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Local Lagrange interpolation using cubic C^1 splines on type-4 cube partitions

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Abstract

We describe a local Lagrange interpolation method using cubic (i.e. non-tensor product) C^1 splines on cube partitions with five tetrahedra in each cube. We show, by applying a complex proof, that the interpolation method is local, stable, has optimal approximation order and linear complexity. Since no numerical results on trivariate cubic C^1 spline interpolation are known from the literature, the steps of the algorithm, which are different from those of the known methods, are focused on its implementation. In this way, we are able to describe the first implementation of a trivariate C^1 spline interpolation method, run numerical tests and visualize the corresponding isosurfaces. These tests with up to 5.5×10^{11} data confirm the efficiency of the algorithm.

Keywords: Trivariate splines; Lagrange interpolation; Type-4 partitions

1. Introduction

In the last few years, a series of papers have appeared on local Lagrange interpolation using bivariate splines (see [11,8,9,7,6]). On the other hand, only a few results are known for this problem in the trivariate case (see [3,4,12,10]). Up to now, no Lagrange interpolation algorithms using trivariate cubic C^1 splines have been implemented.

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In this paper, we describe a local Lagrange interpolation method using cubic C^1 splines on cube partitions with five tetrahedra in each cube, called type-4 tetrahedral partitions. For the first time, a trivariate spline interpolation algorithm is implemented and its efficiency is verified.

A fundamental method for constructing local Lagrange interpolation sets for spaces of cubic C^1 splines on refined arbitrary partitions was developed by [4]. The method is based on decompositions into classes of tetrahedra and the refinement of certain tetrahedra by partial Worsey–Farin splits. A spline space together with a corresponding Lagrange interpolation set is called an interpolation pair (cf. [4]). We note that the above decompositions are not unique.

As regards the aspect of implementation, we investigate type-4 tetrahedral partitions and describe efficient decompositions of these partitions. The tetrahedral partitions are decomposed into classes of cubes such that for each cube, the interpolating splines can be computed by simply repeating the same steps. Moreover, all steps are very similar and - roughly speaking - slight modifications of one single step.

In order to simplify the implementation, we describe in detail which tetrahedra are refined by a partial Worsey–Farin split and which interpolation points are chosen. We note that the algorithm is different from the general method in [4] and from the algorithm in [3] for Freudenthal partitions. In [4], the computation of the interpolating spline is based on chains of tetrahedra with common vertices and common edges, while in [3], a black and white coloring of the tetrahedra is used.

It is proved that the Lagrange interpolating splines can be computed locally and stably, which implies that the method yields optimal approximation order for smooth functions. In addition, the computational complexity of the method is linear in the number of vertices of the cubes.

We implement the algorithm and give numerical results and visualizations of the corresponding isosurfaces. Moreover, we also implement the method of [3] for Freudenthal partitions. The results for our method are slightly better than those for the Freudenthal partition, although fewer tetrahedra are used.

We note, that for trivariate C^1 tensor product spline interpolation, splines of higher degree have to be used.

The paper is organized as follows. In Section 2, we recall the basic Bernstein–Bézier theory of splines. In Section 3, we define type-4 partitions and the classifications of cubes and tetrahedra. We recall the so called partial Worsey–Farin splits in Section 4. In Section 5, we describe an algorithm for refining the given tetrahedral partition and for constructing a corresponding Lagrange interpolation set. Moreover, we give our main result on the locality and the stability of our method by using a complex proof. In Section 6, we establish error bounds for the corresponding interpolation operator. In the last section, we give some numerical tests and visualizations, and we compare them with other methods.

2. Preliminaries

For any given tetrahedral partition Δ, the associated space of C^1 cubic splines is defined by

$$\mathcal{S}_3^1 := \{s \in C^1 : s|_T \in \mathcal{P}_3, \forall T \in \Delta\},$$

where $\mathcal{P}_3$ is the 20-dimensional space of trivariate cubic polynomials. In this paper we use the well-known Bernstein–Bézier techniques (see the book [5], chapter 15.3–15.4). For a collection of tetrahedra $\Delta \subset \mathbb{R}^3$, let

$$\mathcal{D}_\Delta := \bigcup_{T \in \Delta} \mathcal{D}_T,$$

be the set of domain points, where

$$\mathcal{D}_T := \left\{ \xi^T_{i,j,k,l} := \frac{iv_1 + jv_2 + kv_3 + lv_4}{3},\ i + j + k + l = 3 \right\}$$

and $T := \langle v_1, v_2, v_3, v_4 \rangle$.

Definition 2.1. The ball of radius 1 around v_1 is defined by

$$D^T_1(v_1) := \{\xi^T_{i,j,k,l} :\ i \geq 2\},$$

which consists of four domain points. The definition is similar for the other vertices of T. If v is a vertex of a collection of tetrahedra Δ, we define

$$D_1(v) := \bigcup_{\{T \in \Delta:\ v \in T\}} D^T_1(v).$$

The tube of radius 1 around $e := \langle v_1, v_2 \rangle$ is defined by

$$E^T_1(e) := \{\xi^T_{i,j,k,l} :\ k + l \leq 1\},$$

which consists of ten domain points. If e is an edge of a collection of tetrahedra Δ, we define

$$E_1(e) := \bigcup_{\{T \in \Delta:\ e \in T\}} E^T_1(e).$$

We have for every spline $s \in \mathcal{S}^0_3(\Delta)$,

$$s|_T = \sum_{i+j+k+l=3} c^T_{i,j,k,l} B^3_{i,j,k,l}$$

where $B^3_{i,j,k,l} = \frac{3!}{i!j!k!l!} \Phi^i_1 \Phi^j_2 \Phi^k_3 \Phi^l_4$ are the Bernstein polynomials of degree 3 associated with T and $\Phi_\nu \in \mathcal{P}_1,\ \nu = 1, 2, 3, 4$, are the barycentric coordinates of T. Then each spline $s \in \mathcal{S}^0_3(\Delta)$ is uniquely determined by its corresponding set of B-coefficients $\{c_\xi\}_{\xi \in \mathcal{D}_{d,\Delta}}$, with $c_{\xi^T_{i,j,k,l}} := c^T_{i,j,k,l}$.

Suppose $\tilde{T} := \langle v_5, v_2, v_3, v_4 \rangle$ is a further tetrahedron in Δ, and T and $\tilde{T}$ share the face $f := \langle v_2, v_3, v_4 \rangle$. Let p and $\tilde{p}$ be two polynomials of degree 3 with B-coefficients $c^T_{i,j,k,l}$ and $c^{\tilde{T}}_{i,j,k,l}$. Then p and $\tilde{p}$ join with C^r continuity across the face f if and only if

$$c^{\tilde{T}}_{i,j,k,l} = \sum_{\alpha+\beta+\gamma+\delta=i} c^T_{\alpha,j+\beta,k+\gamma,l+\delta} B^i_{\alpha,\beta,\gamma,\delta}(v_5),$$

for $i = 0, \ldots, r,\ \ j + k + l = 3 - i$; see [5], chapter 17.2.

In this paper we also use the concept of minimal determining sets. A set $\mathcal{M}$ of domain points is called a minimal determining set for a spline space $\mathcal{S}$ provided it is the smallest set of points such that the corresponding B-coefficients $\{c_\xi\}_{\xi \in \mathcal{M}}$ can be set independently, and all other B-coefficients of a spline $s \in \mathcal{S}$ are consistently determined from smoothness conditions; see [5], chapter 17.3.

3. Classification of cubes and tetrahedra

Let n be an odd integer and let $\diamond$ be the cube partition of $\Omega = [0, n] \times [0, n] \times [0, n] \subseteq \mathbb{R}^3$ which is obtained by intersecting Ω with $n + 1$ parallel planes in each of the three space

Table 3.1
Classification for $\mathcal{K}_0, \ldots, \mathcal{K}_3$.

	i	j	k
$\mathcal{K}_0$	Even	Even	Even
$\mathcal{K}_1$	Odd	Odd	Even
$\mathcal{K}_2$	Odd	Even	Odd
$\mathcal{K}_3$	Even	Odd	Odd

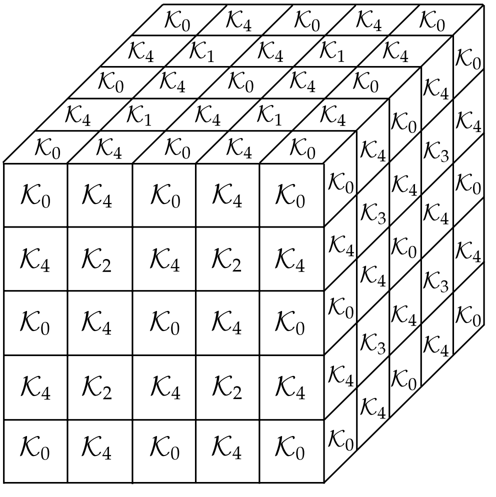

Fig. 3.1. Classification of cubes.

dimensions, i.e.,

$$\diamond = \{Q_{i,j,k} : \; Q_{i,j,k} = [i-1, i] \times [j-1, j] \times [k-1, k], i, j, k = 1, \ldots, n\}.$$

The cubes $\{Q_{i,j,k}\}$ can be classified as $\mathcal{K}_0, \ldots, \mathcal{K}_4$, according to their subscripts, as in Table 3.1. The remaining cubes are in class $\mathcal{K}_4$ (see Fig. 3.1).

These classes have the following properties:

Lemma 3.1. 1. *No two cubes in class $\mathcal{K}_i$, $i = 0, \ldots, 3$, touch each other.*

2. *Each cube Q in class $\mathcal{K}_i$, $i = 1, 2, 3$, touches at most four cubes in class $\mathcal{K}_j$ at the edges of Q, for each $j = 0, \ldots, i-1$.*
3. *Two cubes in class $\mathcal{K}_4$ can have at most one common edge.*
4. *If a cube Q in class $\mathcal{K}_4$ shares an edge e with another cube in class $\mathcal{K}_4$, then Q also shares the two faces containing e with two cubes in the classes $\mathcal{K}_i$, $i = 0, \ldots, 3$ (see Fig. 3.3).*

We now describe the construction of a type-4 tetrahedral partition of $\diamond$ by subdividing each cube Q of $\diamond$ into five tetrahedra, introduced by [12]. Therefore, let $\mathcal{V}$ be the set of vertices of $\diamond$ (see Fig. 3.1).

Lemma 3.2. *The set $\mathcal{V}$ can be divided into two sets $\mathcal{V}_1$ and $\mathcal{V}_2$ such that for every vertex $v \in \mathcal{V}_\nu$, all of its vertices sharing an edge with v are in $\mathcal{V}_\mu$, where $\nu \neq \mu$.*

This partition of $\mathcal{V}$ is not unique in the sense that there can be other partitions of $\mathcal{V}$ into $\mathcal{V}_1$ and $\mathcal{V}_2$ depending on the choice of the vertices.

Without loss of generality we assume that $\{v_{0,0,0}, v_{0,1,1}, v_{1,0,1}, v_{1,1,0}\} \in \mathcal{V}_1$. Thus all other vertices are uniquely classified and can be more easily described.

We say that the vertices in $\mathcal{V}_1$ are of type-1 and those in $\mathcal{V}_2$ are of type-2.

Now it is possible to define the following type-4 tetrahedral partition:

Definition 3.1 (*Type-4 Tetrahedral Partition*). Given a cube partition $\diamond$ in $\mathbb{R}^3$, suppose Δ is the collection of tetrahedra which is obtained by splitting each cube Q of $\diamond$ into five tetrahedra by connecting its four type-2 vertices with each other. Δ is called a type-4 partition of $\diamond$.

For a simpler description of the different tetrahedra in a single cube $Q_{i,j,k}$ we write

$$T^1_{i,j,k} := \langle v_{i,j,k}, v_{i,j,k+1}, v_{i,j+1,k}, v_{i+1,j,k}\rangle,$$
$$T^2_{i,j,k} := \langle v_{i,j,k+1}, v_{i,j+1,k}, v_{i,j+1,k+1}, v_{i+1,j+1,k+1}\rangle,$$
$$T^3_{i,j,k} := \langle v_{i,j,k+1}, v_{i+1,j,k}, v_{i+1,j,k+1}, v_{i+1,j+1,k+1}\rangle,$$
$$T^4_{i,j,k} := \langle v_{i,j+1,k}, v_{i+1,j,k}, v_{i+1,j+1,k}, v_{i+1,j+1,k+1}\rangle,$$
$$T^5_{i,j,k} := \langle v_{i,j,k+1}, v_{i,j+1,k}, v_{i+1,j,k}, v_{i+1,j+1,k+1}\rangle,$$

for the tetrahedra in cubes of the classes $\mathcal{K}_0, \mathcal{K}_1, \mathcal{K}_2$ and $\mathcal{K}_3$ (see Fig. 3.4).

For the different tetrahedra in a cube $Q_{i,j,k}$ in class $\mathcal{K}_4$ (see Fig. 3.5) we use the notation

$$T^1_{i,j,k} := \langle v_{i,j,k}, v_{i,j+1,k}, v_{i,j+1,k+1}, v_{i+1,j+1,k}\rangle,$$
$$T^2_{i,j,k} := \langle v_{i,j,k}, v_{i,j,k+1}, v_{i,j+1,k+1}, v_{i+1,j,k+1}\rangle,$$
$$T^3_{i,j,k} := \langle v_{i,j+1,k}, v_{i+1,j,k+1}, v_{i+1,j+1,k}, v_{i+1,j+1,k+1}\rangle,$$
$$T^4_{i,j,k} := \langle v_{i,j,k}, v_{i+1,j,k}, v_{i+1,j,k+1}, v_{i+1,j+1,k}\rangle,$$
$$T^5_{i,j,k} := \langle v_{i,j,k}, v_{i,j+1,k+1}, v_{i+1,j,k+1}, v_{i+1,j+1,k}\rangle.$$

4. Partial Worsey–Farin splits

In this section we recall the partial Worsey–Farin split. In order to describe a partial Worsey–Farin split we also need the Clough–Tocher split of a triangle. The Clough–Tocher split F_{CT} of a triangle $F := \langle v_1, v_2, v_3\rangle$ with interior point v_F can be obtained by connecting all three vertices of F to v_F. Then F_{CT} consists of the three subtriangles $F_i := \langle v_I, v_{i+1}, v_F\rangle$, $i = 1, 2, 3$, where $v_4 = v_1$.

The following definition can be found in a similar way in [3].

Definition 4.1. Let T be a tetrahedron, and let v_T be its barycenter. Given an integer $1 \le m \le 4$, let $F_1, \ldots, F_m$ be distinct faces of T, and for each $i = 1, \ldots, m$, let v_{F_i} be a point in the interior of F_i. Then we define the m-th-order partial Worsey–Farin split Δ^m_{WF} of T to be the tetrahedral partition obtained by the following steps:

1. connect v_T to each of the four vertices of T,
2. connect v_T to the points v_{F_i} for $i = 1, \ldots, m$,
3. connect v_{F_i} to the three vertices of F_i for $i = 1, \ldots, m$.

The m-th-order partial Worsey–Farin split of a tetrahedron results in $4 + 2m$ subtetrahedra; see Fig. 4.1. The split Δ^4_{WF} is the well-known Worsey–Farin split; see [14]. We need the following result on the space $\mathcal{S}^1_3(\Delta^m_{WF})$, where Δ^m_{WF} is the m-th-order partial Worsey–Farin split of a tetrahedron $T := \langle v_1, v_2, v_3, v_4\rangle$.

Theorem 4.1 *([3], Theorem 6.3). Fix $0 \le m \le 4$. Let $\mathcal{M}_m$ be the union of the following sets of domain points in $D_{\Delta^m_{WF}}$:*

1. *for each $i = 1, \dots, 4$, $D(v_i) \cap T_i$ for some tetrahedron $T_i \in \Delta^m_{WF}$ containing v_i,*
2. *for each face F of T that is not split, the point $\xi^F_{1,1,1}$,*
3. *for each face F of T that has been subjected to a Clough–Tocher split, the points $\{\xi^{F_i}_{1,1,1}\}^3_{i=1}$, where F_1, F_2, F_3 are the subfaces of F.*

Then $\mathcal{M}_m$ is a minimal determining set for $\mathcal{S}^1_3(\Delta^m_{WF})$.

5. Main result

In this section we give an algorithm for the refinement of Δ to Δ^* and develop a local and stable Lagrange interpolation method for $\mathcal{S}^1_3(\Delta^*)$. In this algorithm we split some of the faces of the tetrahedra with a Clough–Tocher split and the tetrahedra with the corresponding partial Worsey–Farin split. To uniquely define these splits we specify the points v_F, where the faces have to be split, in the following way:

1. if F is a face that is shared by two tetrahedra T and $\widetilde{T}$ in Δ, then choose v_F to be the intersection of F with the line connecting the barycenters v_T and $v_{\widetilde{T}}$ of T and $\widetilde{T}$;
2. otherwise choose the barycenter of F to be v_F.

Let $\diamond$ be a cube partition and Δ the corresponding type-4 tetrahedral partition. Moreover, let $\mathcal{K}_i$, $i = 0, \dots, 4$, be the classes of cubes and $T^l_{i,j,k}$, $l = 1, \dots, 5$, the tetrahedra as in Section 3.

Algorithm 5.1. Step 1: For each cube $Q_{i,j,k} \in \mathcal{K}_0$,

(1a) choose the 20 points $\mathcal{D}_{T^1_{i,j,k}}$,

(1b) choose the 10 points $\mathcal{D}_{T^2_{i,j,k}} \setminus E_1^{T^2_{i,j,k}}(\langle v_{i,j,k+1}, v_{i,j+1,k}\rangle)$,

(1c) split $T^3_{i,j,k}$ with a first-order partial Worsey–Farin split at $\langle v_{i,j,k+1}, v_{i+1,j,k}, v_{i+1,j+1,k+1}\rangle$ and choose the four points $D^T_1(v_{i+1,j,k+1})$, $v_{i+1,j,k+1} \in T \subset T^3_{i,j,k}$, $v_F \in \langle v_{i+1,j,k}, v_{i+1,j,k+1}, v_{i+1,j+1,k+1}\rangle$ and $v_{\widetilde{F}} \in \langle v_{i,j,k+1}, v_{i+1,j,k}, v_{i+1,j+1,k+1}\rangle$,

(1d) and split $T^4_{i,j,k}$ with a first-order partial Worsey–Farin split at $\langle v_{i,j+1,k}, v_{i+1,j,k}, v_{i+1,j+1,k+1}\rangle$ and choose the four points $D^T_1(v_{i+1,j+1,k})$, $v_{i+1,j+1,k} \in T \subset T^4_{i,j,k}$.

Step 2: Define all edges of $\Delta \setminus \mathcal{K}_0$ as "unmarked" and all edges in cubes in class $\mathcal{K}_0$ as "marked".

Step 3: For each cube $Q_{i,j,k}$ in $\mathcal{K}_l$, $l = 1, \dots, 4$, for each tetrahedron $T^m_{i,j,k}$, $m = 1, \dots, 4$, in $Q_{i,j,k}$,

3(a) if $T := T^m_{i,j,k}$ has h faces with two or three marked edges, then split these faces with a Clough–Tocher split, T with an h-th-order partial Worsey–Farin split and replace T in Δ by the resulting subtetrahedra,

3(b) if a face $\langle v_1, v_2, v_3\rangle$ of T has no or two marked edges, choose the point v_F,

3(c) mark all edges of T.

Step 4: Split each tetrahedron $T^5_{i,j,k}$ in Δ with a fourth-order partial Worsey–Farin split.

Table 5.1
Possible splits for the different tetrahedra for cubes in $\mathcal{K}_0 \cup \cdots \cup \mathcal{K}_3$.

	No split	1-WF	2-WF	3-WF	4-WF
$\mathcal{K}_0 T^1_{i,j,k}$	×	–	–	–	–
$\mathcal{K}_0 T^2_{i,j,k}$	×	–	–	–	–
$\mathcal{K}_0 T^3_{i,j,k}$	–	×	–	–	–
$\mathcal{K}_0 T^4_{i,j,k}$	–	×	–	–	–
$\mathcal{K}_0 T^5_{i,j,k}$	–	–	–	–	×
$\mathcal{K}_1 T^1_{i,j,k}$	×	–	–	–	–
$\mathcal{K}_1 T^2_{i,j,k}$	–	×	–	–	–
$\mathcal{K}_1 T^3_{i,j,k}$	–	–	×	–	–
$\mathcal{K}_1 T^4_{i,j,k}$	–	–	–	×	–
$\mathcal{K}_1 T^5_{i,j,k}$	–	–	–	–	×
$\mathcal{K}_2 T^1_{i,j,k}$	o	×	–	–	–
$\mathcal{K}_2 T^2_{i,j,k}$	–	o	×	–	–
$\mathcal{K}_2 T^3_{i,j,k}$	–	–	o	×	–
$\mathcal{K}_2 T^4_{i,j,k}$	–	–	–	o	×
$\mathcal{K}_2 T^5_{i,j,k}$	–	–	–	–	×
$\mathcal{K}_3 T^1_{i,j,k}$	o	–	–	×	–
$\mathcal{K}_3 T^2_{i,j,k}$	o	–	–	×	–
$\mathcal{K}_3 T^3_{i,j,k}$	–	–	–	o	×
$\mathcal{K}_3 T^4_{i,j,k}$	–	–	–	o	×
$\mathcal{K}_3 T^5_{i,j,k}$	–	–	–	–	×

Now, let $\mathcal{L}$ be the set of all interpolation points chosen in Algorithm 5.1 and Δ^* be the tetrahedral partition obtained from Algorithm 5.1.

It may be helpful to list the types of the splits that may be applied to the tetrahedra in the different classes—see Tables 5.1 and 5.2. m-WF stands for the m-th-order partial Worsey–Farin split. In the table the symbol "–" indicates that the corresponding tetrahedra are not subdivided with the indicated split. The symbol "o" identifies cases which can occur when the corresponding cube is on the boundary of $\diamond$. The splits for tetrahedra of cubes in the interior of $\diamond$ are identified with the symbol "×".

Note that the tetrahedral partition obtained is not the final partition. Let T and $\widetilde{T}$ be two tetrahedra with a common face F. By Algorithm 5.1, F has not been split as a face of tetrahedron T, but it has been split as a face of $\widetilde{T}$. We first determine the spline on T. Then, we split T at F and represent the spline as a spline on the subdivided tetrahedron T. This representation can be easily obtained by applying the de Casteljau algorithm. Next, the spline can be determined on $\widetilde{T}$. Note that in the Tables 5.1 and 5.2 these additional splits are not considered.

Definition 5.1. A set $\mathcal{L} := \{\xi_i\}_{i=1,\ldots,N}$ is called a Lagrange interpolation set for the spline space $\mathcal{S}^1_3(\Delta^*)$ if N is the dimension of $\mathcal{S}^1_3(\Delta^*)$ and for every choice of real numbers $\{f_i\}_{i=1,\ldots,N}$ there is a unique spline $s \in \mathcal{S}$ satisfying

$$s(\xi_i) = f_i, \quad i = 1, \ldots, N.$$

Table 5.2
Possible splits for the different tetrahedra for cubes in $\mathcal{K}_4$.

	No split	1-WF	2-WF	3-WF	4-WF
$\mathcal{K}_4 T^1_{i,j,k}$	–	o	–	–	×
$\mathcal{K}_4 T^2_{i,j,k}$	–	o	o	–	×
$\mathcal{K}_4 T^3_{i,j,k}$	–	–	–	o	×
$\mathcal{K}_4 T^4_{i,j,k}$	–	–	–	–	×
$\mathcal{K}_4 T^5_{i,j,k}$	–	–	–	–	×

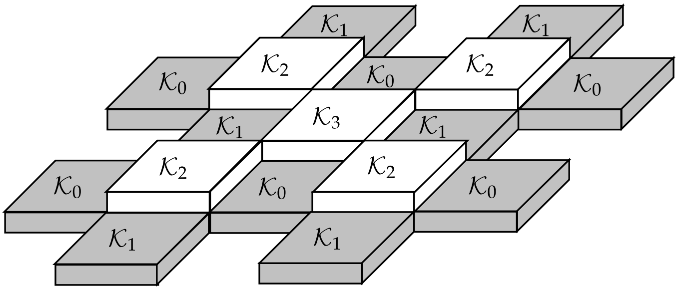

Fig. 3.2. Layers of cubes.

Definition 5.2. A Lagrange interpolation set $\mathcal{L}$ is local if for any tetrahedron T in Δ^* and a spline $s \in \mathcal{S}^1_3(\Delta^*)$, $s|_T$ depends only on values $\{f_\xi\}_{\xi \in \mathcal{L} \cap \Omega_T}$, with $\Omega_T \subset \Omega$.

$\mathcal{L}$ is also stable if

$$|c_\xi| \leq K \max_{\eta \in \Omega_T} |f_\eta| \tag{5.1}$$

holds for the B-coefficients c_ξ of $s|_T$, with an absolute constant K.

Now, we are ready to state the main result of this paper.

Theorem 5.1. *$\mathcal{L}$ is a local and stable Lagrange interpolation set for $S^1_3(\Delta^*)$.*

Before giving the proof, we describe how a spline on $\diamond$ can be computed. Therefore, we argue with layers of cubes; see Fig. 3.2. In Algorithm 5.1, we first choose the interpolation points in tetrahedra of cubes in class $\mathcal{K}_0$. These can be chosen independently, since the cubes in class $\mathcal{K}_0$ are disjoint (cf. Lemma 3.1). Thus, we can compute s on all cubes in class $\mathcal{K}_0$ independently. Moreover, for all cubes in class $\mathcal{K}_0$, these computations are the same.

Next, s can be computed on the cubes in class $\mathcal{K}_1$. Therefore, the interpolation points in tetrahedra of cubes in class $\mathcal{K}_1$ are chosen considering the common edges with cubes in $\mathcal{K}_0$. The computations are very similar to those for the cubes in class $\mathcal{K}_0$. For cubes in class $\mathcal{K}_1$ in the interior of $\diamond$, we always have the same computations, since these cubes are disjoint and each one touches exactly four cubes in class $\mathcal{K}_0$, which lie in the same layer. Thus, s only depends on the values of the interpolation points in each cube in class $\mathcal{K}_1$ and at most four cubes in class $\mathcal{K}_0$ with a common edge in the same layer. In the same way s can be computed in the cubes in the other classes. So we compute s in the cubes of $\mathcal{K}_0, \mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3$ and finally $\mathcal{K}_4$ (see Figs. 3.4 and 3.5).

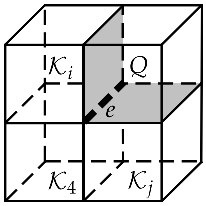

Fig. 3.3. Cube $Q \in \mathcal{K}_4$ sharing e with a cube in class $\mathcal{K}_4$ and sharing two faces with cubes in classes $\mathcal{K}_i$ and $\mathcal{K}_j$, $i, j = 0, \ldots, 3$.

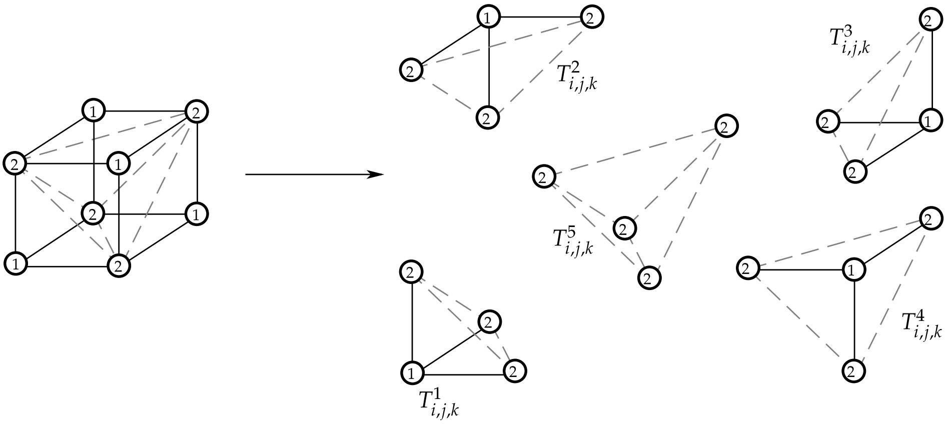

Fig. 3.4. The partition of a cube $Q_{i,j,k} \in \mathcal{K}_0 \cup \mathcal{K}_1 \cup \mathcal{K}_2 \cup \mathcal{K}_3$ into five tetrahedra with marked vertices of type-1 and type-2.

Therefore, a spline s can be computed locally in the sense that $s|_Q$, where Q is a cube in class $\mathcal{K}_4$, only depends on values in Q and the coefficients of s in the surrounding cubes.

For the proof of our main result we need the following two bivariate Lemmas from [3]. Therefore, we also need some bivariate Bernstein–Bézier techniques (cf. [5], chapter 2.3). Let $\mathcal{D}_F := \{\xi^T_{i,j,k} := \frac{iv_1+jv_2+kv_3}{3},\ i+j+k=3\}$ be the set of domain points of a triangle $F := \langle v_1, v_2, v_3 \rangle$. Moreover, let $\{B_{i,j,k}\}_{i+j+k=3}$ be the bivariate Bernstein polynomials associated with F. Then every bivariate polynomial p of degree 3 can be uniquely written as

$$p = \sum_{i+j+k=3} c^F_{i,j,k} B_{i,j,k},$$

where $c^F_{i,j,k}$ are the B-coefficients of p associated with the domain points in F.

Lemma 5.1. *Suppose that we are given all of the coefficients $c^F_{i,j,k}$ of a bivariate cubic polynomial p except for $c^F_{1,1,1}$. Then for any given real number z and any point v_F in the interior of F, there exists a unique $c^F_{1,1,1}$ such that $p(v_F) = z$.*

Let F_{CT} be a triangle which has been subjected to the Clough–Tocher split with subfaces F_i, $i = 1, 2, 3$, as in Section 4. Moreover, let s be a bivariate cubic C^1 spline, with B-coefficients $\{c^{F_l}_{i,j,k}\}_{i+j+k=3,l=1,2,3}$.

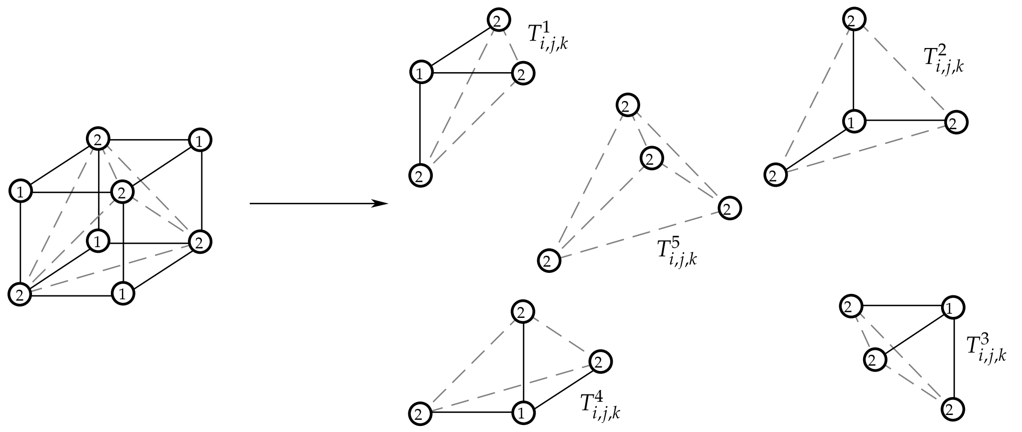

Fig. 3.5. The partition of a cube $Q_{i,j,k} \in \mathcal{K}_4$ into five tetrahedra with marked vertices of type-1 and type-2.

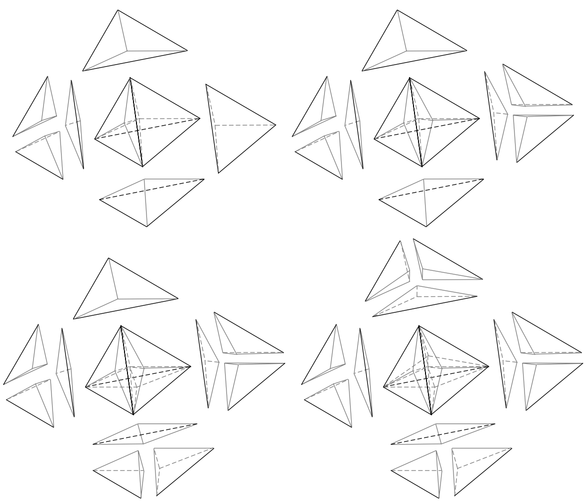

Fig. 4.1. Partial Worsey–Farin splits of m-th order for $m = 1, \dots, 4$.

Lemma 5.2. *Suppose that we are given all of the coefficients of* $s \in S_3^1(F_{CT})$ *except for* $c^{F_1}_{3,0,0}, c^{F_1}_{2,1,0}, c^{F_1}_{2,0,1}, c^{F_1}_{1,1,1}$. *Then for any given real number* z, *there exists a unique choice of these coefficients such that* $s(v_F) = z$.

Note that the spline in Lemma 5.2 is not uniquely determined without the interpolation condition at v_F.

Proof of Theorem 5.1. To show that $\mathcal{L}$ is a local Lagrange interpolation set for $S_3^1(\Delta^*)$, we fix the values $\{z_\xi\}_{\xi \in \mathcal{L}}$ for a spline $s \in S_3^1(\Delta^*)$. Then we show that s is locally, stably and uniquely determined. We have to consider three cases.

Case 1: $Q_{i,j,k} \in \mathcal{K}_0$.

We begin with $s|_{Q_{i,j,k}}$, $Q_{i,j,k} \in \mathcal{K}_0$. By Lemma 3.1 all cubes $Q_{i,j,k} \in \mathcal{K}_0$ are disjoint. Therefore, we only consider one cube $Q \in \mathcal{K}_0$; the remaining cubes in class $\mathcal{K}_0$ can be treated analogously. For simplicity, in the following we set $i = j = k = 0$.

Case 1.1: $T^1_{0,0,0}$.

Since $\mathcal{L}$ contains all the points $\mathcal{D}_{T^1_{0,0,0}}$, the B-coefficients of $s|_{T^1_{0,0,0}}$ can be uniquely and stably computed form the values $\{z_\xi\}_{\xi\in\mathcal{L}\cap T^1_{0,0,0}}$. Thus, all B-coefficients of s associated with domain points in $D_1(v),\ v \in T^1_{0,0,0}$, and $E_1(e),\ e \in T^1_{0,0,0}$, can be uniquely and stably determined using the C^1 smoothness conditions of s at the edges and vertices of $T^1_{0,0,0}$. Since these computations only involve values $\{z_\xi\}_{\xi\in\mathcal{L}\cap T^1_{0,0,0}}$, they are also local.

Case 1.2: $T^2_{0,0,0}$.

Next, we consider the tetrahedron $T^2_{0,0,0}$. The B-coefficients associated with the 10 domain points in $E_1^{T^2_{0,0,0}}(\langle v_{0,0,1}, v_{0,1,0}\rangle)$ are already uniquely determined. Then the remaining undetermined B-coefficients of $s|_{T^2_{0,0,0}}$ can be uniquely and stably computed using the values $\{z_\xi\}_{\xi\in\mathcal{L}\cap T^2_{0,0,0}}$. Thus, the spline $s|_{T^2_{0,0,0}}$ can be computed locally, since the corresponding B-coefficients only depend on the values $\{z_\xi\}_{\xi\in\mathcal{L}\cap(T^1_{0,0,0}\cup T^2_{0,0,0})}$. Now, also all B-coefficients of s associated with domain points in $D_1(v),\ v \in T^2_{0,0,0}$, and $E_1(e),\ e \in T^2_{0,0,0}$, can be uniquely and stably determined using the C^1 smoothness conditions of s at the edges and vertices of $T^2_{0,0,0}$.

Case 1.3: $T^3_{0,0,0}$.

Next, we consider the tetrahedron $T^3_{0,0,0}$, which has been subjected to a first-order partial Worsey–Farin split. The B-coefficients associated with the domain points in $E_1(\langle v_{0,0,1}, v_{1,0,0}\rangle)\cup E_1(\langle v_{0,0,1}, v_{1,1,1}\rangle)$ are already uniquely determined. Those coefficients associated with the domain points in $D_1(v_{1,0,1})$ can be uniquely determined from the values at the interpolation points $\xi \in D_1^T(v_{1,0,1}) \subset \mathcal{L},\ v_{1,0,1} \in T \subset T^3_{0,0,0}$. The undetermined B-coefficients in the faces $\langle v_{1,0,0}, v_{1,0,1}, v_{1,1,1}\rangle$ and $\langle v_{0,0,1}, v_{1,0,0}, v_{1,1,1}\rangle$ can be computed from the values at the two points $v_F \in \mathcal{L}$ and $v_{\widetilde{F}} \in \mathcal{L}$ in these faces using Lemmas 5.1 and 5.2, respectively. Thus, all B-coefficients associated with the domain points in the minimal determining set $\mathcal{M}_1$ from Theorem 4.1 are uniquely and stably determined. Therefore, all other B-coefficients of $s|_{T^3_{0,0,0}}$ are uniquely and stably determined. The computation of these B-coefficients is also local, since they only depend on the values $\{z_\xi\}_{\xi\in\mathcal{L}\cap(T^1_{0,0,0}\cup T^2_{0,0,0}\cup T^3_{0,0,0})}$. So, all B-coefficients of s associated with domain points in $D_1(v),\ v \in T^3_{0,0,0}$, and $E_1(e),\ e \in T^3_{0,0,0}$, can be uniquely and stably determined using the C^1 smoothness conditions of s at the edges and vertices of $T^3_{0,0,0}$.

Case 1.4: $T^4_{0,0,0}$.

Now, we consider the tetrahedron $T^4_{0,0,0}$. This tetrahedron has also been subjected to a first-order partial Worsey–Farin split. Moreover, the B-coefficients associated with the domain points in $E_1(\langle v_{0,1,0}, v_{1,0,0}\rangle) \cup E_1(\langle v_{0,1,0}, v_{1,1,1}\rangle) \cup E_1(\langle v_{1,0,0}, v_{1,1,1}\rangle)$ are already uniquely determined. Thus, all B-coefficients of $s|_{\langle v_{0,1,0}, v_{1,0,0}, v_{1,1,1}\rangle}$ are already uniquely and stably determined, since we already know the B-coefficients corresponding to the domain points in the minimal determining set of the classical Clough–Tocher macro-element (cf. [1]). The remaining B-coefficients of $s|_{T^4_{0,0,0}}$ can be uniquely and stably determined from the values at the interpolation points $\xi \in D_1^T(v_{1,1,0}) \subset \mathcal{L},\ v_{1,1,0} \in T \subset T^4_{0,0,0}$. Thus, all B-coefficients of s associated with domain points in $D_1(v),\ v \in T^4_{0,0,0}$, and $E_1(e),\ e \in T^4_{0,0,0}$, can be uniquely and stably determined using the C^1 smoothness conditions of s at the edges and vertices of $T^4_{0,0,0}$.

At this point s is uniquely and stably determined on all edges of Δ^*.

Case 1.5: $T^5_{0,0,0}$.

Next, we consider the last tetrahedron in Q, $T^5_{0,0,0}$. By the construction of Δ, $T^5_{0,0,0}$ has one common face with each of the four tetrahedra $T^l_{0,0,0}$, $l = 1, \ldots, 4$. Furthermore, by Algorithm 5.1 $T^5_{0,0,0}$ has been subjected to the fourth-order partial Worsey–Farin split. Since $s|_{T^l_{0,0,0}}$, $l = 1, \ldots, 4$, is already uniquely determined, we use the de Casteljau Algorithm to subdivide the tetrahedra $T^l_{0,0,0}$, $l = 1, 2, 3$, with a partial Worsey–Farin split with a split face at the common face with $T^5_{0,0,0}$ and the C^1 smoothness conditions at the edges and vertices of Δ^*, to uniquely and stably determine the B-coefficients associated with the domain points on the faces of $T^5_{0,0,0}$. Now, we have determined all B-coefficients associated with the domain points in the minimal determining set $\mathcal{M}_4$ for a spline s over the fourth-order partial Worsey–Farin split and the remaining undetermined B-coefficients of $s|_{T^5_{0,0,0}}$ can be uniquely and stably computed (see Theorem 4.1). These computations are also local, since the B-coefficients of $s|_{T^5_{0,0,0}}$ only depend on the values $\{z_\xi\}_{\xi \in \mathcal{L} \cap Q}$.

Case 2: $Q_{i,j,k} \in (\mathcal{K}_1 \cup \mathcal{K}_2 \cup \mathcal{K}_3)$.

In the following we consider the cubes $Q_{i,j,k} \in (\mathcal{K}_1 \cup \mathcal{K}_2 \cup \mathcal{K}_3)$. The spline s is determined on these cubes according to the ordering imposed by their classes. So, first s can be determined on all cubes in the class $\mathcal{K}_1$, then on all cubes in class $\mathcal{K}_2$ and afterwards on all cubes in class $\mathcal{K}_3$. By Lemma 3.1, no two cubes in class $\mathcal{K}_i$, $i = 0, \ldots, 3$, touch each other. Therefore, we can compute s on each cube $Q_{i,j,k} \in \mathcal{K}_l$, $l = 1, \ldots, 3$, in the same way. These computations are also very similar to those for determining $s|_{\mathcal{K}_0}$. Let $\widetilde{Q}$ be a cube in $(\mathcal{K}_1 \cup \mathcal{K}_2 \cup \mathcal{K}_3)$ and $\widetilde{T}^l_{i,j,k}$, $l = 1, \ldots, 5$, the tetrahedra in $\widetilde{Q}$. Since s is already uniquely determined on the edges of Δ^*, the only undetermined B-coefficients of $s|_{\widetilde{Q}}$ are associated with domain points on the faces of the tetrahedra in $\widetilde{Q}$.

Case 2.1: $T^1_{i,j,k}, T^2_{i,j,k}, T^3_{i,j,k}, T^4_{i,j,k}$.

We first consider the faces of the tetrahedron $\widetilde{T}^1_{i,j,k}$. Let F be a face of $\widetilde{T}^1_{i,j,k}$. We have to distinguish four cases:

(1) If F has no marked edges, then $\mathcal{L}$ contains the barycenter v_F of F and the remaining undetermined B-coefficient of $s|_F$ can be uniquely and stably determined using Lemma 5.1.
(2) If F has one marked edge, then the remaining undetermined B-coefficient of $s|_F$ can be uniquely and stably determined using the C^1 smoothness conditions at the marked edge.
(3) If F has two marked edges, then F is split with a Clough–Tocher split and $\mathcal{L}$ contains the barycenter v_F of F and the remaining undetermined B-coefficients of $s|_F$ can be uniquely and stably determined using Lemma 5.2.
(4) If F has three marked edges, then F is split with a Clough–Tocher split and the remaining undetermined B-coefficients of $s|_F$ can be uniquely and stably computed using the C^1 smoothness conditions at the edges of F, since $s|_F$ is just a classical C^1 Clough–Tocher macro-element (cf. [1]).

If none of the faces of $\widetilde{T}^1_{i,j,k}$ is subdivided by a Clough–Tocher split, $s|_{\widetilde{T}^1_{i,j,k}}$ is uniquely and stably determined. If one or more of the faces of $\widetilde{T}^1_{i,j,k}$ is subdivided, we can use Theorem 4.1 to uniquely and stably determine $s|_{\widetilde{T}^1_{i,j,k}}$. All these computations are also local. The spline $s|_{\widetilde{T}^1_{i,j,k}}$ can be uniquely determined from values corresponding to interpolation points in $\widetilde{T}^1_{i,j,k}$, the B-coefficients of s associated with domain points in the cubes in class $\mathcal{K}_0$ sharing vertices with $\widetilde{Q}$ and the maximal four cubes in class $\mathcal{K}_{i-1}$ touching $\widetilde{Q}$ at the edges, where $\widetilde{Q}$ is in class $\mathcal{K}_i$. The cube $\widetilde{Q}$ can touch fewer cubes in class $\mathcal{K}_{i-1}$ if it lies on the boundary of $\diamond$.

On the tetrahedra $\widetilde{T}^l_{i,j,k},\ l = 2, 3, 4,\ s$ can be determined in the same way as on $\widetilde{T}^1_{i,j,k}$. The only difference is, that $s|_{\widetilde{T}^l_{i,j,k}},\ l = 2, 3, 4$, then also depends on the B-coefficients of s associated with domain points in the tetrahedra $\widetilde{T}^m_{i,j,k},\ m = 1, \ldots, l$.

Case 2.2: $T^5_{i,j,k}$.

Now, we can determine $s|_{\widetilde{T}^5_{i,j,k}}$. The tetrahedron $\widetilde{T}^5_{i,j,k}$ has exactly one common face with each of the four tetrahedra $\widetilde{T}^1_{i,j,k}, \ldots, \widetilde{T}^4_{i,j,k}$ and by Algorithm 5.1 $\widetilde{T}^5_{i,j,k}$ has been subjected to the fourth order partial Worsey–Farin split. So, we use the de Casteljau Algorithm to subdivide the tetrahedra $\widetilde{T}^1_{i,j,k}, \ldots, \widetilde{T}^4_{i,j,k}$ with a partial Worsey–Farin split with a split face at the common face with $T^5_{i,j,k}$, if not done earlier, and the C^1 smoothness conditions at the edges and vertices of Δ^*, to uniquely and stably determine the B-coefficients associated with the domain points on the faces of $\widetilde{T}^5_{i,j,k}$. Thus, we have determined all B-coefficients associated with the domain points in the minimal determining set $\mathcal{M}_4$ for a spline s over the fourth-order partial Worsey–Farin split and the remaining undetermined B-coefficients of $s|_{\widetilde{T}^5_{i,j,k}}$ can be uniquely and stably computed (see Theorem 4.1). These computations are also local, since the B-coefficients of $s|_{\widetilde{T}^5_{i,j,k}}$ depend on the same values and already determined B-coefficients as the previous tetrahedra $\widetilde{T}^1_{i,j,k}, \ldots, \widetilde{T}^4_{i,j,k}$.

Case 3: $Q_{i,j,k} \in \mathcal{K}_4$.

Finally, we determine $s|_{Q_{i,j,k}},\ Q_{i,j,k} \in \mathcal{K}_4$. Let $\hat{Q}$ be a cube in class $\mathcal{K}_4$. Then the tetrahedra in $\hat{Q}$ can be determined in the same way and in the same ordering as the tetrahedra in the cubes in $\mathcal{K}_0, \ldots, \mathcal{K}_3$. Note that for each tetrahedron T sharing a face with a tetrahedron $\hat{T} \in \hat{Q}$ we also split these faces in the neighboring tetrahedra, if this has not already happened, and use the de Casteljau Algorithm to subdivide $s|_T$ with the corresponding partial Worsey–Farin split.

These computations are also local and stable and $s|_{\hat{Q}}$ depends only on the values associated with the interpolation points inside $\hat{Q}$ and the B-coefficients of s associated with the domain points in the cubes sharing an edge or a vertex with $\hat{Q}$, which are not in $\mathcal{K}_4$. Moreover, none of the C^1 smoothness conditions is violated. By Lemma 3.1 two cubes in class $\mathcal{K}_4$ can only touch on edges and moreover $\hat{Q}$ must also touch two cubes in the classes $\mathcal{K}_i,\ i = 0, \ldots, 3$, at these edges. But s is already determined on the cubes in these classes. □

In the proof it is shown that $\mathcal{L}$ is a local and stable Lagrange interpolation method for $\mathcal{S}^1_3(\Delta^*)$. Since the spline is computed locally on each cube, it can easily be seen that for any tetrahedron T of a cube in class $\mathcal{K}_4$, $s|_T$ depends only on values $\{z_\xi\}_{\xi \in \mathcal{L} \cap \Omega_T}$, where Ω_T is the collection of at most $5 \times 5 \times 7$ cubes, where T lies in the middle of Ω_T. But we emphasize that s only depends on the cubes in Ω_T of lower class. Thus, for tetrahedra in cubes of lower class, Ω_T is still smaller. Moreover, since we only use the C^1 smoothness conditions and some small systems of linear equations to determine s, all computations needed to uniquely determine s are stable in the sense that

$$|c_\xi| \le \max_{\eta \in \Omega_T} |z_\eta| \tag{5.2}$$

holds with an absolute constant K, since the angles of Δ are bounded away from zero by an absolute constant independent of the mesh size of Δ. Therefore, also

$$\|s\|_T \le \widetilde{K} \|f\|_{\Omega_T} \tag{5.3}$$

holds with an absolute constant $\widetilde{K}$, where s is the interpolating spline of the function $f \in C(\Omega)$.

Remark 5.1. In the notation of [5], $\mathcal{L}$ and $\mathcal{S}_3^1(\Delta^*)$ form a Lagrange interpolation pair. Moreover, from the proof of Theorem 5.1 it is easy to see that $\mathcal{L}$ is a minimal determining set for $\mathcal{S}_3^1(\Delta^*)$ and that the set $\{\epsilon_\xi\}_{\xi\in\mathcal{L}}$ is also a nodal minimal determining set for $\mathcal{S}_3^1(\Delta^*)$, where ϵ_ξ denotes the point evaluation at the point ξ in Ω.

6. Bounds on the error of the interpolant

In this section we want to provide a bound on the error $\|f - s\|_\Omega$ for smooth functions, where the error is measured in the maximum norm on Ω.

Let $\mathcal{L}$ be the Lagrange interpolation method constructed in Section 5 associated with the spline space $\mathcal{S}_3^1(\Delta^*)$. Then for every $f \in C(\Omega)$, there is a unique spline $\mathcal{I}f \in \mathcal{S}_3^1(\Delta^*)$ such that

$$\mathcal{I}f(\xi) = f(\xi), \quad \xi \in \mathcal{L}. \tag{6.1}$$

This defines a linear projector $\mathcal{I}$ mapping $C(\Omega)$ onto $\mathcal{S}_3^1(\Delta^*)$.

Now for a compact set $B \subseteq \Omega$ and an integer $m \geq 1$, let $W_\infty^m(B)$ be the usual Sobolev space defined on B with seminorm

$$|f|_{m,B} := \sum_{|\alpha|=m} \|D^\alpha f\|_B,$$

where $\|\cdot\|_B$ denotes the infinity norm on B and $D^\alpha := D_x^{\alpha_1} D_y^{\alpha_2} D_z^{\alpha_3}$ with $\alpha = (\alpha_1, \alpha_2, \alpha_3)$. Let $|\Delta^*|$ be the mesh size of Δ^*, i.e. the maximum diameter of the tetrahedra in Δ^*.

Theorem 6.1. *Let $f \in W_\infty^{m+1}(\Omega)$ for some $0 \leq m \leq 3$. Then there exists an absolute constant K such that*

$$\|D^\alpha (f - \mathcal{I}f)\|_\Omega \leq K|\Delta^*|^{m+1-|\alpha|}|f|_{m+1,\Omega}, \tag{6.2}$$

for all multi-indices α with $0 \leq |\alpha| \leq m$.

Proof. Fix m, and let $f \in W_\infty^{m+1}(\Omega)$. Fix $T \in \Delta^*$, and let Ω_T be as in Section 5. By Lemma 4.3.8 of [2], there exists a cubic polynomial p such that

$$\|D^\beta (f - p)\|_{\Omega_T} \leq K_2|\Omega_T|^{m+1-|\beta|}|f|_{m+1,\Omega_T}, \tag{6.3}$$

for all $0 \leq |\beta| \leq m$, where $|\Omega_T|$ is the diameter of Ω_T. Since $\mathcal{I}p = p$, it follows that

$$\|D^\alpha (f - \mathcal{I}f)\|_T \leq \|D^\alpha (f - p)\|_T + \|D^\alpha \mathcal{I}(f - p)\|_T.$$

Due to (6.3) with $\beta = \alpha$, it suffices to estimate the second term $\|D^\alpha \mathcal{I}(f - p)\|_T$. By the Markov inequality [13] and (5.3)

$$\|D^\alpha \mathcal{I}(f - p)\|_T \leq K_3|T|^{-|\alpha|}\|\mathcal{I}(f - p)\|_T \leq K_4|T|^{-|\alpha|}\|f - p\|_{\Omega_T},$$

where $|T|$ is the diameter of T. Because of the geometry of the partition, two absolute constants K_5 and K_6 exist with $\Omega_T \leq K_5|T|$ and $|T| \leq K_6|\Delta^*|$. In view of this and by inserting (6.3) with $\beta = 0$ for $\|f - p\|_{\Omega_T}$, we get

$$\|D^\alpha (f - \mathcal{I}f)\|_T \leq K_1|\Omega_T|^{m+1-|\alpha|}|f|_{m+1,\Omega_T}.$$

Now taking the maximum over all tetrahedra in Δ^* leads to (6.2). □

Thus, it is also shown that the Lagrange interpolation method constructed in this paper yields optimal approximation order.

Table 7.1
Results for the type-4 partition.

n	Dim	Error_E	Error_F	Error_T	Decay
128^3	17 073 158	4.84×10^{-5}	2.08×10^{-4}	2.313×10^{-4}	–
256^3	135 399 430	3.07×10^{-6}	1.11×10^{-5}	1.268×10^{-5}	4.19
512^3	1 078 464 518	1.88×10^{-7}	6.02×10^{-7}	6.952×10^{-7}	4.19
1024^3	8 608 817 158	1.18×10^{-8}	3.24×10^{-8}	3.591×10^{-8}	4.27
2048^3	68 794 990 598	7.36×10^{-10}	2.01×10^{-9}	2.252×10^{-9}	3.99
4096^3	550 057 836 550	4.43×10^{-11}	1.23×10^{-10}	1.407×10^{-10}	4.00

Table 7.2
Results for the Freudenthal partition.

n	Dim	Error_E	Error_F	Error_T	Decay
128^3	17 138 451	5.00×10^{-5}	2.43×10^{-4}	2.811×10^{-4}	–
256^3	135 661 075	3.18×10^{-6}	1.63×10^{-5}	1.864×10^{-5}	3.91
512^3	1 079 512 083	1.99×10^{-7}	1.03×10^{-6}	1.065×10^{-6}	4.13
1024^3	8 613 009 427	1.23×10^{-8}	6.43×10^{-8}	6.630×10^{-8}	4.01
2048^3	68 811 763 731	6.89×10^{-10}	3.57×10^{-9}	4.128×10^{-9}	4.01
4096^3	550 124 937 235	4.45×10^{-11}	2.23×10^{-10}	2.577×10^{-10}	4.00

7. Numerical tests and visualizations

In this section we illustrate our method by interpolating the Marschner–Lobb test function

$$p(x, y, z) := \frac{1 - \sin(\pi \frac{z}{2}) + \alpha(1 + \rho_r(x^2 + y^2))}{2(1 + \alpha)},$$

with $\rho_r := \cos(2\pi f_M \cos(\pi \frac{r}{2}))$, where $f_M = 6$ and $\alpha = 0.25$.

For the following numerical tests and visualizations, we interpolate the function on $\Omega = [[-0.1,\ 0.9], [-0.1,\ 0.9], [-0.6,\ 0.4]]$.

7.1. Numerical tests

In Table 7.1, we list the number of cubes n, the dimension of the spline space (dim), the error on the edges (Error_E), on the faces (Error_F), in the tetrahedra (Error_T) and the decay exponent (decay). Therefore, we compute the error of the spline, using 8 points per edge, 36 points per subface and 56 points per subtetrahedron. This confirms that our method yields optimal approximation order.

To compare our results to those from other methods, we also compute the errors and the decay exponent for the method described in [3] for the same test function. These can be seen in Table 7.2.

It can easily be seen that the results for our method are slightly better, with the same order of magnitude, although we use one tetrahedron less per cube.

We finally compare the errors obtained by our method with the errors for interpolation using linear splines, for the case when the exact values of the Marschner–Lobb test function are only interpolated at the vertices of the cubes in both cases. The values at the additional interpolation points needed for our method are taken from linear splines with about the same errors (for sufficiently many cubes) as our method (for far fewer cubes)—see Table 7.3, where s is a spline

Table 7.3
Results for a spline from linear data.

n	Error s	N	Error $\widetilde{s}_{lin}$	n	Error $\widetilde{s}$
256^3	1.2678×10^{-5}	5632^3	2.72×10^{-6}	256^3	3.57×10^{-5}
360^3	3.86×10^{-6}	10080^3	8.39×10^{-7}	360^3	4.15×10^{-6}
512^3	6.95×10^{-7}	22528^3	1.67×10^{-7}	512^3	7.98×10^{-7}

Table 7.4
Data reduction for linear splines.

n	Dimension cubic spline	N	Dimension linear spline	Quotient
256^3	135 399 430	1600^3	4 096 000 000	30.25
360^3	375 583 686	4600^3	97 336 000 000	259.16
512^3	1 078 464 518	10800^3	1 259 712 000 000	1168.06

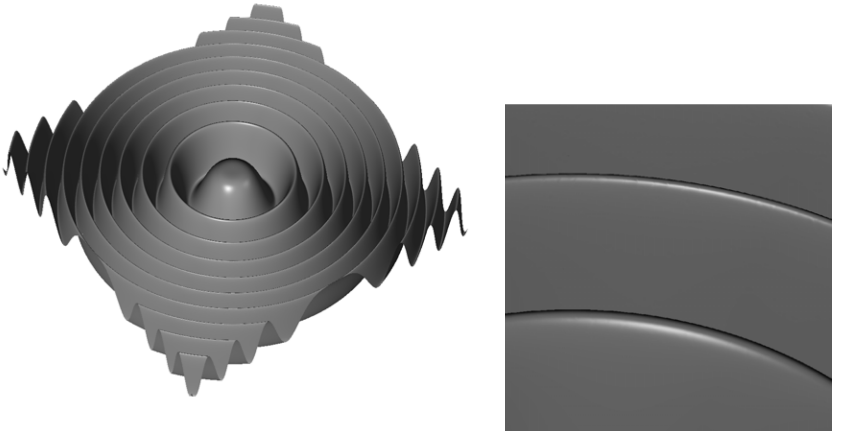

Fig. 7.1. Isosurface with value 0.5 of the Marschner–Lobb test function with 200 × 200 × 200 cubes, visualized by our method.

constructed from exact data at all Lagrange interpolation points, $\widetilde{s}_{lin}$ a linear spline and $\widetilde{s}$ a spline with exact data only at the vertices of the cubes. Moreover, n is the number of cubes for the cubic splines and N the number of cubes for the linear splines.

The results show that if we compare the dimensions of the space of cubic C^1 splines used for our method with the dimensions of the spaces of linear splines which yield approximately the same errors, we obtain data reductions up to factor 10^3—see Table 7.4, where n is the number of cubes for the cubic splines and N the number of cubes for the linear splines. Moreover, we also list the quotient of the dimension of the linear spline and the cubic spline.

7.2. *Visualizations*

In the following we also illustrate the application of our method. Therefore, we visualize an isosurface with value 0.5 of the Marschner–Lobb test function with 200 × 200 × 200 cubes. To show the smoothness of the interpolant more clearly, we also visualize an enlargement of a small section of the isosurfaces (see Fig. 7.1).

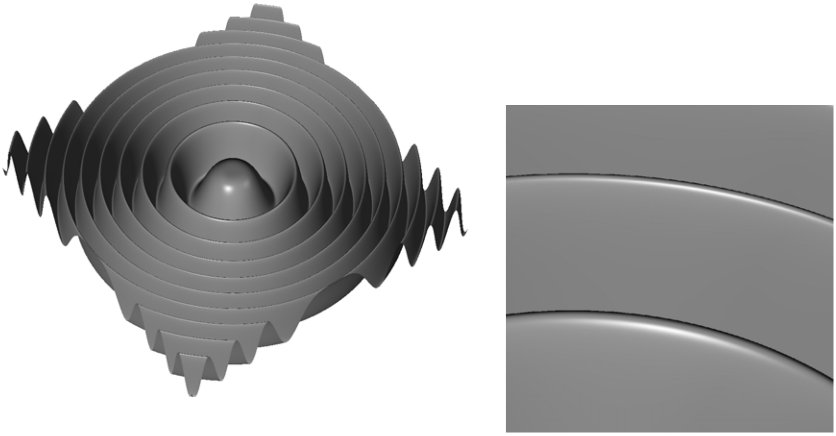

Fig. 7.2. Isosurface with value 0.5 of the Marschner–Lobb test function with $200 \times 200 \times 200$ cubes, visualized by the method described in [3].

In order to compare the results visually, we also illustrate the method described in [3] (see Fig. 7.2).

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References

[1] P. Alfeld, L.L. Schumaker, Smooth macro-elements based on Clough–Tocher triangle splits, Numer. Math. 90 (2002) 597–616.

[2] S.C. Brenner, L.R. Scott, The Mathematical Theory of Finite Element Methods, in: Texts in Applied Mathematics, vol. 15, Springer-Verlag, New York, 1994.

[3] G. Hecklin, G. Nürnberger, L.L. Schumaker, F. Zeilfelder, A local Lagrange interpolation method based on C^1 cubic splines on Freudenthal partitions, Math. Comp. 77 (2008) 1017–1036.

[4] G. Hecklin, G. Nürnberger, L.L. Schumaker, F. Zeilfelder, Local Lagrange interpolation with cubic C^1 splines on tetrahedral partitions, J. Approx. Theory, 2008, in press (doi:10.1016/j.jat.2008.01.012).

[5] M.-J. Lai, L.L. Schumaker, Spline Functions on Triangulations, Cambridge University Press, 2007.

[6] G. Nürnberger, F. Zeilfelder, Local Lagrange interpolation by cubic splines on a class of triangulations, in: K. Kopotun, T. Lyche, M. Neamtu (Eds.), Trends in Approximation Theory, Vanderbilt University Press, Nashville, TN, 2001, pp. 341–350.

[7] G. Nürnberger, F. Zeilfelder, Lagrange interpolation by bivariate C^1-splines with optimal approximation order, Adv. Comput. Math. 21 (2004) 381–419.

[8] G. Nürnberger, V. Rayevskaya, L.L. Schumaker, F. Zeilfelder, Local Lagrange interpolation with C^2-splines of degree seven on triangulations, in: M. Neamtu, E. Saff (Eds.), Advances in Constructive Approximation, Vanderbilt 2003, Nashboro Press, Brentwood, TN, 2004, pp. 345–370.

[9] G. Nürnberger, L.L. Schumaker, F. Zeilfelder, Lagrange interpolation by C^1 cubic splines on triangulated quadrangulations, Adv. Comput. Math. 21 (3–4) (2004) 357–380.

[10] G. Nürnberger, L.L. Schumaker, F. Zeilfelder, Two Lagrange interpolation methods based on C^1 splines on tetrahedral partitions, in: C. Chui, M. Neamtu, L.L. Schumaker (Eds.), Approximation Theory XI: Gatlinburg 2004, Nashboro Press, Brentwood, TN, 2005, pp. 327–344.

[11] G. Nürnberger, V. Rayevskaya, L.L. Schumaker, F. Zeilfelder, Local Lagrange interpolation with bivariate splines of arbitrary smoothness, Constr. Approx. 23 (2006) 33–59.

[12] L.L. Schumaker, T. Sorokina, C^1 quintic splines on type-4 tetrahedral partitions, Adv. Comput. Math. 21 (3–4) (2004) 421–444.
[13] D.R. Wilhelmsen, A Markov inequality in several dimensions, J. Approx. Theory 11 (3) (1974) 216–220.
[14] A.J. Worsey, G. Farin, An n-dimensional Clough–Tocher interpolant, Constr. Approx. 3 (1) (1987) 99–110.