. As $t_2 <_{vertex} t$, t_2 satisfies (P) and there exists a decomposition $t_2 = \tau'[\ldots(u'_j,v'_j)\ldots](\ldots s'_j[w_j^i \leftarrow \theta_j^i | j \in L_i]\ldots)$. Because of the minimal choice of s, there is no (u'_i,v'_i) such that $u'_i v'_i \leq_{prefix} v$. Thus the insertion of $\sigma[w'''_l \leftarrow \theta'''_l | l \in L''']$ in t_2 is performed either at a vertex for which no u'_i is a prefix, or inside a pattern s'_i and we can prove, like above, by Lemma 4.20, that the new tree satisfies (P). Using Lemma 4.20, when we substitute in this new tree, at vertices $w \in K$, the trees t/w, we get a tree that satisfies (P). Thus, the tree $\tau[u \leftarrow s[w_1 \leftarrow \theta_1, w_2 \leftarrow \theta_2]]$ satisfies (P). A similar proof shows that $t = \tau[u \leftarrow s[w_i \leftarrow \theta_i | 1 \leq i \leq n]]$ satisfies (P). In conclusion, for every tree $t \in R_k^*[S_k]$, $RBH(t) \leq k+1$.

Theorem 4.22

If S is factor-unavoidable with bound k, then $T(\Sigma) \subset R_k^{\|k+1\|}[S_k]$.

Proof. This result is a consequence of Theorem 4.14 and Proposition 4.21.

5 QUASI-ORDERINGS ON T(Σ,ω) AND THEIR PROPERTIES.

We define a quasi-ordering related to the insertion of trees belonging to a given subset of $T(\Sigma,\omega)$. If S is a subset of $T(\Sigma,\omega) - T(\Sigma)$, we define the tree insertion ordering TIO(S), denoted $\leq_S$, over $T(\Sigma,\omega)$ by $t \leq_S t'$ if and only if t' is built from t by insertion of trees from S. Some leaves of t' may therefore be labelled by ω. In order to compare trees belonging to T(Σ) we define another relation $\leq_{S\omega}$.

Definition 5.1 (Relation I_S and quasi-ordering $\leq_S$ on T(Σ,ω)) .

Let S be a subset of T(Σ,ω) -T(Σ) and t and t' trees in T(Σ,ω). t I_S t' if and only if t=t' or there exists s in S such that t' ∈ t[ ]s. $\leq_S$ is the transitive closure of I_S, i.e. t $\leq_S$ t' if and only if there exists a finite sequence $t_0,\ldots,t_n$ such that $t_0 = t$, $t_n = t'$ and for every i $(0 \leq i \leq n-1)$ $t_i\ I_S\ t_{i+1}$.

Definition 5.2 (Relation I_ω and quasi-ordering $\leq_\omega$ on T(Σ,ω)) .

Let τ_i and θ_i be trees in T(Σ,ω) and f an element of the alphabet Σ. Define I_ω by

a) ω I_ω t for every tree t in T(Σ,ω).

b) $\tau_i\ I_\omega\ \theta_i$ implies f... τ_i ... I_ω f... θ_i ...

$\leq_\omega$ is the transitive closure of I_ω.

Definition 5.3 (Quasi-ordering $\leq_{S\omega}$ on T(Σ,ω)) .

Let S be a subset of T(Σ,ω) -T(Σ) and t and t' trees in T(Σ,ω). t$\leq_{S\omega}$ t' if and only if there exists t'' in T(Σ,ω) such that t $\leq_S$ t''$\leq_\omega$ t'.

It is proved below that $\leq_{S\omega}$ is a quasi-ordering.

Example 5.4

Let Σ = {f, g, y } with arities 2, 1, 0 respectively.

Let s_1 = f(f(ω,ω),ω), s_2 = f(ω,f(ω,ω)), s_3 = g(g(ω)), s_4 = g(f(ω,ω)).

Let S = {s_1, s_2, s_3, s_4}. Let t = f(y,y), t' = f(g(g(y)),y) and t'' = f(f(y,f(y,g(g(y)))),y).

Obviously t'∈ t [ ] s_3 and thus t I_S t'.

Then t $\leq_S$ t'' because t I_S f(f(ω,f(y,ω)),y) I_S f(f(ω,f(y,g(g(ω)))),y).

Let t''' = f(f(y,f(y,g(g(y)))),y). t $\leq_{s\omega}$ t''' (Figure 5.1) because

t I_S f(f(ω,f(y,ω)),y) I_S f(f(ω,f(y,g(g(ω)))),y) I_ω f(f(y,f(y,f(y,g(g(ω))))),y) I_ω t'''.

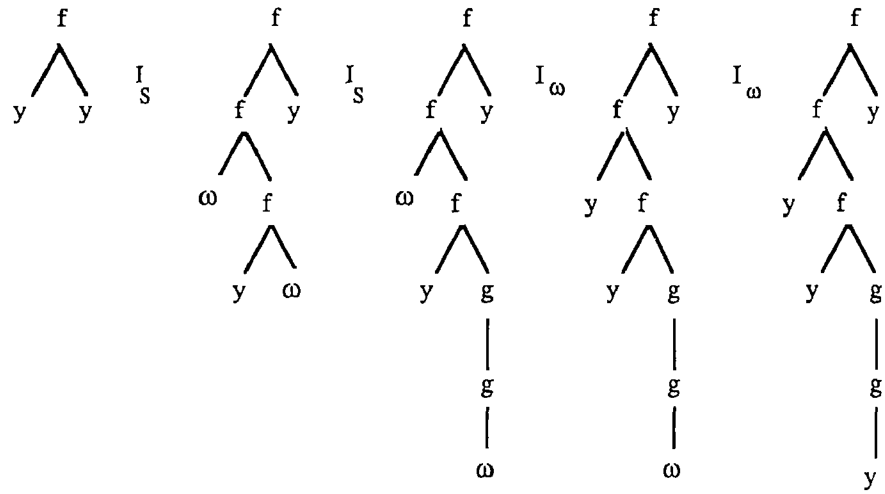


Figure 5.1

Lemma 5.5 Commutation of I_ω and I_S

Let S be a subset of $T(\Sigma,\omega)$ -$T(\Sigma)$ and t, t' and t'' trees in $T(\Sigma,\omega)$. If $t\ I_\omega\ t' I_S t''$ then there exists τ in $T(\Sigma,\omega)$ such that $t\ I_S\ \tau\ I_\omega\ t''$.

Proof. If $t\ I_\omega\ t'\ I_S\ t''$ there are u in Vertex(t) with $t(u) = \omega$, v in Vertex(t'), θ in $T(\Sigma,\omega)$, s in S and w in Vertex (s) with $s(w)=\omega$ such that $t' = t[u\leftarrow\theta]$ and $t'' = t'[v\leftarrow s[w\leftarrow t'/v]]$.

– If u and v are incomparable then $t[u\leftarrow\theta]/v = t/v$ and thus

$$\begin{aligned} t'' &= t[u\leftarrow\theta]\ [v\leftarrow s[w\leftarrow t[u\leftarrow\theta]/v]] \\ &= t\ [v\leftarrow s[w\leftarrow t/v]][u\leftarrow\theta] . \end{aligned}$$

– If $v \leq_{prefix} u$ (i.e. $u = vv'$) then $t[u\leftarrow\theta]/v = t/v[v'\leftarrow\theta]$

$$\begin{aligned} t'' &= t[u\leftarrow\theta]\ [v\leftarrow s[w\leftarrow t[u\leftarrow\theta]/v]] \\ &= t[v\leftarrow s[w\leftarrow t/v[v'\leftarrow\theta]]] \\ &= t[v\leftarrow s[w\leftarrow t/v]][vwv'\leftarrow\theta] . \end{aligned}$$

In these both cases, if $\tau = t[v\leftarrow s[w\leftarrow t/v]]$ then $t\ I_S\ \tau\ I_\omega\ t''$.

– If $u \leq_{prefix} v$ (i.e. $v = uu'$) then $t[u\leftarrow\theta]/v = \theta/u'$

$$\begin{aligned} t'' &= t[u\leftarrow\theta]\ [v\leftarrow s[w\leftarrow t[u\leftarrow\theta]/v]] \\ &= t[u\leftarrow\theta]\ [v\leftarrow s[w\leftarrow\theta/u']] \\ &= t[u\leftarrow\theta[u'\leftarrow s[w\leftarrow\theta/u']]] \end{aligned}$$
and thus $t\ I_S\ t\ I_\omega\ t''$.

Lemma 5.6 Commutation. of $\leq_S$ and $\leq_\omega$

Let t, t' and t'' trees in $T(\Sigma,\omega)$. If $t \leq_\omega t' \leq_S t''$ then there exists τ in $T(\Sigma,\omega)$ such that $t \leq_S \tau \leq_\omega t''$.

Proof. It is a direct consequence of the definitions and Lemma 5.5.

Lemma 5.7

The relation $\leq_{S\omega}$ is a quasi-ordering on $T(\Sigma,\omega)$.

Proof. Clearly this relation is reflexive, so we only need to prove the property of transitivity. Let t_1, t_2 and t_3 be trees such that $t_1 \leq_{S\omega} t_2 \leq_{S\omega} t_3$. By definition, there are t' and t'' such that $t_1 \leq_S t' \leq_\omega t_2 \leq_S t'' \leq_\omega t_3$. There exists a tree τ such that $t' \leq_S \tau \leq_\omega t''$ (Lemma 5.6). Then by transitivity of $\leq_S$ and $\leq_\omega$, $t_1 \leq_S \tau \leq_\omega t_3$ and thus $t_1 \leq_{S\omega} t_3$.

Lemma 5.8

Let t, t' and t'' be in $T(\Sigma,\omega)$, v in Vertex(t'') such that $t \; I_S \; t'$ (resp. $t \leq_S t'$, $t \leq_{S\omega} t'$). Then $t''[v\leftarrow t] \; I_S \; t''[v\leftarrow t']$ (resp. $\leq_S$, $\leq_{S\omega}$).

Proof. As $t \; I_S \; t'$ there exists s in S, μ in vertex(t) and ν in vertex(s) such that $t'=t[(\mu, \nu)]s$. Inserting t' in t'' at vertex v, we get the following tree:

$t''[v\leftarrow t'] = t''[v\leftarrow t[(\mu, \nu)]s] = (t''[v\leftarrow t])[(v\mu, \nu)]s$ and thus $t''[v\leftarrow t] \; I_S \; t''[v\leftarrow t']$. By iteration of this proof, we get the analogous property for $\leq_S$ and $\leq_{S\omega}$.

Because of Lemma 5.8 these three relations are said to be stable under grafting.

Lemma 5.9

Let $t_1,\ldots, t_l, t'_1,\ldots, t'_l$ and t'' be in $T(\Sigma,\omega)$. Let $u_1,\ldots, u_l$ be l elements in Vertex(t'') pairwise incomparable under prefix ordering. If for every integer i ($1\leq i\leq l$) $t_i I_S t'_i$ (resp. $t_i \leq_S t'_i$, $t_i \leq_{S\omega} t'_i$) then $t''[u_i\leftarrow t_i | 1\leq i\leq l] \; I_S$ (resp. $\leq_S$, $\leq_{S\omega}$) $t''[u_i\leftarrow t'_i | 1\leq i\leq l]$.

Proof. This proof is analogous to the preceding proof.

Lemma 5.10

Let t, t' and t'' be in $T(\Sigma,\omega)$, s in S, u and w in Vertex(t), u' in Vertex(t') and v in Vertex(s). If $t' = t[(w,v)]s$ so $t \; I_S \; t'$, then $t[u\leftarrow t''] \; I_S \; t'[u'\leftarrow t'']$ if one of the following case holds:

i) $w \leq_{prefix} u$ (i.e. $u = ww'$) and $u' = wvw'$
ii) w is not a prefix of u and $u' = u$.

Proof. Intuitively, $t[u \leftarrow t''] \ I_S \ t'[u' \leftarrow t'']$ if the insertion vertices in t and t' are the same after erasing s in t', where s is the tree inserted in t to get t'. The proof is only a computation of different substitutions in a tree.

Let S be a finite subset of $T(\Sigma,\omega)$ -$T(\Sigma)$ and Q a finite subset of $T(\Sigma,\omega)$. In section 6 we prove that $\leq_S$ is a well quasi-ordering on S_n, then on $Q[S_n]$ as defined in section 4. In order to show by induction on n that S_n is well quasi-ordered by $\leq_S$, the trees have to be split into subtrees belonging to T_n as in the definition of S_n. Then these subtrees have to be considered separately and the initial trees rebuilt. But at this point there is a technical problem. Roughly speaking, the relation $\leq_S$ is not stable under the operation ♦ (Lemma 5.10), so we have to keep track of the vertices where the trees were split. For this purpose these vertices are relabelled by new special constants in such a way that each new constant occurs at most once in a tree, and we extend the quasi-orderings defined above to the trees built on the new alphabet to get the following property: If t and t' are two trees with only one occurence each of a new constant, ω' for example, and $t \leq_S t'$ then $t[\omega' \leftarrow t''] \leq_S t'[\omega' \leftarrow t'']$.

We add new constants $\omega', \omega_1, \ldots, \omega_k$ to the alphabet $\Sigma \cup \{\omega\}$ and we extend the relations I_S, $\leq_S$, and $\leq_{S\omega}$ to $T(\Sigma \cup \{\omega, \omega', \omega_1, \ldots, \omega_k\})$ in the following way.

Definition 5.11 (Relation I_S and quasi-ordering $\leq_S$ on $T(\Sigma,\omega,\omega',\omega_1,\ldots,\omega_k)$).

Let S be a subset of $T(\Sigma,\omega)$ -$T(\Sigma)$ and t and t'trees in $T(\Sigma,\omega,\omega', \omega_1,\ldots,\omega_k)$. $t \ I_S \ t'$ if and only if $t = t'$ or there exists s in S such that $t' \in t[\ ]s$. $\leq_S$ is the transitive closure of I_S.

Definition 5.12 (Relation I and quasi-ordering $\leq_\omega$ on $T(\Sigma,\omega,\omega',\omega_1,\ldots,\omega_k)$).

Let τ_i and θ_i be trees in $T(\Sigma,\omega,\omega', \omega_1,\ldots,\omega_k)$ and f an element of the alphabet Σ. Define I_ω by

a) $\omega \ I_\omega \ t$ for every tree t in $T(\Sigma,\omega)$.

b) $\tau_i \ I_\omega \ \theta_i$ implies $f\ldots\tau_i\ldots \ I_\omega \ f\ldots\theta_i\ldots$

$\leq_\omega$ is the reflexive transitive closure of I_ω.

Definition 5.13 (Relation $\leq_{S\omega}$ on $T(\Sigma,\omega,\omega', \omega_1,\ldots,\omega_k)$).

Let t and t' be trees in $T(\Sigma,\omega,\omega', \omega_1,\ldots,\omega_k)$ and S a subset of $T(\Sigma,\omega)$. We define the relation $\leq_{S\omega}$ by $t \leq_{S\omega} t'$ if and only if there is t'' in $T(\Sigma,\omega,\omega',\omega_1,\ldots,\omega_k)$ such that $t \leq_S t'' \leq_\omega t'$.

Remarks.

1) The relations I_ω and I_S commute in $T(\Sigma,\omega,\omega', \omega_1,\ldots,\omega_k)$. The proof is the same as for Lemma 5.5. It suffices to notice that, when $u \leq_{prefix} v$, θ belongs to $T(\Sigma,\omega)$ and thus $\theta[u' \leftarrow s[w \leftarrow \theta/u']]$ belongs to $T(\Sigma,\omega)$.

2)We use the same notation for the relations on $T(\Sigma,\omega,\omega', \omega_1,\ldots,\omega_k)$ and$T(\Sigma,\omega)$ because these relations are the same on $T(\Sigma,\omega)$.

3) The operation of insertion does not introduce new vertices labelled by one of the new constant $\omega',\omega_1,\ldots$ or ω_k because S does not contain any of $\omega',\omega_1,\ldots$ or ω_k. So let t and t' be trees in $T(\Sigma,\omega,\omega', \omega_1,\ldots,\omega_k)$ with only one vertex labelled by ω'. Then t I_S (resp. $\leq_S$, $\leq_{S\omega}$) t' implies t [$\omega' \leftarrow t''$] I_S (resp. $\leq_S$, $\leq_{S\omega}$)t'[$\omega' \leftarrow t''$] because the vertices of insertion are exactly the same in both cases. More generally, we get the following Lemma:

Lemma 5.14

Let t , t', τ and τ' be trees in $T(\Sigma,\omega,\omega', \omega_1,\ldots,\omega_k)$ such that in t and t' there is only one vertex labelled by ω'.

$$t \leq_S t' \text{ and } \tau \leq_S \tau' \text{ implies } t[\omega' \leftarrow \tau] \leq_S t'[\omega' \leftarrow \tau']$$
$$t \leq_{S\omega} t' \text{ and } \tau \leq_{S\omega} \tau' \text{ implies } t[\omega' \leftarrow \tau] \leq_{S\omega} t'[\omega' \leftarrow \tau']$$

Proof. First we prove this property for I_S and I_ω. Let u and u' be the vertices where ω' occurs in t and t' respectively. These vertices satisfy hypothesis of Lemma 5.10, so $t[\omega' \leftarrow \tau] I_S t'[\omega' \leftarrow \tau]$. By a same argument over ω'-vertices we prove $t[\omega' \leftarrow \tau] I_S t[\omega' \leftarrow \tau']$ $t[\omega' \leftarrow \tau] I_\omega t'[\omega' \leftarrow \tau]$ $t[\omega' \leftarrow \tau] I_\omega t[\omega' \leftarrow \tau']$. Then we get the result by induction on the length of the insertion sequence.

Lemma 5.15

Let t and t' be trees in $T(\Sigma,\omega,\omega', \omega_1,\ldots,\omega_k)$ with only one node labelled by ω'. Then t I_S (resp. $\leq_S$, $\leq_{S\omega}$)t' implies t $[\omega' \leftarrow \omega]$ I_S (resp. $\leq_S$, $\leq_{S\omega}$) t'$[\omega' \leftarrow \omega]$.

Proof. It is a consequence of the definition of the relations.

6 WELL QUASI-ORDERINGS

Suppose that s is a finite subset of $T(\Sigma,\omega)-T(\Sigma)$ and Q a finite subset of $T(\Sigma,\omega)$. We prove that $\leq_S$ is a well quasi-ordering on S_n and on $Q[S_n]$ for every integer n. Unfortunately, however, even $Q^{||1||}[S_n]$ is usually not well quasi-ordered under $\leq_S$. If Q contains trees belonging to T(Σ), then $Q^{||1||}[S_n]$ includes infinite subsets which are each pairwise incomparable under the relation $\leq_S$. For example, let Q = {a} and S = {f(f(ω, ω), ω)}. The set $\{t_i \mid i \in \mathbf{N}\} \subset Q^{||1||}[S_n]$ where $t_0 = f(f(a,\omega), \omega)$ and for every positive integer i, $t_{i+1} = t_i[(\varepsilon,12)]\ t_0$ is an infinite set of pairwise incomparable trees (Figure 6.1). That is the reason for which we introduced the relation $\leq_{S\omega}$ above and the notion of closure of a set by another one below.

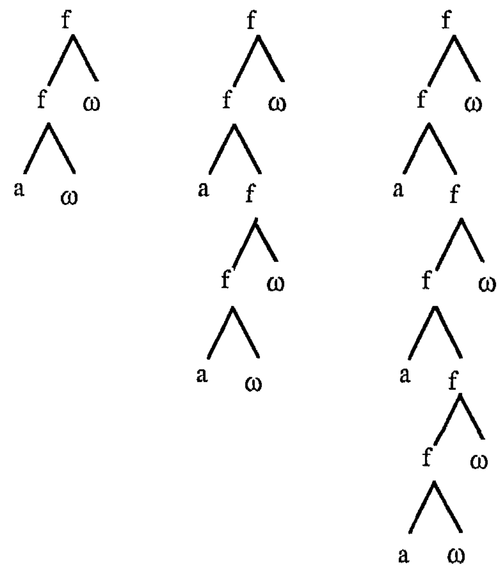


Figure 6.1

Definition 6.1

Let E and G be subsets of T(Σ,ω). Let E(G) be the subset of T(Σ,ω) whose elements are those of E in which trees from G have been substituted for occurrences of ω.

$E(G) = \{t \in T(\Sigma,\omega) \mid \exists t' \in E,\ u_i\ (1 \leq i \leq l)$ ω-vertex of t', $\tau_i\ (1 \leq i \leq l) \in G$ such that $t = t'[u_i \leftarrow \tau_i \mid 1 \leq i \leq l]\}$

Remarks.

1. E ⊂ E(G) because l can be equal to 0.

2. With this notation, $Q[S_n](Q) \subset Q^*[S_n]$ is the subset of trees built with elements of Q and only one element of $Q[S_n]$ - Q.

Notation

Let $\Omega=\{\omega_1,\ldots,\omega_k\}$ be a set of k new constant symbols which do not occur in $\Sigma\cup\{\omega\}$ and which can be used as ω in insertion operation. We denote by $t[\leftarrow\omega_1,\ldots,\omega_k]$ or $t[\leftarrow\Omega]$ the set of trees of the form $t[u_i\leftarrow\omega_i \mid 1\leq i\leq k]$ where t belongs to $T(\Sigma, \omega)$ and u_i are k terminal vertices labelled by ω in t. By extension $E[\leftarrow\Omega]=E[\leftarrow\omega_1,\ldots,\omega_k] = \cup_{t\in E} t[\leftarrow\omega_1,\ldots,\omega_k]$ where E is a subset of $T(\Sigma, \omega)$. Each term in $E[\leftarrow\Omega]$ contains each ω_i at most once. Notice that the only case where all k symbols $\omega_1,\ldots,\omega_k$ do not occur in a tree $t'\in E[\leftarrow\omega_1,\ldots,\omega_k]$ is when the tree t from which t' has been built contains less than k occurences of ω.

Lemma 6.2

If, for every integer k, $T_n[\leftarrow\omega_1,\ldots,\omega_k]$ is well quasi-ordered by $\leq_S$, then for every integer k, $S_n[\leftarrow\omega_1,\ldots,\omega_k]$ is also well quasi-ordered by $\leq_S$.

Proof. The proof is analogous to Higman's proof that words are well quasi-ordered. Let (t_i) be a minimal counterexample in $S_n[\leftarrow\omega_1,\ldots,\omega_k]$ (i.e. (t_i) is a sequence minimal with respect to $\leq_{vertex}$ that does not contain an infinite increasing subsequence). We may suppose that every tree t_i contains all the constants $\omega_1,\ldots,\omega_k$, since otherwise there exists an infinite subsequence such that all its elements contain the same subset Ω of constants, and then it is sufficient to consider this subset Ω. For each tree t_i there exist a tree τ_i in $T_n[\leftarrow\omega_1,\ldots,\omega_k, \omega']$ and a tree θ_i in $S_n[\leftarrow\omega_1,\ldots,\omega_k]$ such that $t_i=\tau_i[\omega'\leftarrow\theta_i]$. As $T_n[\leftarrow\omega_1,\ldots,\omega_k, \omega']$ is well quasi-ordered by $\leq_S$, all but a finite number of the θ_i are different from the trivial tree ω (if not, we can get a counterexample in $T_n[\leftarrow\omega_1,\ldots,\omega_k, \omega']$). Thus we can extract from (t_i) a subsequence $(t_{\gamma(i)})$ such that $\theta_{\gamma(i)}\neq\omega$ for every i and $(\tau_{\gamma(i)})$ is an ascending sequence in $T_n[\leftarrow\omega_1,\ldots,\omega_k, \omega']$. The corresponding set of $\theta_{\gamma(i)}$ is well quasi-ordered by $\leq_S$ because if not, we can find a counterexample θ'' and extract from it a subsequence $(\theta_{\delta(i)})$ which is also a subsequence of $(\theta_{\gamma(i)})$. Then the sequence $(\theta_{\delta(i)})$ is a counterexample and the sequence $t_1,\ldots,t_{\delta(1)-1},\theta_{\delta(1)},\theta_{\delta(2)},\ldots$ is also a counterexample, because if there exists an integer $j<\delta(1)$ and an integer i such that $t_j\leq_S\theta_{\delta(i)}$, then $\theta_{\delta(i)}$ contains all the constants $\omega_1,\ldots,\omega_k$, $\theta_{\delta(i)}\leq_S t_{\delta(i)}$ because in that case $t_{\delta(i)}$ is built from $\theta_{\delta(i)}$ by insertion of elements belonging only to S_n and thus $t_j\leq_S t_{\delta(i)}$. Furthermore this counterexample is smaller than (t_i). So, the set of $\theta_{\gamma(i)}$ being well quasi-ordered by $\leq_S$ there are two elements $\theta_{\gamma(i)}$ and $\theta_{\gamma(j)}$ such that $\theta_{\gamma(i)}\leq_S\theta_{\gamma(j)}$. As the sequence $(\tau_{\gamma(i)})$ is an ascending sequence in $T_n[\leftarrow\omega']$, $\tau_{\gamma(i)}[\omega'\leftarrow\theta_{\gamma(i)}]\leq_S\tau_{\gamma(j)}[\omega'\leftarrow\theta_{\gamma(j)}]$ by Lemma 5.15 and we get an ascending subsequence in $S_n[\leftarrow\omega_1,\ldots,\omega_k]$.

Lemma 6.3

If, for every integer k, $T_n[\leftarrow\omega_1,\ldots,\omega_k](G)$ is well quasi-ordered by $\leq_{S\omega}$, then, for every integer k, $S_n[\leftarrow\omega_1,\ldots,\omega_k](G)$ is also well quasi-ordered by $\leq_{S\omega}$.

Proof. We get this result by the same argument as in the previous Lemma. The only modification is the following one: $\theta_{\delta(i)} \leq_{S\omega} t_{\delta(i)}$ because in that case $t_{\delta(i)}$ is built from $\theta_{\delta(i)}$, not only by insertion of elements from S_n, but also by insertion of elements belonging to S_n in which elements from G have been substituted for ω.

Lemma 6.4

Let S be a finite subset of $T(\Sigma,\omega)$. For every integer n and for every integer k, $T_n[\leftarrow\omega_1,\ldots,\omega_k]$ and $S_n\,[\leftarrow\omega_1,\ldots,\omega_k]$ are well quasi-ordered by $\leq_S$.

Proof.

– case n=0: For every integer k,$T_0[\leftarrow\omega_1,\ldots,\omega_k] = S[\leftarrow\omega_1,\ldots,\omega_k]$ is finite and hence well quasi-ordered by $\leq_S$. Thus, as a consequence of Lemma 6.2, $S_0[\leftarrow\omega_1,\ldots,\omega_k]$ is well quasi-ordered by $\leq_S$.

– induction case: Let us suppose that for every integer n'<n and for every integer k $T_{n'}[\leftarrow\omega_1,\ldots,\omega_k]$ is well quasi-ordered by $\leq_S$. Then (Lemma 6.2) for each n' < n and each k, $S_{n'}[\leftarrow\omega_1,\ldots,\omega_k]$ is well quasi-ordered by $\leq_S$. Let (t_i) be an infinite sequence in $T_n[\leftarrow\omega_1,\ldots,\omega_k]$. As in the previous Lemma, we suppose that every tree t_i contains all the constants $\omega_1,\ldots,\omega_k$. As S is a finite set there exist a partition $\{\Omega_j \mid 0 \leq j \leq q\}$ of Ω, a tree $s_1 \in S[\leftarrow\Omega_0]$ and q vertices $u_i \in \mathrm{Vertex}(s_1)$ and p vertices $w_i \in \mathrm{Vertex}(s_1)$ such that we can extract from (t_i) an infinite subsequence $(t_{\gamma(i)})$ with the following property:

For every integer i there are

q trees $\theta_{j,\gamma(i)}$ in $S_{n-1}[\leftarrow\Omega_j]$ $(1 \leq j \leq q)$

p trees $\theta_{q+l,\gamma(i)}$ in S_{n-1} $(1 \leq l \leq p)$

with for each of these trees $\theta_{j,\gamma(i)}$ a vertex $v_{j,\gamma(i)}$ $(1 \leq j \leq q+p)$ such that

$t_{\gamma(i)} = s_1[\ldots(u_j,v_{j,\gamma(i)})\ldots(w_l,v_{l,\gamma(i)})\ldots](\ldots\theta_{j,\gamma(i)}\ldots\theta_{l,\gamma(i)}\ldots)$.

Let $\theta'_{j,\gamma(i)} = \theta_{j,\gamma(i)}[v_{j,\gamma(i)}\leftarrow\omega']$ for every j $(1\leq j \leq q)$.

Let $\tau'_{l,\gamma(i)} = \theta_{l,\gamma(i)}[w_{l,\gamma(i)}\leftarrow\omega']$ for every l $(1 \leq l \leq p)$.

Using the induction hypothesis and a generalization of Proposition 1.4, the product $S_{n-1}[\leftarrow\Omega_1\cup\{\omega'\}]\times\ldots\times S_{n-1}[\leftarrow\Omega_q\cup\{\omega'\}]\times S_{n-1}[\leftarrow\omega']\times\ldots\times S_{n-1}[\leftarrow\omega']$ is well quasi-ordered by the quasi-ordering product generated by $\leq_S$. So, there exist $\gamma(i) < \gamma(i')$ such that $(\theta'_{1,\gamma(i)},\ldots,\theta'_{q+p,\gamma(i)},\tau'_{1,\gamma(i)}\ldots,\tau'_{p,\gamma(i)}) \leq_S (\theta'_{1,\gamma(i')},\ldots,\theta'_{p,\gamma(i')},,\tau'_{1,\gamma(i')}\ldots,\tau'_{p,\gamma(i')})$. Using now the definition of T_n and Lemmas 5.8 and 5.9, we can conclude that $t_{\gamma(i)} \leq_S t_{\gamma(i')}$ and thus $T_n[\leftarrow\omega_1,\ldots,\omega_k]$ is well quasi-ordered by $\leq_S$. From Lemma 6.2, we conclude that for every integer k, $S_n[\leftarrow\omega_1,\ldots,\omega_k]$ is well quasi-ordered by $\leq_S$.

Lemma 6.5

Let S be a finite subset of $T(\Sigma,\omega)$ and G a subset of $T(\Sigma,\omega)$. If G is well quasi-ordered by $\leq_{S\omega}$, for every integer n and for every integer k, $T_n[\leftarrow\omega_1,\ldots,\omega_k](G)$ and $S_n[\leftarrow\omega_1,\ldots,\omega_k](G)$ are well quasi-ordered by $\leq_{S\omega}$.

Proof. case n=0 Let (t_i) be an infinite sequence in $T_0[\leftarrow\omega_1,\ldots,\omega_k](G)$ which is equal to $S[\leftarrow\omega_1,\ldots,\omega_k](G)$. $t_i = s_i[u_j^i\leftarrow\omega_j \mid 1\leq j\leq k][\, v_j^i\leftarrow\theta_j^i \mid j\in J_i]$ with $s_i\in S$, $\theta_j^i \in G$. As S is a finite set, there exist an element s of S and an infinite subsequence of (t_i), still denoted (t_i), such that $t_i = s[u_j\leftarrow\omega_j \mid 1\leq j\leq k][\, v_j\leftarrow\theta_j^i \mid j\in J]$ with $\theta_j^i \in G$. As $\leq_{S\omega}$ is a well quasi-ordering on G, the product $G^{|J|}$ is well quasi-ordered by $\leq_{S\omega}$. Thus, there are two integers q<r such that, for every $j\in J$, $\theta_j^q \leq_{S\omega} \theta_j^r$ .We deduce that $t_q \leq_{S\omega} t_r$, which proves $T_0[\leftarrow\omega_1,\ldots,\omega_k](G)$ well quasi-ordered by $\leq_{S\omega}$.
Induction case: In this part, we can use the same argument as in Lemma 6.4.

Proposition 6.6

Let S be a finite subset of $T(\Sigma,\omega)$ and G a subset of $T(\Sigma,\omega)$ well quasi-ordered by $\leq_{S\omega}$. For every integer n, S_n is well quasi-ordered by $\leq_S$ and $S_n(G)$ is well quasi-ordered by $\leq_{S\omega}$

Proof. It is an immediate consequence of Lemmas 6.4 and 6.5 where we take k=0. Thus $S_n=S_n[\leftarrow\varnothing]$ and $S_n(G)=S_n(G)[\leftarrow\varnothing]$ are well quasi-ordered by $\leq_S$ and $\leq_{S\omega}$ respectively.

Using the same kind of proof as the one used for Lemma 6.4 we prove that $Q[S_n] = \cup_{t\in Q} \cup_{U\subset \mathrm{Vertex}(t)} \cup_{\rho:U\to S_n} t[U]\rho(U)$ is well quasi-ordered by $\leq_S$ when Q is a finite subset of $T(\Sigma,\omega)$.

Lemma 6.7

Let S and Q be finite subsets of $T(\Sigma,\omega)$. For every integer n, $Q[S_n]$ is well quasi-ordered by $\leq_S$.

Proof. Let (t_i) an infinite sequence in $Q[S_n]$. As Q is a finite set, there exists an infinite subsequence $(t_{\gamma(i)})$ of (t_i) such that $t_{\gamma(i)} = t[\ldots(u_j,v_{j,\gamma(i)})\ldots](\ldots\theta_{j,\gamma(i)}\ldots)$ for suitable chosen

$t \in Q$, integer q, $u_j \in$ Vertex(t) $(1 \le j \le q)$, trees $\theta_{j,\gamma(i)}$ in S_n and $v_{j,\gamma(i)} \in$ Vertex($\theta_{j,\gamma(i)}$). Let $\theta'_{j,\gamma(i)} = \theta_{j,\gamma(i)}[v_{j,\gamma(i)} \leftarrow \omega']$. As $\le_S$ is a well quasi-ordering on $S_n[\leftarrow \omega']$, the product quasi-ordering induced by $\le_S$ is a well quasi-ordering on $S_n[\leftarrow \omega'] \times \ldots \times S_n[\leftarrow \omega']$. Thus, there exist $\gamma(i) < \gamma(i')$ such that $(\theta'_{1,\gamma(i)}, \ldots, \theta'_{q,\gamma(i)}) \le_S (\theta'_{1,\gamma(i')}, \ldots, \theta'_{q,\gamma(i')})$. Using now the definition of $Q[S_n]$ and Lemmas 5.8 and 5.9 we can conclude that $t_{\gamma(i)} \le_S t_{\gamma(i')}$ and thus $Q[S_n]$ is well quasi-ordered by $\le_S$.

As a particular case we get the following corollary:

Corollary 6.8

Let Σ be finite ranked alphabet and S a finite subset of $T(\Sigma, \omega)$. For every k and n, $R_k[S_n]$ is well quasi-ordered by $\le_S$.

Lemma 6.9

Let S be a finite subset of $T(\Sigma,\omega)$, G a subset of $T(\Sigma,\omega)$ and Q a finite subset of $T(\Sigma)$. If G is well quasi-ordered by $\le_{S\omega}$, then, for every integer n, $Q[S_n](G)$ is well quasi-ordered by $\le_{S\omega}$.

Proof. The proof is the same as the proof of Lemma 6.7 where $\le_{S\omega}$ is substituted for $\le_S$ and $Q[S_n](G)$ for $Q[S_n]$. We could state a similar lemma with $Q \subset T(\Sigma,\omega)$ but in that case we have to avoid the substitution of elements of G for leaves of Q.

Theorem 6.10

Let Q and S be finite subsets of $T(\Sigma)$ and $T(\Sigma,\omega)$-$T(\Sigma)$-$\{\omega\}$respectively. For every pair of integers m and n, $Q^{||m||}[S_n]$ is well quasi-ordered by $\le_{S\omega}$.

Proof. Let n and m be two integers. Let (t_i) be a minimal (with respect to $\le_{vertex}$) counterexample in $Q^{||m||}[S_n]$ (i.e. there is no i<j such that $t_i \le_{S\omega} t_j$). There are two cases according to the nature of the decomposition of the t_i .which defines their residual branch height.
1) There exists an infinite subsequence (t_i) where $t_i = \theta_i[w_j^i \leftarrow \theta_j^i \mid j \in L_i]$ with $\theta_i \in S_n$. Let I denotes the set of indices i for which t_i satisfies this property. The set $\cup_{i \in I} \cup_{j \in L_i} \{\theta_j^i\}$ is well quasi-ordered by $\le_{S\omega}$. Otherwise, let (θ'_l) be a counterexample in this set with $\theta'_1 = \theta_j^i$. The sequence $t_1, \ldots, t_{i-1}, \theta_j^i = \theta'_1, \theta'_2, \ldots$ is a counterexample because if $t_l \le_{S\omega} \theta'_{l'} = \theta_{j'}^{i'}$ then $t_l \le_{S\omega} t_{i'} = \theta_{i'}[w_j^{i'} \leftarrow \theta_j^{i'} \mid j \in L_{i'}]$ which contradicts the hypothesis that (t_i) is a counterexample. Furthermore this counterexample (θ'_l) is smaller than (t_i) with respect to $\le_{vertex}$ which contradicts the hypothesis that (t_i) is minimal. We deduce now, from Proposition 6.6, that $\le_{S\omega}$ is a well quasi-ordering on $S_n(\cup_{i \in I} \cup_{j \in L_i} \{\theta_j^i\})$ and that there exist i<j such that $t_i \le_{S\omega} t_j$

which contradicts the hypothesis that (t_i) is a counterexample.

2) There exists an infinite subsequence (t_i) where the head symbol of each t_i belongs to the head residual tree (there is no element of S_n inserted at the root of the head residual tree). As the alphabet is finite , there exist an infinite subsequence, still denoted (t_i), where the t_i have the same head symbol, f for example. Thus $t_i = f(t_1{}^i,\ldots,t_l{}^i)$ and RBH ($t_j{}^i$) $\leq$ RBH (t_i) - 1 for $1 \leq j \leq l$. We prove, by induction on m that (t_i) cannot be a counterexample. When m=1, the arguments $t_j{}^i$ satisfy case 1) and there exist i<j such that $(t_1{}^i,\ldots,t_l{}^i)$ WEO($\leq_{S\omega}$) $(t_1{}^j,\ldots,t_l{}^j)$ and thus $f(t_1{}^i,\ldots,t_l{}^i) \leq_{S\omega} f(t_1{}^j,\ldots,t_l{}^j)$ (Lemma 5.9). Let us suppose now that for every integer m'<m and for every integer n, $Q^{||m'||}[S_n]$ is well quasi-ordered by $\leq_{S\omega}$. $t_i = f(t_1{}^i,\ldots,t_l{}^i) \in Q^{||m||}[S_n]$ implies $t_j{}^i \in Q^{||m-1||}[S_n]$, which is well quasi-ordered by the induction hypothesis. We conclude by using Lemma 5.9 as in the previous case.

7 MAIN Theorem

We are now able to state the relation between the unavoidability property of a set S and the property for the related quasi-ordering $\leq_{S\omega}$ which is the main result of this paper.

Proposition 7.1

Let $S \subset T(\Sigma,\omega)$. If $\leq_S$ is a well quasi-ordering on $T(\Sigma,\omega)$, then $S \cap (T(\Sigma,\omega)-T(\Sigma))-\{\omega\}$ is factor-unavoidable.

Proof. If $S' = S \cap (T(\Sigma,\omega)-T(\Sigma))- \{\omega\}$ is not factor-unavoidable, there exists an infinite subset T of $T(\Sigma,\omega)$ such that every tree t in T has no factor in S'. We show that the trees of T are pairwise incomparable with respect to $\leq_S$, and thus T contradicts the hypothesis. If $t <_S t'$ in T there are $t_0,\ldots,t_n$ in $T(\Sigma,\omega)$ with $t_0 = t$, $t_n = t'$ and for every i $(1 \leq i \leq n)$ $t_i \; I_S \; t_{i+1}$. Let l the greatest index such that $t_l \neq t_n$. There is s in S' such that $t_n \in t_l[\;]s$, which implies $s <_{factor} t'$, a contradiction.

Proposition 7.2

Let $S \subset T(\Sigma,\omega)$. If $\leq_{S\omega}$ is a well quasi-ordering on $T(\Sigma)$, then $S \cap (T(\Sigma,\omega) -T(\Sigma)) - \{\omega\}$ is factor-unavoidable.

Proof. If $S' = S \cap (T(\Sigma,\omega) -T(\Sigma)) - \{\omega\}$ is not factor-unavoidable, there exists an infinite subset T of $T(\Sigma,\omega)$ such that every tree t in T has no factor in S'. By well quasi-ordering, there exists two trees t and t' be from T such that $t <_{S\omega} t'$. If T is included in $T(\Sigma)$, it is not possible that $t \leq_\omega t'$. Thus there exists a tree t'' such that $t <_S t'' \leq_\omega t'$ and therefore a tree s in S' such that $s <_{factor} t''$. This implies $s <_{factor} t'$ which is a contradiction. On the other hand, if T is not included in T(Σ), let T' be T in which all the occurrences of ω have been replaced by constants from Σ. T' is included in T(Σ) and thus there is a tree t' in T' and a tree s in S' such that $s <_{factor} t'$. Removing the constants added in place of ω does not disturb the factor s. So, if t is the tree of T from which t' has been built, $s <_{factor} t$. We get a contradiction and conclude that S is factor-unavoidable.

Theorem 7.3 (Main Theorem)

Let Σ be a finite ranked alphabet, ω a constant not belonging to Σ and S a subset of $T(\Sigma,\omega)-T(\Sigma)-\{\omega\}$. $\leq_{S\omega}$ is a well quasi-ordering on T(Σ) if and only if S is factor-unavoidable.

Proof. If $\leq_{S\omega}$ is a well quasi-ordering on T(Σ) then S is factor-unavoidable (Proposition 7.2). In order to prove that S factor-unavoidable implies T(Σ) well quasi-ordered by $\leq_{S\omega}$, we put together the results obtained in previous sections. Let S be an unavoidable subset of T(Σ,ω) with avoidance bound k and R_k the set of trees in T(Σ) whose depth is less than or equal to k. R_k is finite. There is a finite subset F of S that is unavoidable with the same bound (Theorem 3.3). The second structure theorem (Theorem 4.22) implies $T(\Sigma) \subset R_k^{||k+1||}(F, k)$ which is well quasi-ordered by $\leq_{F\omega}$ (Theorem 6.10). As $\leq_{F\omega} \subset \leq_{S\omega}$, $R_k^{||k+1||}(F, k)$ and thus T(Σ) are also well quasi-ordered by $\leq_{S\omega}$.

Application

To illustrate the value of our main theorem, we use it here to prove termination of a rewriting system that contains only one rule, namely, " $f(s(x)) \to *(s(x), f(p(s(x))))$ ". With the usual orderings it is not possible to prove the termination of this system, because the left-hand side of the rule is embedded in the right-hand one. Let Σ be the ranked alphabet {*, p, s, f} with arities 2, 1, 1, 1 respectively. Let $<_1$ be the transitive closure of the relation $<_r$ on $T(\Sigma \cup \{x\})$ defined by $t[u \leftarrow *(s(\tau), f(p(s(\tau))))] <_r t[u \leftarrow f(s(\tau))]$ for every trees t and τ and every vertex u. Let S be the unavoidable set { *(ω,ω), s(ω), f(ω), p(p(ω)) } with avoidance bound 3 and $\leq_{S\omega}$ the quasi-ordering defined in section 5 related to S. Obviously $<_1$ is irreflexive. The transitive closure of $(<_{S\omega} \cup <_1)$, denoted <, is irreflexive too because it is necessary to build a factor f(p(s(ω))) only by insertion of trees from S, i.e. with p(p(ω)), f(ω) and s(ω). But the additional f and s that have to be inserted never disappear. Thus ≤, including the well quasi-ordering $\leq_{S\omega}$, is a well quasi-ordering. So, < is well founded and $<_1$, included in <, is also well founded. This property implies the termination of the rewriting system considered.

8 POSSIBLE EXTENSIONS

In this paper we defined in the case of a finite ranked alphabet a relation on trees that we proved to be a well quasi-ordering. The restriction of this quasi-ordering on words is the relation of insertion defined by Erhenfeucht et al.[2]. This result can be extended to an infinite well quasi-ordered alphabet. In that case, we keep the idea of insertion of trees belonging to an unavoidable set but in a less restrictive sense: we allow the insertion of a tree split in several pieces. The definition of this relation is much more close to the general definition of the embedding and can also be extended in a kind of recursive path ordering used to prove automatically the termination of term rewriting system. This results will be given in a forthcoming paper.

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