Fixed points on pairs of nilmanifolds

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Received 13 August 1993; revised 20 January 1994

Abstract

A theorem of D. Anosov states that, for any selfmap \( f : M \to M \) of a compact nilmanifold \( M \), \( N(f) = |L(f)| \) where \( N(f) \) and \( L(f) \) denote the Nielsen and the Lefschetz numbers of \( f \), respectively. We generalize this result for relative Nielsen type numbers to selfmaps of pairs of nilmanifolds. As an application, we estimate the minimal number of periodic points of prime power period.

Keywords: Relative Nielsen fixed point theory; Nilmanifolds; Periodic points

AMS (MOS) Subj. Class.: Primary 55M20; secondary 57F99

1. Introduction

One of the central questions in Nielsen fixed point theory is the computation of the Nielsen number. Anosov [1] showed that the Nielsen number of a selfmap on a compact nilmanifold is the absolute value of its Lefschetz number. In other words, the essential Nielsen classes are of the same index of value +1 or −1. For relative Nielsen theory, the computation is more difficult (see [7]). The purpose of this paper is to generalize Anosov’s theorem, similar to the generalization in [11] for map extensions, for other relative Nielsen type numbers. As an application, we employ the approach of [12] to study periodic points on nilmanifolds. We assume that the reader is familiar with the basics of Nielsen theory as presented in [5] as well as relative Nielsen theory in [8,9,13]. We follow the notations of [11].

First we recall the definitions of some of the relative Nielsen type numbers. For any selfmap \( f : (X,A) \to (X,A) \) of a pair of compact connected polyhedra, there is a well-defined function \( \text{FPC} : \text{FPC}_0(f_A) \to \text{FPC}_0(f) \) where \( \text{FPC}_0(f_A) \) and \( \text{FPC}_0(f) \) denote...
the sets of Nielsen (nonempty fixed point) classes of $f_A = f|A$ and $f$ respectively. This function $\text{FPC}_c$ simply sends a Nielsen class of $f_A$ to the unique Nielsen class of $f$ containing it. If $A$ has more than one component, we let $\text{FPC}_0(f_A) = \bigcup_j \text{FPC}_0(f_j)$ where the union is taken over all components $A_j$ of $A$ which are mapped under $f$ to themselves, and $f_j$ is the restriction of $f_A$ on the component $A_j$. In the case where $f(A_j) \not\subset A_j$ for all $j$, then $\text{FPC}_0(f_A) = \emptyset$.

The relative Nielsen number [8] is defined by

$$N(f; X, A) = N(f) - N(f, f_A) + N(f_A)$$

where $N(f, f_A) = \#\{\mathcal{H} \in \text{FPC}_0(f) \mid i(X, f, \mathcal{H}) \neq 0 \text{ and } \mathcal{H} = \text{FPC}_0(\mathcal{F}) \text{ for some } \mathcal{F} \in \text{FPC}_0(f_A) \text{ with } i(A, f_A, \mathcal{F}) \neq 0\}. (i(Z, \varphi, T) \text{ is the fixed point index of the class } T \text{ of } \varphi \text{ in } Z.)$

A Nielsen class $\mathcal{H} \in \text{FPC}_0(f)$ of $f$ does not assume its index in $A$ [9] if $i(X, f, \mathcal{H}) \neq i(A, f_A, \mathcal{H} \cap A)$. The relative Nielsen number of the closure [10] (same as $\bar{N}(f; X, A)$ defined in [9]) is defined to be

$$N(f; \overline{X - A}) = \#\{\mathcal{H} \in \text{FPC}_0(f) \mid \mathcal{H} \text{ does not assume its index in } A\}$$

and the (topological) extension Nielsen number [2] is defined to be

$$N(f|f_A) = \#\{\mathcal{H} \in \text{FPC}_0(f) \mid \mathcal{H} \text{ does not assume its index in } A$$

$$\text{ and } \mathcal{H} \cap \partial A = \emptyset\}$$

where $\partial A$ denotes the boundary of $A$ in $X$.

Using the covering space approach [5, Ch.1], all fixed point classes (including empty ones) are determined by the conjugacy classes of liftings to the universal cover. Let $\tilde{f} : \tilde{X} \rightarrow X$ and $\tilde{i}_j : \tilde{A}_j \rightarrow \tilde{X}$ be liftings of $f$ and the inclusion $i_j : A_j \hookrightarrow X$ to the respective universal covers where $A_j$ is a component of $A$. We say that the fixed point class $\eta \text{Fix } \alpha \tilde{f}$ does not contain any fixed point class of $f_A$ if $\tilde{i}_j \circ \tilde{f} \neq \alpha \tilde{f} \circ \tilde{i}_j$ for all liftings $\tilde{f}_j$ of $f_j$, where $\alpha$ is an element of the group of deck transformations of the universal covering map $\eta : \tilde{X} \rightarrow X$. The relative Nielsen number of $f$ on the complement [13] is defined by

$$N(f; X - A) = \#\{\mathcal{H} \in \text{FPC}_0(f) \mid i(X, f, \mathcal{H}) \neq 0 \text{ and } \mathcal{H} \text{ does not contain any fixed point class of } f_A\}.$$

The numbers $N(f; X, A), N(f; \overline{X - A}), N(f|f_A)$ and $N(f; X - A)$ have the usual properties of the ordinary Nielsen number. In particular, they have the homotopy (type) invariance and the lower bound properties. For further details, see [2,8,9,13]. We denote by $L(f, f_A) = L(f) - L(f_A)$ the relative Lefschetz number of $f : (X, A) \rightarrow (X, A)$ and by $R(f)$ the Reidemeister number of $f$ [5, p.6].
2. Main results

It is well known that every nilmanifold $M$ admits a principal torus bundle $T \rightarrow M \xrightarrow{p} N$ such that $T$ is a torus and $N$ is a nilmanifold of lower dimension. Furthermore, every selfmap $f : M \rightarrow M$ is homotopic to a fiber preserving map $f'$ which induces the following commutative diagram

$$
\begin{array}{ccc}
T & \xrightarrow{f'} & T \\
\downarrow & & \downarrow \\
M & \xrightarrow{f'} & M \\
\downarrow p & & \downarrow p \\
N & \xrightarrow{f'} & N
\end{array}
$$

Note that if $L(f') \neq 0$ then $i^\text{FPC} : \text{FPC}_0(f') \rightarrow \text{FPC}_0(f')$ is injective. Moreover, a product formula for the generalized Lefschetz numbers can be established (see [6]) so that we obtain the following strengthened version of Anosov's theorem.

**Theorem 2.1** ([6, Corollary 9.4]). Let $M$ be a compact nilmanifold and $f : M \rightarrow M$ a map. If $L(f) \neq 0$ then $N(f) = |L(f)| = R(f)$; otherwise $N(f) = 0$.

As an immediate consequence of this result and the homotopy type invariance of $N(f)$, $L(f)$ and $R(f)$, we have the following

**Proposition 2.2.** Let $(X, A)$ be a compact polyhedral pair of the homotopy type of a compact nilmanifold pair with $X$ connected and $f : (X, A) \rightarrow (X, A)$ be a map. If $L(f) \cdot L(f_j) \neq 0$ then all Nielsen classes of $f$ and $f_j$ are essential, and hence $i(X, f, i^\text{FPC}(F)) \neq 0$ if and only if $i(A, f^0, F) \neq 0$ for all $F \in \text{FPC}_0(f_j)$.

Recall that a point $x$ in a space $X$ is said to be a local cut point if there is a connected neighborhood $U$ of $x$ such that $U - \{x\}$ is not connected. A subspace $A \subset X$ can be bypassed if the inclusion $i : X - A \hookrightarrow X$ induces an epimorphism $i_2 : \pi_1(X - A) \twoheadrightarrow \pi_1(X)$ on fundamental groups.

Here is our main theorem.

**Theorem 2.3.** Let $(X, A)$ be a compact polyhedral pair of the homotopy type of a compact nilmanifold pair such that $X$ is connected, $X - A$ has no local cut points and is not a 2-manifold; $A$ can be bypassed. Suppose that $f : (X, A) \rightarrow (X, A)$ is a map such that $i^\text{FPC} : \text{FPC}_0(f_A) \rightarrow \text{FPC}_0(f)$ is injective.

1. If $L(f) \neq 0$ then $N(f; X, A) = |L(f)|$; otherwise, $N(f; X, A) = \sum_j |L(f_j)|$.
2. If $L(f) \neq 0$ and $L(f) \cdot L(f_j) \geq 0$ for all $j$, then $N(f; X - A) = |L(f, f_A)|$. If $L(f) = 0$, then $N(f; X - A) = \sum_j |L(f_j)|$.
3. If $L(f) \cdot \prod_j L(f_j) \neq 0$, then $N(f; X - A) = |L(f)| - \sum_j |L(f_j)|$. 


Proof. (1) The assertion follows from definition in the cases where \( L(f) = 0 = N(f) \) or \( L(f_j) = 0 \) for all \( j \), i.e., \( N(f_A) = 0 \). Suppose that \( L(f) \cdot L(f_j) \neq 0 \) for some \( j \). By Proposition 2.2 and the injectivity of \( i^{\text{FPC}} \), there are \( |L(f_j)| \) essential Nielsen classes of \( f \) containing the essential Nielsen classes of \( f_j \). Applying this argument to all \( f_j \) with \( L(f_j) \neq 0 \), we conclude that \( N(f, f_A) = \sum_j N(f_j) = N(f_A) \) and hence \( N(f;X,A) = N(f) = |L(f)| \).

(2) Following the proof of Theorem 2.4 of [11] (see also [8, Theorem 6.2]), we may homotope \( f \) relative to \( A \) to a map \( f' : (X,A) \to (X,A) \) such that \( f' \) retracts some open neighborhood \( U \) of \( A \) onto \( A \) so that \( i(X, f', \mathcal{H}) = i(A, f_A, \mathcal{H} \cap A) \) for \( \mathcal{H} \in \text{FPC}_0(f') \); that each of the fixed point classes of \( f'|X - A \) is a distinct essential Nielsen class of \( f \) in \( X \). Suppose that \( L(f) \cdot L(f_j) > 0 \) for some \( j \). Then \( i(X, f', \mathcal{H}) = i(A_j, f_j, \mathcal{F}_j) (= \pm 1) \) for any \( \mathcal{H} \in \text{FPC}_0(f') \) and \( \mathcal{F}_j \in \text{FPC}_0(f_j) \). Thus, if \( x \in \text{Fix} f' \cap (X - A) \), and \( \mathcal{F} \in \text{FPC}_0(f_k) \) (for any \( k \) with \( L(f_k) \neq 0 \)) were Nielsen equivalent, then \( \mathcal{F} \cup \{x\} \in \text{FPC}_0(f') \) and hence \( i(X, f', x) = 0 \) contradicting the assumption on \( \text{Fix} f' \cap (X - A) \). In other words, \( \text{Fix} f' \cap (X - A) \) consists of essential Nielsen classes of \( f' \) that do not assume their index in \( A \). Since \( \#(\text{Fix} f' \cap (X - A)) = |L(f') - L(f_A)| \), we have \( N(f;X - A) = N(f',X - A) = |L(f', f_A)| = |L(f, f_A)| \). If \( L(f_j) = 0 \) for all \( j \), then \( N(f;X - A) = \#(\text{Fix} f' \cap (X - A)) = |L(f)| = |L(f, f_A)| \). If \( L(f) = 0 \), then all Nielsen classes of \( f \) are inessential. The only Nielsen classes of \( f \) that do not assume their index in \( A \) must contain essential classes of \( f_j \) with \( L(f_j) \neq 0 \). Therefore, there must be \( \sum_j |L(f_j)| \) of them.

(3) Suppose that \( L(f) \cdot \prod_j L(f_j) \neq 0 \). It follows from Proposition 2.2 that all fixed point classes of \( f_j \) and \( f \) are essential. By the injectivity of \( i^{\text{FPC}} \), there are \( |L(f)| - \sum_j |L(f_j)| \) essential classes that do not contain any fixed point class of \( f_A \). Hence the assertion follows. \( \Box \)

Note that when \( A = \emptyset, L(f_j) = 0 \) for all \( j \) so that in this case, (1) and (2) of Theorem 2.3 reduce to Anosov's theorem. Furthermore, if in addition to the assumption on \( i^{\text{FPC}} \), the interior \( \text{int} A \) of \( A \) in \( X \) is empty and \( L(f) \cdot \prod_j L(f_j) \neq 0 \) then one can show that \( N(f|f_A) = |L(f)| - \sum_j |L(f_j)| \) (same argument as in the proof of Theorem 2.4 of [11]) and hence by (3) of Theorem 2.3, we have

**Theorem 2.4.** Let \( f : (X,A) \to (X,A) \) be as in Theorem 2.3 such that \( i^{\text{FPC}} \) is injective. If \( \text{int} A = \emptyset \) and \( L(f) \cdot \prod_j L(f_j) \neq 0 \) then \( N(f|f_A) = N(f;X - A) = |L(f)| - \sum_j |L(f_j)| \).

(Compare [11, Theorem 2.4].)

The assumptions in Theorem 2.3 may not be relaxed as we illustrate in the following examples.

**Example 2.5.** Let \( X = S^1 \times S^1 \times S^1 \) be the three-dimensional torus and \( A \) be the circle imbedded as the first component of \( X \). Take \( f = f_A \times f_1 \times f_2 \) where \( f_A, f_1 \) and \( f_2 \) are maps of degree \(-1, 2 \) and \( 0 \) respectively. It is easy to see that \( f \) has two essential
Nielsen classes which lie in \( A \) so that \( N(f; \overline{X-A}) = 2; L(f) = -2; L(f_A) = 2 \) and thus \(|L(f, f_A)| = 4\). Note that \( N(f; X, A) = |L(f)| = 2 \) and \( N(f; X - A) = |L(f)| - |L(f_A)| = 0 \). Here, (1) and (3) of Theorem 2.3 hold but (2) does not.

Example 2.6. Let \( X = D^2 \times S^1 \) be the three-dimensional solid torus and \( A = \partial D^2 \times \{1\} \) be a meridian on the boundary of \( X \). Consider \( f = f_1 \times f_2 : D^2 \times S^1 \rightarrow D^2 \times S^1 \) where \( f_1(re^{i\theta}) = re^{-i\theta} \) and \( f_2(z) = \bar{z} \), the complex conjugate of \( z \in S^1 \subset \mathbb{C} \). Then \( L(f) = 2, L(f_A) = 2 \) and thus \( L(f) \cdot L(f_A) > 0 \) but \( f^{\text{FPC}} \) is not injective. It is easy to see that \( N(f; X, A) = 3 > |L(f)| = 2; N(f; X - A) = 2 > |L(f, f_A)| = 0 \) and \( N(f; X - A) = 1 > |L(f)| - |L(f_A)| = 0 \). Here (1)-(3) of Theorem 2.3 fail to hold.

Example 2.7. This is Example 2.5 of [11]. Consider the 4-torus \( X = T^4 = S^1 \times S^1 \times S^1 \times S^1 \) and \( A = \{(z_1, z_2, z_3, z_4) \mid z_1 = z_2 = z_3 = z_4\} \approx S^1 \). Let \( f : (X, A) \rightarrow (X, A) \) be given by \( f(z_1, z_2, z_3, z_4) = (\bar{z}_4, \bar{z}_1, \bar{z}_2, \bar{z}_3) \). It follows that

\[
L(f_A) = 2 = N(f_A)
\]

and

\[
\text{Fix } f = \{(z, \varphi(z), \varphi^2(z), \varphi^3(z)) \mid z \in \text{Fix } \varphi^4\}
= \{(z, \bar{z}, z, \bar{z}) \mid z \in S^1\}
\]

where \( \varphi : S^1 \rightarrow S^1 \) is given by \( \varphi(z) = \bar{z} \). Since \( L(f) = L(\varphi^4) = L(\text{identity}) = 0 \), it follows from definition that \( N(f; X, A) = N(f_A) = 2 \) and \( N(f; X - A) = 0 \). Moreover Fix \( f \) is connected so that \( N(f; \overline{X-A}) \leq 1 < 2 = |L(f, f_A)| \). In this example, \( L(f) \cdot L(f_A) = 0 \) and \( f^{\text{FPC}} \) is not injective.

Remark 2.8. The invariant \( N(f; \overline{X-A}) \) is a lower bound for the number of fixed points in the closure of \( X - A \), which coincides with \( X \) when \( \text{int } A = \emptyset \) in which case the number coincides with the ordinary Nielsen number of \( f \). In light of this, the subspace \( A \) in all of the above examples may be thickened by taking a tubular neighborhood in \( X \).

3. Periodic points

Relative Nielsen theory can be employed [12] to study periodic points. In this section, we will apply our results from Section 2 to estimate and to compute the Nielsen type invariants \( N\Phi_n(f) \) (or \( NF_n(f) \) in [5]) and \( NF_n(f) \). Recall that \( NF_n(f) \) is defined to be \( n \) times the number of \( f \)-orbits of irreducible (i.e., not containing any fixed point class of \( f^m \) for \( m|n \) essential fixed point classes of \( f^m \). The number \( N\Phi_n(f) \) is defined to be \( \min\{h(S)\} \) where \( S \) is \( f \)-invariant and every essential fixed point class of \( f^m \) with \( m|n \) contains at least one class from \( S \) and \( h(S) \) is the sum of the periods of the \( f \)-orbits of \( S \) (see [5, p.69], [3] or [4]). We compute these invariants for selfmaps on a compact nilmanifold with period \( n = p^r \) where \( p \) is prime and \( r \) a positive integer.
We first recall the setup in [12]. Let $M$ be a compact connected nilmanifold and $n = p'$. Denote by $Y_n = M \times \cdots \times M$ the $n$-fold product of $M$. For any $f : M \to M$, we define $g_f : Y_n \to Y_n$ by $g_f(x_1, \ldots, x_n) = (f(x_1), f(x_2), \ldots, f(x_{n-1}))$. The cyclic group $\mathbb{Z}_n$ of order $n$ acts on $Y_n$ via $(x_1, \ldots, x_n) \mapsto (x_n, x_1, \ldots, x_{n-1})$. If $Y_n = \{ y \in Y_n \mid \text{stab}(y) \neq 1 \}$ where \( \text{stab}(y) \) is the stabilizer of $y$ in $\mathbb{Z}_n$, then $g_f : (Y_n, \bar{Y}_n) \to (Y_n, \bar{Y}_n)$ is a map of a pair. Moreover, $\bar{Y}_n$ is homeomorphic to the $p'^{-1}$-fold product of $M$, i.e., $\bar{Y}_n \cong Y_{n/p}$, and thus $g_f$ is a map of a pair of compact nilmanifolds. More generally, for any $m = p^k$, $k \leq r$,

$$Y_n^z = \{ y \in Y_n \mid \sigma y = y, \forall \sigma \in \mathbb{Z}_m \subset \mathbb{Z}_n \} \cong Y_{n/m}$$

and $g_f^z = g_f|Y_n^z : Y_n^z \to Y_n^z$. We therefore have a well-defined function $i_{l,m} : \Gamma \text{PC}_n(g_f^z) \to \Gamma \text{PC}_0(g_f^z)$ for every $l|m$. Since the fixed point classes of $g_f^z$ are in one to one correspondence with those of $f^{m/n}$ [12, Theorem 2.1], $i_{l,m}$ can be thought of as the function which sends an $n/l$ periodic point class to the unique $n/m$ periodic point class containing it.

**Theorem 3.1.** Let $n = p'$ and $m = p'^{-1}$. Suppose that $i_{m,n}$ is injective. If $L(f^n) \cdot L(f^m) \neq 0$, then

$$|L(f^n)| - |L(f^m)| \leq NP_n(f) \leq n \cdot (|L(f^n)| - |L(f^m)|).$$

**Proof.** It follows from Theorem 3.1 of [12] that $N(g_f; Y_n - \bar{Y}_n) \leq NP_n(f) \leq n \cdot N(g_f; Y_n - \bar{Y}_n)$. By (3) of Theorem 2.3, $N(g_f; Y_n - \bar{Y}_n) = |L(g_f)| - |L(g_f|\bar{Y}_n)| = N(g_f) - N(g_f^{Z^m})$. The assertion follows from the fact that $N(g_f^{Z^m}) = N(f^{m/k})$ ([12, Theorem 2.1]).

In Theorem 3.7 of [3], $NP_n(f)$ is computed using Möbius inversion in terms of \( \{ N(f^{m}) \} \) under some conditions among which is the $n$-toral condition. Recall that a map $f : X \to X$ is $n$-toral if (i) for every $m|n$ and every fixed point class $F$ of $f^n$, the depth of $F$ (the smallest integer $d$ such that $F$ contains a fixed point class of $f^d$) is equal to the length of $F$ (the number of elements in the $f$-orbit of $F$), and (ii) for every $m|n$, no two fixed point classes of $f^m$ belong to the same fixed point class of $f^n$ (see [3]).

**Theorem 3.2.** Suppose that $n = p'$ and $L(f^n) \neq 0$ for all $m$ with $m|n$. If $i_{m,n}$ is injective, then

$$|L(f^n)| \leq NP_n(f) \leq n \cdot |L(f^n)|.$$

If, further, $f$ is $n$-toral, then

$$NP_n(f) = |L(f^n)| - |L(f^{n/p})|$$

and

$$NP_n(f) = |L(f^n)|.$$
Proof. Since $L(f^m) \neq 0$ for all $m|n$, all Nielsen classes of $f^m$ are essential and by Theorem 4.2 of [4], we have

\[ N\Phi_n(f) = \sum_{m|n} NP_m(f). \]

The first assertion follows immediately from the inequality in Theorem 3.1. Suppose that $f$ is $n$-toral. Thus $f$ is also $m$-toral for $m|n$ and $i_{m,n}$ is injective. It follows from the proof of [12, Theorem 4.2] that $N(g_{f^m}; Y - Y_m) = NP_m(f)$. Applying Theorem 2.3(3), we obtain $NP_m(f) = |L(f^m)| - |L(f^{m/p})|$ and so the second assertion follows with $m = n$. Hence

\[ N\Phi_n(f) = \sum_{m|n} NP_m(f) = \sum_{m|n} (|L(f^m)| - |L(f^{m/p})|) = |L(f^n)|. \]

\[ \square \]

Remark 3.3. In Theorem 3.2, $f$ is not required to be eventually commutative or to satisfy the Jiang condition, $J(f) = \pi$. Compare the similar results obtained in [4] (see Corollaries 4.4, 4.11 and 4.12 of [4]).

Acknowledgement

I would like to thank Professor Helga Schirmer for helpful suggestions and comments concerning this paper.

References