Journal of Computational and Applied Mathematics 12&13 (1985) 221–232
North-Holland

A review of multivariate Padé approximation theory

Annie A.M. CUYT
Departement Wiskunde, Universitaire Instelling Antwerpen, B-2610 Wilrijk, Belgium

Received 3 September 1984

Keywords: Padé approximation, multivariate functions, Toeplitz system, recurrence relations, continued fractions.

In the past decade a lot of work has been done on the subject of multivariate Padé approximation theory and so it would be interesting to compare some different definitions. Let us first take a look at the univariate case. It is well known that univariate Padé approximants can be obtained in several equivalent ways and that they satisfy some typical properties. We will briefly repeat the definition together with these characteristic properties and indicate some of the methods to calculate them.

Consider a real-valued function f of one real variable x given by its Taylor series expansion at the origin

$$f(x)=\sum_{i=0}^{\infty} c_i x^i. \tag{1}$$

The Padé approximation problem of order (m, n) then consists in finding polynomials

$$p(x)=\sum_{i=0}^{m} a_i x^i$$

and

$$q(x)=\sum_{i=0}^{n} b_i x^i,$$

such that in the power series $(f\cdot q-p)(x)$ the first $m+n+1$ terms disappear:

$$(f\cdot q-p)(x)=\sum_{i>m+n} d_i x^i. \tag{2}$$

We can introduce the notion of interpolationset E in $\mathbb{N}$ to reformulate (2) as

$$(f\cdot q-p)^{(i)}(0)=0 \quad \text{for } i\in E=\{0,1,\ldots,m+n\}.$$

Since (2) results in a homogeneous system of $m+n+1$ linear equations for the $m+n+2$ unknowns a_i and b_i we know that a nontrivial solution of (2) is always possible.

Also it is easy to prove that for fixed m and n and given f different solutions p_1, q_1 and p_2,

q_2 of (2) are equivalent meaning that

$$(p_1q_2)(x) = (p_2q_1)(x).$$

So they all have the same irreducible form which we shall denote by $r_{m,n}^{(f)}(x)$. This irreducible rational function $r_{m,n}^{(f)}(x)$ is now called the (m, n) *Padé approximant to* f and we immediately have our first theorem.

Theorem 1. *For all m and n in* $\mathbb{N}$ *and for every series* (1) *the* (m, n) *Padé approximant exists and is unique.*

Next consider the following problem. Suppose we have some operator ϕ transforming the function f into the function ϕf, is it then possible to calculate the Padé approximants to ϕf from the knowledge of the Padé approximants to f? Yes, in some cases it is possible and the next theorem describes these *co-variance* properties.

Theorem 2. (a) *For* $\phi f = 1/f$ *and* $r_{m,n}^{(f)}(x) = p(x)/q(x)$ *we have* $r_{n,m}^{(\phi f)}(x) = q(x)/p(x)$.

(b) *For* $\phi f = (af + b)/(cf + d)$ *with* a, b, c, d *in* $\mathbb{R}$ *and* $r_{n,n}^{(f)}(x) = p(x)/q(x)$ *we have* $r_{n,n}^{(\phi f)}(x) = (ap + bq)/(cp + dq)$.

(c) *For* $(\phi f)(x) = f(ax/(1 + bx))$ *and* $r_{n,n}^{(f)}(x) = p(x)/q(x)$ *we have* $r_{n,n}^{(\phi f)}(x) = p(ax/(1 + bx))/q(ax/(1 + bx))$.

Remark that the second and third covariance properties are only valid for diagonal approximants, i.e. $m = n$. Let us now turn to the question of how to compute an (m, n) Padé approximant to f. The defining equations (2) can be split up into a linear system of equations determining the numerator coefficients a_i and a homogeneous linear system of equations completely determining the denominator coefficients b_i

$$\begin{cases} c_0 \cdot b_0 = a_0, \\ c_1 \cdot b_0 + c_0 \cdot b_1 = a_1, \\ \vdots \\ c_m \cdot b_0 + \cdots + c_0 \cdot b_m = a_m, \end{cases} \qquad \text{with } b_i = 0 \text{ if } i > n.$$

$$\begin{cases} c_{m+1} \cdot b_0 + \cdots + c_{m+1-n} \cdot b_n = 0, \\ \vdots \\ c_{m+n} \cdot b_0 + \cdots + c_m \cdot b_n = 0, \end{cases} \qquad \text{with } c_i = 0 \text{ if } i < 0.$$

In fact one only has to solve the homogeneous system for then the a_i can be obtained by substitution of the b_i in the left hand side of the nonhomogeneous system. What's more the coefficient matrix of the homogeneous system is a *Toeplitz* matrix. Consequently these equations can be solved in $O(n^2)$ operations instead of $O(n^3)$ operations for an arbitrary system of n linear equations. So computing a solution of (3) already solves the "coefficient" problem by which we mean the calculation of the a_i and b_i.

Now there are more interesting ways to solve the "value" problem which consists in computing $r_{m,n}^{(f)}(x)$ for some particular x. To this end we introduce the epsilon-algorithm. Put

$$
\begin{array}{lllll}
 & \epsilon_0^{(-1)}=0 & & \epsilon_2^{(-2)}=0 & \dots \\
\epsilon_{-1}^{(0)}=0 & & \epsilon_1^{(-1)} & & \\
 & \epsilon_0^{(0)}=c_0 & & \epsilon_2^{(-1)} & \dots \\
\epsilon_{-1}^{(1)}=0 & & \epsilon_1^{(0)} & & \\
 & \epsilon_0^{(1)}=c_0+c_1x & & \dots & \\
\epsilon_{-1}^{(2)}=0 & & \dots & & \\
 & \epsilon_0^{(2)}=c_0+c_1x+c_2x^2 & & & \\
\vdots & & & & \\
 & \vdots & & &
\end{array}
$$

and compute

$$\epsilon_{i+1}^{(j)}=\epsilon_{i-1}^{(j+1)}+1/\left(\epsilon_i^{(j+1)}-\epsilon_i^{(j)}\right),$$
$$i=0,1,2,\dots,\qquad j=-\lfloor i/2\rfloor-1,\ -\lfloor i/2\rfloor,\dots. \tag{4}$$

In [1] is shown that the value

$$r_{m,n}^{(f)}(x)=\epsilon_{2n}^{(m-n)}.$$

Clearly (4) is a recursive way to compute Padé approximants. A third method is based on a *continued fraction* representation of rational functions.

Consider a continued fraction of the form

$$c_0+\cdots+c_{m-n}x^{m-n}+\frac{c_{m-n+1}x^{m-n+1}|}{|\ 1}$$
$$-\frac{q_1^{(m-n+1)}x|}{|\ 1}-\frac{e_1^{(m-n+1)}x|}{|\ 1}-\frac{q_2^{(m-n+1)}x|}{|\ 1}-\frac{e_2^{(m-n+1)}x|}{|\ 1}\cdots. \tag{5}$$

If the coefficients $q_i^{(m-n+1)}$ and $e_i^{(m-n+1)}$ are computed using the following rhombus rules then $r_{m,n}^{(f)}(x)$ appears to be the $(2n)$th convergent of (5):

$$
\begin{array}{lllll}
e_0^{(1)}=0 & & & & \\
 & q_1^{(1)}=c_2/c_1 & & & \\
e_0^{(2)}=0 & & e_1^{(1)} & & \\
 & q_1^{(2)}=c_3/c_2 & & q_2^{(1)} & \\
e_0^{(3)}=0 & & e_1^{(2)} & & e_2^{(1)} \\
 & q_1^{(3)}=c_4/c_3 & & q_2^{(2)} & \\
\vdots & & e_1^{(3)} & & e_2^{(2)} \\
 & \vdots & & q_2^{(3)} & \\
 & & \vdots & & e_2^{(3)} \\
 & & & \vdots & \\
 & & & & \vdots
\end{array}
$$

The q-values are calculated such that the indicated products are equal and the e-values such that the indicated sums are equal. This results in:

$$q_i^{(j)} = q_{i-1}^{(j+1)}\left(e_{i-1}^{(j+1)}/e_{i-1}^{(j)}\right), \qquad i=2,3,\ldots, \quad j=1,2,\ldots,$$

$$e_i^{(j)} = e_{i-1}^{(j+1)} + q_i^{(j+1)} - q_i^{(j)}, \qquad i=1,2,\ldots, \quad j=1,2,\ldots.$$

In view of the fact that mainly three methods exist for the computation of univariate Padé approximants, we can consider three main types of generalizations for multivariate functions: a class of definitions based on the notion of interpolationset, some definitions using different continued fraction representations of multivariate functions, and a multivariate generalization of the epsilon-algorithm. Each author of a multivariate definition of course tries to preserve some of the interesting covariance properties and to add a projection property and a symmetry property which are more or less obvious to expect for your multivariate Padé approximant. The projection property enables you to equate one of the variables to zero both in the function and the approximant without disturbing the order of approximation and the symmetry property tells you which approximants will be symmetric in case your work with a symmetric function.

In order to avoid notational difficulties we will restrict ourselves to the case of a bivariate function; the generalization to more than two variables is straightforward.

The first type of generalization we will consider is the group of definitions based on the idea to set up a system of defining equations for the multivariate approximant such that it copies some of the univariate properties. We will describe this way of working in a very general setting that covers the definitions introduced by Chisholm and his group in Canterbury [2,9], by Lutterodt [12,13], by Karlsson and Wallin [10] and by Levin [11].

If we define bivariate polynomials by choosing index sets in $\mathbb{N}*\mathbb{N}$, then the bivariate Padé approximation problem to

$$f(x, y) = \sum_{i,j=0}^{\infty} c_{ij}x^iy^j,$$

consists in finding polynomials

$$p(x, y) = \sum_{(i,j)\in N\subseteq \mathbb{N}*\mathbb{N}} a_{ij}x^ix^j \quad (N \text{ from 'numerator'}),$$

$$q(x, y) = \sum_{(i,j)\in D\subseteq \mathbb{N}*\mathbb{N}} b_{ij}x^iy^j \quad (D \text{ from 'denominator'}),$$

and a bivariate interpolationset E such that

$$(f\cdot q - p)(x, y) = \sum_{(i,j)\in \mathbb{N}*\mathbb{N}\setminus E} d_{ij}x^iy^j \quad (E \text{ from 'equations'}), \tag{6}$$

with

$$N \subseteq E, \tag{7a}$$

$$\#(E\setminus N) = \#D - 1. \tag{7b}$$

A typical situation would be the one given in Fig. 1.

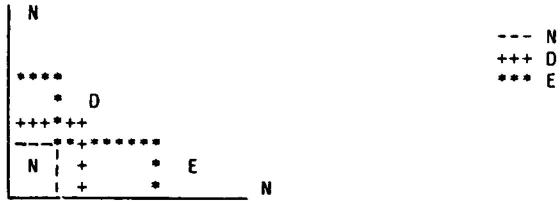

Fig. 1.

Clearly the equations (6) can be rewritten as

$$\left.\frac{\partial^{(i+j)}(f\cdot q-p)}{\partial x^i \partial y^j}\right|_{(0,0)}=0 \quad \text{for } (i,\ j) \text{ in } E$$

and this clarifies the terminology of *interpolationset*.

Condition (7a) allows you to split up the system of defining equations in a nonhomogeneous part completely determining the coefficients a_{ij} and a homogeneous part for the b_{ij}, while (7b) guarantees the existence of a nontrivial solution. In general unicity of the Pade approximant is not guaranteed unless $E \backslash N$ supplies a homogeneous system of linearly independent equations. The structure of the homogeneous system of equations depends on the choice of E. Users interested in obtaining a system that can be solved in a fairly easy way, such as the Toeplitz system in the univariate case, are referred to the prong structure for the Canterbury approximants [8] and the Lutterodt approximants of type B1 [13].

If we want our covariance properties to hold then E must satisfy the rectangle rule:

$$\text{if } (i,\ j) \in E \text{ then for } 0 \leqslant k \leqslant i \text{ and } 0 \leqslant l \leqslant j \text{ also } (k,\ l) \in E.$$

The necessity of this can be seen as follows. If $f(0, 0)=c_{00}$ is nonzero then $1/f$ can be constructed:

$$(1/f)(x,\ y)=\sum_{(i,j)=0}^{\infty} e_{ij}x^iy^j$$

with

$$e_{00}=1/c_{00}.$$

Now take a look at

$$\left(\frac{1}{f}\cdot p-q\right)(x,\ y)=[(-1/f)\cdot(f\cdot q-p)](x,\ y)=\sum_{i,j=0}^{\infty} g_{ij}x^iy^j$$

with

$$g_{ij}=-\sum_{k=0}^{i}\sum_{l=0}^{j} d_{kl}e_{i-k,j-l}.$$

A coefficient g_{ij} will certainly disappear if all the d_{kl} with $0 \leqslant k \leqslant i$ and $0 \leqslant l \leqslant j$ vanish. If we want the projection property to hold then E must certainly contain the univariate interpolationsets

$$\{(i, 0) \mid 0 \leqslant i \leqslant m+n\} \cup \{(0, j) \mid 0 \leqslant j \leqslant m+n\}.$$

In order for symmetry to hold the index sets N and D and the interpolationset E must be symmetric configurations in $\mathbb{N} * \mathbb{N}$. As a conclusion we can say that the properties that are valid for the bivariate approximant depend very much on the choice of the numerator, denominator and the set of equations. For the Canterbury approximants the choice of N, D and E is described in a very precise way while Karlsson and Wallin, Lutterodt and Levin only set up some general requirements. It is important to emphasize that in none of the cases there is a clear link with continued fraction theory nor is an easy recursive scheme such as the epsilon-algorithm valid.

A second important type of definitions are those who generalize the idea to represent the considered function as a continued fraction and consider convergents of that continued fraction. For bivariate functions one can use *branched continued fractions*

$$b_0 + \sum_{i=1}^{\infty} \frac{a_i|}{|b_i}$$

where the b_i are infinite expressions themselves and are called the branches of the continued fraction. Siemaszko [15] uses a representation of $f(x, y)$ of the form

$$f(x, y) = K_0(x \cdot y) + \sum_{i=1}^{\infty} \frac{a_i x \quad |}{|K_i(x \cdot y)} + \sum_{i=1}^{\infty} \frac{b_i y \quad |}{|L_i(x \cdot y)} \tag{8}$$

with

$$K_0(x \cdot y) = d_0 + \sum_{j=1}^{\infty} \frac{d_j xy|}{|\ 1},$$

$$K_i(x \cdot y) = 1 + \sum_{j=1}^{\infty} \frac{a_j^{(i)} x \cdot y|}{|\ 1}, \quad i = 1, 2, \ldots,$$

$$L_i(x \cdot y) = 1 + \sum_{j=1}^{\infty} \frac{b_j^{(i)} xy|}{|\ 1}, \quad i = 1, 2, \ldots$$

The branches $K_i(x \cdot y)$ are constructed using univariate Viskovatov algorithms so that $K_i(x \cdot y)$ is a corresponding continued fraction to the univariate series

$$\sum_{k=0}^{\infty} c_{i+k,k} x^{i+k} y^k, \quad i = 0, 1, \ldots,$$

while the $L_i(x \cdot y)$ will be corresponding continued fractions to the series

$$\sum_{k=0}^{\infty} c_{k,i+k} x^k y^{i+k}, \quad i = 1, 2, \ldots.$$

So the series

$$f(x, y) = \sum_{(i,j) \in \mathbb{N} * \mathbb{N}} c_{ij} x^i y^j$$

is rewritten as a sum of univariate power series

$$\sum_{k=0}^{\infty} c_{kk} x^k y^k + \sum_{i=1}^{\infty} \left\{ \sum_{k=0}^{\infty} c_{i+k,k} x^{i+k} y^k \right\} + \sum_{i=1}^{\infty} \left\{ \sum_{k=0}^{\infty} c_{k,i+k} x^k y^{i+k} \right\}$$

which is illustrated in Fig. 2. As a result of this convergents of (8) will correspond to some partial sum

$$\sum_{(i,j) \in I \subseteq \mathbb{N} * \mathbb{N}} c_{ij} x^i y^j$$

of $f(x, y)$.

Murphy and O'Donohoe [14] used branched continued fractions of the form

$$f(x, y) = \frac{e_0 \quad |}{|1 + G_0(x) + H_0(y)} + \sum_{i=1}^{\infty} \frac{e_i x \cdot y \quad |}{|1 + G_i(x) + H_i(y)} \tag{9}$$

where

$$G_i(x) = \sum_{j=1}^{\infty} \frac{g_j^{(i)} x |}{| \ 1}, \quad i = 0, 1, 2, \ldots,$$

$$H_i(y) = \sum_{j=1}^{\infty} \frac{h_j^{(i)} y |}{| \ 1}, \quad i = 0, 1, 2, \ldots,$$

Again the branches $1 + G_i(x) + H_i(y)$ are computed using univariate techniques. In fact the power series

$$f(x, y) = \sum_{(i,j) \in \mathbb{N} * \mathbb{N}} c_{ij} x^i y^j$$

has been split up here as

$$\sum_{i=0}^{\infty} \left\{ c_{ii} x^i y^i + \sum_{k=1}^{\infty} c_{i+k,i} x^{i+k} y^i + \sum_{k=1}^{\infty} c_{i,i+k} x^i y^{i+k} \right\}$$

just as indicated in Fig. 3.

Existence of the branched continued fractions (8) and (9) depends on the coefficients of the given power series $f(x, y)$ and usually conditions for the existence involve the nontriviality of some determinants. If they exist then the representations (8) and (9) are unique. Bivariate Padé approximants constructed from the use of branched continued fractions have the pojection property and the symmetry property if the convergents are defined in a symmetric way with respect to x and y, but they don't satisfy any of the covariance properties.

If we reformulate the continued fraction approach using the index sets N and D, which

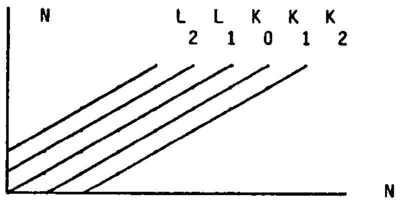

Fig. 2.

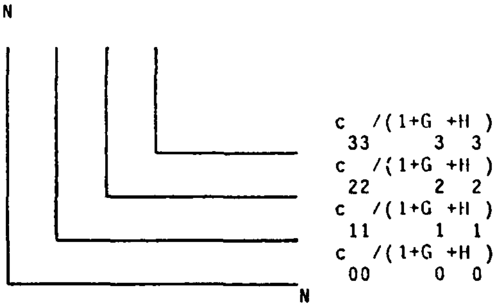

Fig. 3.

indicate the degree of the numerator and denominator, and using the interpolationset E then we remark that N and D blow up very rapidly while E does not satisfy the conditions (7) anymore. For instance consider the convergent

$$\frac{p(x, y)}{q(x, y)} = d_0 + d_1 xy + \frac{a_1 x}{1 + a_1^{(1)} xy + a_2 x/(1 + a_3 x)} + \frac{b_1 y}{1 + b_1^{(1)} xy + b_2 y/(1 + b_3 y)}$$

of (8). Then it's easy to check that

$$p(x, y) = \sum_{(i,j) \in N} a_{ij} x^i y^j, \qquad q(x, y) = \sum_{(i,j) \in D} b_{ij} x^i y^j,$$

$$\left. \frac{\partial^{i+j}(f \cdot q - p)}{\partial x^i \partial y^j} \right|_{(0,0)} = 0 \quad \text{for } (i, j) \text{ in } E$$

with N, D and E as given in Fig. 4. If we do the same for the convergent

$$\frac{p(x, y)}{q(x, y)} = \frac{e_0}{1 + \cfrac{g_1^{(0)} x}{1 + \cfrac{g_2^{(0)} x}{1 + g_3^{(0)} x}} + \cfrac{h_1^{(0)} y}{1 + \cfrac{h_2^{(0)} y}{1 + h_3^{(0)} y}}} + \frac{e_1 xy}{1 + g_2^{(1)} x + h_2^{(1)} y}$$

of (9) then the sets N, D, and E can be found in Fig. 5. So there is no point in rewriting the definition of bivariate Padé approximants using continued fractions in order to obtain an equivalent definition using a linear system of equations.

A third way to define multivariate Padé approximants is to set up a recursive scheme. To this end we will let the epsilon-algorithm inspire us, or more precisely we will exploit the fact that a

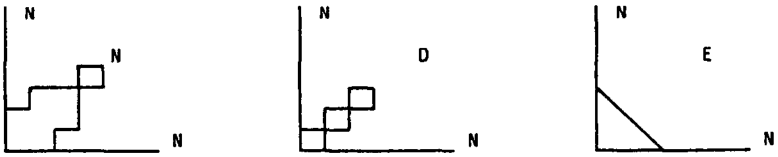

Fig. 4.

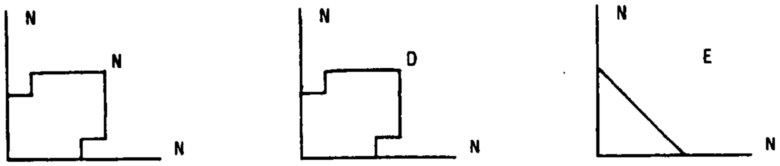

Fig. 5.

value in the *epsilon-table* is a quotient of two determinants. Solving the system of equations (3) explicitly it is easy to see that

$$r_{m,n}^{(f)} = \frac{\begin{vmatrix} \sum_{i=0}^{m} c_i x^i & \sum_{i=0}^{m-1} c_i x^i & \cdots & \sum_{i=0}^{m-n} c_i x^i \\ c_{m+1}x^{m+1} & c_m x^m & \cdots & c_{m+1-n}x^{m+1-n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m+n}x^{m+n} & c_{m+n-1}x^{m+n-1} & \cdots & c_m x^m \end{vmatrix}}{\begin{vmatrix} 1 & 1 & \cdots & 1 \\ c_{m+1}x^{m+1} & c_m x^m & \cdots & c_{m+1-n}x^{m+1-n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m+n}x^{m+n} & c_{m+n-1}x^{m+n-1} & \cdots & c_m x^m \end{vmatrix}} = \frac{p(x)}{q(x)}$$

and since $r_{m,n}^{(f)} = \epsilon_{2n}^{(m-n)}$, this determinant representation can be used to define multivariate Padé approximants in a recursive way. We replace the partial sums $\sum_{i=0}^{k} c_i x^i$ of $f(x)$ by partial sums $\sum_{i+j=0}^{k} c_{ij} x^i y^j$ of $f(x, y)$ and the terms $c_1 x^1$ of degree 1 in the univariate series by the group of terms $\sum_{i+j=1} c_{ij} x^i y^j$ of degree 1 in the bivariate series. In this way we obtain

$$\frac{p(x, y)}{q(x, y)} = \frac{\begin{vmatrix} \sum\limits_{i+j=0}^{m} c_{ij}x^i y^j & \cdots & & \sum\limits_{i+j=0}^{m-n} c_{ij}x^i y^j \\ \sum\limits_{i+j=m+1} c_{ij}x^i y^j & \sum\limits_{i+j=m} c_{ij}x^i y^j & & \\ \vdots & & \ddots & \\ \sum\limits_{i+j=m+n} c_{ij}x^i y^j & \cdots & & \sum\limits_{i+j=m} c_{ij}x^i y^j \end{vmatrix}}{\begin{vmatrix} 1 & \cdots & & 1 \\ \sum\limits_{i+j=m+1} c_{ij}x^i y^j & \sum\limits_{i+j=m} c_{ij}x^i y^j & & \\ \vdots & & \ddots & \\ \sum\limits_{i+j=m+n} c_{ij}x^i y^j & \cdots & & \sum\limits_{i+j=m} c_{ij}x^i y^j \end{vmatrix}}.$$

If we develop the determinants by their first row, we can prove that $p(x, y)$ and $q(x, y)$ are of the form

$$p(x, y) = \sum_{i+j=mn}^{mn+m} a_{ij} x^i y^j, \qquad q(x, y) = \sum_{i+j=mn}^{mn+n} b_{ij} x^i y^j$$

and that they satisfy [3]

$$(f \cdot q - p)(x, y) = \sum_{i+j>mn+m+n} d_{ij} x^i y^j. \tag{10}$$

In fact we can see that what we have done is shifted all the degrees in the univariate Padé approximation problem of order (m, n) over mn. We also emphasize that condition (10) can be used as the defining set of equations for these multivariate Padé approximants and that it defines them *uniquely* in that sense that different solutions of (10) supply equivalent rational functions and consequently the same irreducible form. What's more the system of equations (10) turns out to have a near-Toeplitz structure so that it can be solved in $O(\alpha N_e^2)$ operations where α is the displacement rank of the coefficient matrix and N_e is the number of equations [4]. Because of the great analogy with the univariate case one can now also immediately prove the covariance properties listed in Theorem 2, as well as a projection property and a symmetry property. Even the *qd*-algorithm as given in (5) remains valid if the factors x are included in the $q_i^{(j)}$ and $e_i^{(j)}$ coefficients. Let us first rewrite the univariate version in this way. We consider a fraction of the form

$$c_0 + \cdots + c_{m-n} x^{m-n} + \frac{c_{m-n+1} x^{m-n+1} |}{| \quad 1} - \frac{Q_1^{(m-n+1)} |}{| \quad 1} - \frac{E_1^{(m-n+1)} |}{| \quad 1} - \cdots$$

where

$$E_0^{(i)} = 0, \quad i = 0, 1, 2, \ldots,$$

$$Q_1^{(i)} = c_{i+1} x^{i+1} / c_i x^i, \quad i = 1, 2, 3, \ldots,$$

$$Q_i^{(j)} = Q_{i-1}^{(j+1)} E_{i-1}^{(j+1)} / E_{i-1}^{(j)}, \qquad i = 2, 3, \ldots, \quad j = 1, 2, \ldots,$$

$$E_i^{(j)} = E_{i-1}^{(j+1)} + Q_i^{(j+1)} - Q_i^{(j)}, \qquad i = 1, 2, \ldots, \quad j = 1, 2, \ldots.$$

If we redefine

$$E_0^{(i)} = 0, \quad i = 1, 2, 3, \ldots,$$

$$Q_1^{(i)} = \sum_{j+k=i+1} c_{jk} x^j y^k / \sum_{j+k=i} c_{jk} x^j y^k, \quad i = 1, 2, 3, \ldots,$$

$$Q_i^{(j)} = Q_{i-1}^{(j+1)} E_{i-1}^{(j+1)} / E_{i-1}^{(j)}, \qquad i = 2, 3, \ldots, \quad j = 1, 2, \ldots,$$

$$E_i^{(j)} = E_{i-1}^{(j+1)} + Q_i^{(j+1)} - Q_i^{(j)}, \qquad i = 1, 2, \ldots, \quad j = 1, 2, \ldots,$$

then $(p/q)(x, y)$ satisfying (10) is the $(2n)$th convergent of the continued fraction

$$c_0 + \cdots + \sum_{j+k=m-n} c_{jk} x^j y^k + \frac{\sum_{j+k=m-n+1} c_{jk} x^j y^k |}{| \quad 1} - \frac{Q_1^{(m-n+1)} |}{| \quad 1} - \frac{E_1^{(m-n+1)} |}{| \quad 1} - \cdots.$$

Before we proceed let us give a review of the properties satisfied by the different types of approximants.

Property	Type of approximant		
	Interpolationset	Continued fraction	Recursion
Unicity	Under certain conditions on $f(x, y) = \sum_{(i,j)} c_{ij}x^iy^j$	Under certain conditions on $f(x, y) = \sum_{(i,j)} c_{ij}x^iy^j$	Yes
Covariance	Yes if E satisfies the rectangle rule	No	Yes
Special structure of system of eqs.	Yes for Chisholm approx. and Lutterodt approx. of type B1	No	Yes
Epsilon-algorithm	No	No	Yes
qd-algorithm or Viskovatov	No	Yes	Yes
Projection property	Yes if E contains univariate interpolationsets	Yes	Yes
Symmetry property	Yes if E is symmetric	Yes	Yes

Now that the development of the theory of multivariate Pade approximants during the past ten years is somewhat more clear we can indicate some possibilities for the generalization of the concept of rational interpolant to the multivariate case. Here the interpolation conditions will be spread over several points instead of having all the approximation conditions in the origin.

Rational interpolants associated with index sets in $\mathbb{N} * \mathbb{N}$ and defined by setting up a system of linear equations are introduced in [6] while Thiele type branched continued fractions and their convergents are discussed in [7] and [16]. Very recently an attempt has also been made to define rational interpolants by means of a multivariate version of Claessens' generalized epsilon-algorithm; this approach would then be based on a recursive scheme.

Acknowledgement

The author is ultimately grateful to the Alexander von Humboldt Foundation for their willingness to support future research with respect to multivariate rational interpolation.

References

[1] C. Brezinski, Algoritmes d'acceleration de la convergence, Editions Technip, Paris, 1978.
[2] J. Chisholm, N-variable rational approximants, in: E. Saff and R. Varga, *Padé and Rational Approximation: Theory and Applications* (Academic Press, London, 1977) 23–42.

[3] A. Cuyt, Multivariate Padé approximants, *J. Math. Anal. Appl.* **96**(1) (1983) 283–293.
[4] A. Cuyt, A comparison of some multivariate Padé approximants, *SIAM J. Math. Anal.* **14**(1) (1983) 195–202.
[5] A. Cuyt, The QD-algorithm and multivariate Padé approximants, *Numer. Math.* **42** (1983) 259–269.
[6] A. Cuyt and B. Verdonk, General order Newton–Padé approximants for multivariate functions, *Numer. Math.* **43** (1984) 293–307.
[7] A. Cuyt and B. Verdonk, Multivariate rational interpolation, *Computing*, to appear.
[8] P. Graves-Morris, R. Hughes-Jones and G. Makinson, The calculation of some rational approximants in two variables, *J. Inst. Math. Appl.* **13** (1974) 311–320.
[9] R. Hughes-Jones, General rational approximants in *n* variables, *J. Approx. Theory* **16** (1976) 201–233.
[10] J. Karlsson and H. Wallin, Rational approximation by an interpolation procedure in several variables, in: E. Saff and R. Varga, *Padé and Rational Approximation: Theory and Applications* (Academic Press, London, 1977) 83–100.
[11] D. Levin, General order Padé type rational approximants defined from double power series, *J. Inst. Math. Appl.* **18** (1976) 1–8.
[12] C. Lutterodt, Rational approximants to holomorphic functions in *n* dimensions, *J. Math. Anal. Appl.* **53** (1976) 89–98.
[13] C. Lutterodt, A two-dimensional analogue of Padé approximant theory, *J. Phys. A* **7**(9) (1974) 1027–1037.
[14] J. Murphy and M. O'Donohoe, A two-variable generalization of the Stieltjes-type continued fraction, *J. Comput. Appl. Math.* **4**(3) (1978) 181–190.
[15] W. Siemaszko, Branched continued fractions for double power series, *J. Comput. Appl. Math.* **6**(2) (1980) 121–125.
[16] W. Siemaszko, Thiele type branched continued fractions for two-variable functions, *J. Comput. Appl. Math.* **9** (1983) 137–153.