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Aristotle's Thesis between paraconsistency and modalization

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Abstract

If the arrow $\rightarrow$ stands for classical relevant implication, Aristotle's Thesis $\neg(A \rightarrow \neg A)$ is inconsistent with the Law of Simplification $(A \wedge B) \rightarrow B$ accepted by relevantists, but yields an inconsistent non-trivial extension of the system of entailment **E**. Such paraconsistent extensions of relevant logics have been studied by R. Routley, C. Mortensen and R. Brady. After examining the semantics associated to such systems, it is stressed that there are nonclassical treatments of relevance which do not support Simplification. The paper aims at showing that Aristotle's Thesis may receive a sense if the arrow is defined as strict implication endowed with the proviso that the clauses of the conditional have the same modal status, i.e. the same position in the Aristotelian square. It is so grasped, in different form, the basic idea of relevant logic that the clauses of a true conditional should have something in common. It is proved that thanks to such definition of the arrow Aristotle's Thesis subjoined to the minimal normal system **K** yields a system equivalent to the deontic system **KD**.

Keywords: Connexive; Implication; Aristotle's square; Relevance; Paraconsistency

1. Aristotle's Thesis is the cornerstone of the logics belonging to the family of so-called connexive logics worked out by Angell and McCall in the Sixties.[1] If $\rightarrow$ is the symbol of some non-truthfunctional notion of implication, Aristotle's Thesis is symbolized by

(**AT**) $\qquad \neg(A \rightarrow \neg A)$.[2]

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[1] See [3,4,10,11].

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If $\rightarrow$ is transitive, reflexive and contrapositive, and we have Modus Ponens for $\supset$, Aristotle's Thesis is interdeducible with so-called *Boethius' Rule*

(**BR**) $A \rightarrow B \vdash \neg(A \rightarrow \neg B)$.[3]

Two variants of such a rule are what we shall call here *Boethius' Thesis*

(**BT**) $(A \rightarrow B) \supset \neg(A \rightarrow \neg B)$[4]

and *Strong Boethius' Thesis*

(**SBT**) $(A \rightarrow B) \rightarrow \neg(A \rightarrow \neg B)$.[5]

It is clear that if $\rightarrow$ is $\supset$, Aristotle's Thesis is inconsistent with the standard Propositional Calculus **PC** since in **PC** $\neg(A \supset \neg A)$ is equivalent to $A \wedge A$ and A. Furthermore, if $\rightarrow$ is strict implication (—3), $\neg(A$ —3 $\neg A)$ equals $\Diamond A$, from which in any Lewis' system of strict implication we would have the absurdity $\Diamond\bot$[6] as a theorem. It is also clear that any logic closed under Uniform Substitution containing Simplification in the forms $(A \wedge B) \rightarrow A$ and $(A \wedge B) \rightarrow B$ cannot contain Boethius' Rule. In fact $(A \wedge \neg A) \rightarrow A$ and $(A \wedge \neg A) \rightarrow \neg A$ are both instances of Simplification, and they are jointly a counterexample to Boethius' Rule, their logical form being $B \rightarrow A$ and $B \rightarrow \neg A$, respectively.

In connexive systems in which having Simplification $(A \wedge B) \rightarrow A$ as a theorem is equivalent to having Addition $A \rightarrow (A \vee B)$ as a theorem the proof that excludes Aristotle's Thesis is even more simple. In fact $(A \wedge \neg A) \rightarrow A$ coinjoined with $A \rightarrow (A \vee \neg A)$ leads to $(A \wedge \neg A) \rightarrow (A \vee \neg A)$, so to $\bot \rightarrow \neg\bot$, which is a counterexample to Aristotle's Thesis $\neg(A \rightarrow \neg A)$.

In classical logical negation-inconsistency and triviality (i.e. having every wff A as a theorem) are coincident, while it is well known that such identification is not obvious in some of the systems which are known as *non-Scotian*, i.e. systems which do not contain the classical law $P \rightarrow (\neg P \rightarrow Q)$. Relevant logics in the Anderson–Belnap tradition (in this paper we shall call them *classical* relevant logics) are non-Scotian logics whose key-principle, as is well known, is that antecedent and consequent of a valid implication should share a common variable.[7]

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[2] In a passage of *Prior Analytics* (ii 4.57b3) Aristotle uses in a proof the principle $\neg(\neg A \rightarrow A)$, which is equivalent to AT in any system containing Double Negation.

[3] In fact, a substitution instance of (BR) is $A \rightarrow A \vdash \neg(A \rightarrow \neg A)$ and thanks to $\vdash A \rightarrow A$, $\neg(A \rightarrow \neg A)$ becomes a theorem by Modus Ponens. In the reverse direction, from the instance of transitivity which is $((A \rightarrow B) \wedge (B \rightarrow \neg A)) \supset (A \rightarrow \neg A)$ we have $\neg(A \rightarrow \neg A) \supset \neg((A \rightarrow B) \wedge (B \rightarrow \neg A))$ and, given $\vdash \neg(A \rightarrow \neg A)$, by Modus Ponens we have as a theorem $\vdash \neg((A \rightarrow B) \wedge (B \rightarrow \neg A))$, i.e. $\vdash (A \rightarrow B) \supset \neg(B \rightarrow \neg A)$, so the derived rule $A \rightarrow B \vdash \neg(B \rightarrow \neg A)$. But if $\rightarrow$ is contrapositive $\neg(B \rightarrow \neg A)$ implies $\neg(A \rightarrow \neg B)$. For the name "Boethius' Rule" see [8, p. 53].

[4] In the literature the wff $\neg((A \rightarrow B) \wedge (A \rightarrow \neg B))$ has received the name of Strawson's Thesis. (BT) and Strawson's Thesis are obviously equivalent if the reference system is the standard propositional calculus.

[5] *Strong Boethius' Thesis* is currently qualified as *Boethius' Thesis* in the original Angell's and McCall's papers of the Sixties.

[6] Here and afterwards, $\bot$ stands for $A \wedge \neg A$ and $\top$ stands for $A \vee \neg A$.

[7] The literature on this topic is immensely wide. For a bibliography see [1,2]. See also [24].

What is now interesting is the fact that the addition of Aristotle's Thesis to different systems of relevant logics such as **B**, **E**, **R** gives rise to different results studied by Chris Mortensen [14].

We recall that **B** is the system which is so axiomatized:

(A1) $A \to A$.
(A2) $(A \wedge B) \to A$.
(A3) $(A \wedge B) \to B$.
(A4) $((A \wedge B) \wedge (A \wedge C)) \to (A \to (B \wedge C))$.
(A5) $A \to (A \vee B)$.
(A6) $B \to (A \vee B)$.
(A7) $((A \to C) \wedge (B \to C)) \to ((A \vee B) \to C)$.
(A8) $(A \wedge (B \vee C)) \to ((A \wedge B) \vee (A \wedge C))$.
(A9) $\neg\neg A \to A$.

Rules:

R1: $A, A \to B/B$.
R2: $A, B/A \wedge B$.
R3: $A \to B, C \to D/(B \to C) \to (A \to D)$.
R4: $A \to \neg B/B \to \neg A$.

The system **E** of entailment amounts to **B** with the addition of

(A10) $(A \to B) \to ((B \to C) \to (A \to C))$.
(A11) $(A \to B) \to ((C \to A) \to (C \to B))$.
(A12) $(A \to (A \to B)) \to (A \to B)$.
(A13) $(A \to \neg A) \to \neg A$.
(A14) $(A \to \neg B) \to (B \to \neg A)$

along with the following rule

R5: $A/(A \to B) \to B$.

R is the strongest system and is obtained by adding to **E**:

(A15) $A \to ((A \to B) \to B)$.

Note that, thanks to the definition of the box as $\Box A =_{\mathrm{Df}} (A \to A) \to A$, (A15) yields $A \to ((A \to A) \to A)$, so $A \to \Box A$. Rule R5 grants deriving from $A \to \Box A$ and $A \to A$ the wff $\Box A \to A$, so **R** contains the collapse-equivalence $\Box A \rightleftarrows A$ and is not modally meaningful.

E contains the law $(A \to B) \to \Box(A \to B)$ stating that every implication is necessary. It is well known that the modal fragment of **E** is **S4**,[8] so $\Box(A \to B) \to (A \to B)$ is also a thesis and such is the equivalence $(A \to B) \rightleftarrows \Box(A \to B)$.

The results which are obtained by extending such systems with Aristotle's Thesis are as follows (see [14, p. 109]).

(1) **B** + **AT** is negation-inconsistent but not trivial.
(2) **R** + **AT** is trivial, i.e. it contains B, for every B.
(3) **E** + **AT** contains $\Diamond B$, where of course $\Diamond B$ is $\neg((\neg B \to \neg B) \to \neg B)$.

Given the equivalence between $A \to B$ and $\Diamond(A \to B)$ which is provable in **E**, we have as theorems $\neg(A \to B) \rightleftarrows \Diamond\neg(A \to B)$ and $\Diamond\neg(A \to B)$, so by Modus Ponens also $\neg(A \to B)$. **AT** is the special case of $\neg(A \to B)$ in which B is just $\neg A$, and the absurdity $\neg(A \to A)$ is the special case in which B is A. Since they are both provable, **E** + **AT** is then negation-inconsistent, but it may be proved that is not trivial.

A suggestion which is natural to entertain given the preceding remarks is that **E** + **AT** may be treated as a paraconsistent system. It is interesting however to grasp the sense in which **E** + **AT** may be classified as a paraconsistent logic. There are at least three senses in which a logic may be qualified as paraconsistent. The first sense is weak: a paraconsistent logic is a non-trivial inconsistent theory. The second is strong: there is at least one inconsistent true proposition. The third is very strong: there is at least one inconsistent proposition which is logically true. This is the case of **E** + **AT** and **B** + **AT**.

It follows that we may treat such extended systems as **E** + **AT** as paraconsistent systems. The first attempt in this direction has been made by Routley [23]. The key-idea is to introduce models for connexive logics as extended models for relevant logics which can be sketched in the following way. A basic model M for connexive logic is a 8-ple $\langle T, O, K, R, S, *, \boldsymbol{G}, v\rangle$ where

(1) O and K are sets of set-ups such that $O \subseteq K$.
(2) T is an element of O (intuitively, the "real" set-up).
(3) R and S are three-place relations on **K** in the Routley–Meyer style.[9]
(4) * is one-place involution on **K** (intuitively, $*a$ is a weakened image of a, i.e. a set-up where the propositions which are negated in a are not true).
(5) $\boldsymbol{G}$ is a relation between wffs and worlds.
(6) v is a function from couples of wffs and set-ups to the set $\{0, 1\}$.

The novelty with respect to standard relevant models is given by the existence of a second access relations S beyond R, and by the relation $\boldsymbol{G}$. The role of S is to evaluate conjunction (which thus turns out to be an intensional connective), while the role of $\boldsymbol{G}$ is to grant a special relation between formulas and set-ups. The conditions which define the interpretation function v are as follows:

[8] See for instance [8, p. 244].
[9] See for instance [8, pp. 306 ff].

Writing $a \leqslant b$ in place of $\exists x \in O$ such that $Rxbc$.[10]

(I) If $v(P, a) = 1$ and $a \leqslant b$ then $v(P, b) = 1$, for every P.
(II) $v(B \to C, a) = 1$ iff, for every b and c such that $Rabc$, if $v(B, b) = 1$ then $v(C, c) = 1$.
(III) $v(B \wedge C, a)$ iff, for some b and c such that $Sbca$, $v(B, b) = 1$ and $v(C, c) = 1$.
(IV) $v(\neg A, a) = 1$ iff $v(A, a^*) \neq 1$.
(V) If $A\boldsymbol{G}b$ then $v(A, b) = 1$.

A formula A is *true in a connexive model* M if $v(A, T) = 1$ where T belongs to the O in M. A formula A is *valid* iff it is true in all connexive models M.

The intuitive meaning of $\boldsymbol{G}$ in $A\boldsymbol{G}b$ is "A generates set-up b", which means that everything that holds in set-up b is implied by A. Routley proves that every system **CB** + **x** is sound and complete for the given semantics, where **CB** + **x** is **CB** extended with one or more connexively accepted principle **x**. Each one of the **x** has a semantics counterpart in terms of some condition on the model. For instance Aristotle's Thesis **AT** is mirrored by the condition

(**CAT**) $\exists y(RT^*yy^* \wedge A\boldsymbol{G}y)$ for every wff A.

Recovering the subject in 1984, Mortensen found that Routley's semantic conditions "are not particularly intuitively enlightening" and reserved his attention to the condition (V), which Routley himself qualified as "non entirely desirable since it is not inductively defined" [23, p. 399]. Furthermore, the generation relation is a relation between a point of a frame and a formula, so it is a restriction on the evaluation relation (i.e. on the model), not on the frame.

Mortensen concentrated his attention on the inconsistent non-trivial **E** + **AT** and found a simple modeling for it by employing the notion of a non-normal world. In reminiscence of Kripke semantics for non-normal modal systems, a non-normal set-up is a set-up in which no implicative formula is true:

(**DNN**) $\neg Na$ iff for every A, B, $v(A \to B, a) \neq 1$.

Since **AT** is a substitution instance of $\neg(A \to B)$, we introduce the condition that the opposite of the "real set-up" is a non-normal one

(**CNN**) $\neg NT^*$.

This claim amounts to the claim that the denial of every implicative proposition is true at the "real" set-up T, so that in T we have $\neg(A \to B)$, and in particular $\neg(A \to \neg A)$. Soundness and Completeness of **E** + **AT** are easily proved thanks to the mentioned semantic restriction. The disadvantage of this condition is that it is too generous: Aristotle's Thesis turns out to be valid simply since every negated conditional turns out to be valid.

[10] The axiomatic properties of $\leqslant$ are: $a = a^{**}$; $a \leqslant a$; if $a < d$ and $Rdbd$, then $Rabc$; $a \leqslant b$ only if $b^* \leqslant a^*$; $a \leqslant b \wedge Scda \supset Scdb$.

Mortensen showed that a different device which amounts to the same result is to require the condition

$$(\mathbf{MC}) \qquad \mathbf{C}(A, M) \neq \emptyset,$$

where $\mathbf{C}(A, M)$ is the set of set-ups $\mathbf{x}$ where $\mathbf{x}$ is an element of the model M such that $v(A, \mathbf{x}) = 1$ and $v(\neg A, \mathbf{x}) \neq 1$. But we have again to consider that this condition is, so to speak, semantically spurious since the condition of non normality is again not a condition *via* a restriction on frames but a condition on the class of models *via* a restriction on the assignment function.

2. Ross Brady in [6] tried to correct the defects of the preceding approaches by imposing to the models what he calls a “regulatory structure”. Technically speaking, Brady’s idea is to distinguish between propositions and set of set-ups by introducing a function connecting the two sets.

Let us call *Brady frame* a 10-ple $\langle T, O, K, F, \mathscr{I}, ^*, R, -, \cap, \Rightarrow\rangle$ where $T, O, K, ^*, R$ are as in Mortensen’s model structures, F is a set of propositions, $-, \cap, \Rightarrow$ are operations on F, $\mathscr{I}$ is a function from the set of propositions F to the power set $\Pi(K)$ of K.

We have of course that $a \in \mathscr{I}(-f)$ iff $a^* \notin \mathscr{I}(f)$, $a \in \mathscr{I}(f \cap g)$ iff $a \in \mathscr{I}(f)$ and $a \in \mathscr{I}(g)$, and furthermore that

(i) $a \in \mathscr{I}(f \Rightarrow g)$ iff for every b, c, in K, if $Rabc$ and $b \in \mathscr{I}(f)$ then $c \in \mathscr{I}(g)$,
(ii) if $a \leqslant b$ and $a \in \mathscr{I}(f)$ then $b \in \mathscr{I}(f)$.

We distinguish here between interpretation functions and evaluation functions. The notion of interpretation I is defined by putting $I(p) \in F$ and $I(\neg A) = -I(A)$, $I(A \wedge B) = I(A) \wedge I(B)$, $I(A \to B) = I(A) \Rightarrow I(B)$. The evaluation function v is then defined for atomic wffs by

$$v(p, a) = 1 \quad \text{iff} \quad a \in \mathscr{I}(Ip),$$

an equivalence which is recursively extended to arbitrary propositions. The definition of validity is standard.

It is straightforward to prove by induction an Interpretation Lemma which Brady states as follows:

IL. *For any Brady frame F and each interpretation I on F, $v(A, a) = 1$ iff $a \in \mathscr{I}(I(a))$ for any formula A and any set-up a.*

The mentioned semantics is associated to the basic system **B**. Different connexive formulas are mirrored by different conditions on the models. The condition which models Aristotle’s Thesis is the following:

$$(\mathbf{cAT}) \qquad \text{If } a \in O \text{ then } \exists x, y \in \mathscr{I}(f) \text{ such } Ra^*xy^*, \text{ for any } f \text{ in } F.$$

The way in which such a condition is specular to **AT** is shown by the following soundness result.

(**TAT**) **AT** is valid in all Brady's frames.

(The proof may be sketched as follows. Let A be a formula and I an interpretation on a Brady frame. Let f be $I(A)$ and then, by the Interpretation Lemma, $v(A,x)=1$ and $v(A,y)=1$ for the x and y mentioned in (**cAT**). Hence RT^*xy^*, $v(A,x)=1$ and $v(\neg A,y^*)=0$ and so $v(A\to\neg A,T^*)=0$. Then $v(\neg(A\to\neg A,T))=1$, as required.)

Brady's inquiries show that after all it is possible to give an unequivocal semantics for a paraconsistent logic including both relevant logic and Aristotle's Thesis. However, there is something perplexing in this construction even from the viewpoint of philosophers who have a positive attitude towards paraconsistent logics. The point is that in this semantic framework negation is treated in a semantically uniform way, while classical relevantists deny that connexive negation has the same meaning as relevant negation.

In fact, in an important essay written by chief exponents of classical relevantism,[11] it is argued that connexivism is committed to a particular theory of negation which they call "cancellation theory of negation". According to the cancellation theory of negation, due to Strawson, to deny something means to cancel it from the logical space. Now we may agree that A implies B iff the content of A includes the content of B. Since $\neg A$ cancels out A, the content of $A\wedge\neg A$ is zero, so the content of $A\wedge\neg A$ is less than the content of A and less than the content of $\neg A$; so it cannot be true that its content includes the content of A or the content of $\neg A$. This aspect turns out to be important in evaluating the different attitude of relevantists and connexivists toward Simplification. A connexivist can argue in fact that the content of $A\wedge B$ is not in general greater than the content of A and the content of B.

However, in looking more deeply to the notion of relevance, we realize that classical relevantists have not a monopoly on the notion of relevance, and that there are different notions of relevance known in the literature according to which Simplification is invalid.[12] The main reference in this connection is to the Körner–Weingartner–Schurtz theory of relevance.[13] In this perspective what is relevant or not relevant, strictly speaking, are not formulas but sentential variables inside a formula. According to Körner's original proposal, a relevance criterion could be stated as follows. If in a classical theorem a variable p may be *singularly* (i.e. not uniformly) replaced by its negation $\neg p$ without destroying the validity of the formula, the variable p can be considered not to be relevant. A paradigm example is just given by Simplification $(p\wedge\underline{q})\supset p$. q may be replaced by $\neg q$ and the result $(p\wedge\neg\underline{q})\supset p$ is also valid, so q is not a relevant variable.[14] Schurz generalizes Körner's criterion by proposing that if a component can be singularly substituted by any formula *salva validitate*, it is irrelevant. He distinguishes then between conclusion-irrelevance and

[11] See [24, p. 89 ff].

[12] Notice that in most systems having Simplification as a theorem is equivalent to having Addition as a theorem, and the refusal or the acceptance of one goes hand in hand with the refusal or the acceptance of the other. However, there are systems such as Parry's analytic implication which admit Simplification but not Addition (see [16]).

[13] See [9,25,27]. We will call it KWS-relevantism.

[14] Here and afterwards the underlined wffs are the wffs which are irrelevant in the defined sense.

premise-irrelevance. In the sequent $p \wedge \underline{q} \vdash p$, the variable q gives an example of premise-irrelevance, while in $p \vdash p \vee \underline{q}$ q gives an example of conclusion-irrelevance.

In this perspective the aim is not to work out an axiomatic system, but simply to identify the classical truths which are relevant and the ones which are irrelevant and, being such, cannot be used in reasoning. The key slogan of the KWS-relevantism is then: Reasoning = Logic + Relevance.

The authors have not completely firm intuitions about more complex cases. The final proposal in [25] is that the irrelevance in premises concerns components which are singularly replaceable unless there are inconsistent disjuncts. So the following are instances of premise-irrelevance:

$$(\underline{q} \wedge \neg \underline{q}),\ p \supset r \vdash r; \qquad p \vee \underline{q},\ p \vee \neg \underline{q} \vdash p.$$

However, there are odd consequences in Schurz's original proposal. The most strange is that the following are cases of c-irrelevant conclusions: $p \vee \neg p \vdash \underline{p} \vee \neg \underline{p}$; $p \vee \neg p \vdash \underline{q} \vee \neg \underline{q}$; $p \wedge \neg p \vdash \underline{q} \wedge \neg \underline{q}$. Here we are clearly out of ordinary intuitions, since the wffs at the left and the ones at the right are equivalent and transmit the same informations (which in the case of tautologies is no information at all).

Interestingly enough, it turns out that while irrelevance is closed under Uniform Substitution, KSW-relevance is not (just for this reason it is impossible to build a system axiomatizing relevance). Suffice it to remark that Disjunctive Syllogism is relevant in the form $\neg p,\ p \vee q \vdash q$ but its substitution instance $\neg p,\ p \vee p \vdash p$ is not such since $\neg p$ is an irrelevant formula. (Incidentally, this is an advantage of this theory with respect to classical relevantism, in which Disjunctive Syllogism is invalid.)

KWS-relevantism is not the only kind of relevantism which rises doubts about Simplification. We neglect here considering various other notion of relevantism which might be seen as implying a partial or total refusal of Simplification. Bolzano's notion of consequence for instance might be viewed as an anticipation of relevantism inasmuch it allows drawing consequences only from consistent sets of premises.[15]

What it turns out from this short overview is that there are reasons to refuse Simplification which have nothing to do with the cancellation theory of negation. As a matter of fact, the basic intuitions which are at the ground of connexivism are not concerned with negation but with implication, which according to connexivist has to describe some consequential nexus between the clauses. In this perspective, simplification is indeed a source of puzzlement. As Thompson remarked in [26], a reason to reject Simplification which connexivists might endorse is that, in $(A \wedge B) \to A$, B may stand for the negation of the very principle which sets a bridge from A to A, i.e. $A \to A$. The wff $(A \wedge \neg(A \to A)) \to A$ is indeed intuitively astonishing, and we have no reason to say that $\neg(A \to A)$ cancels A.

On the other hand, we may agree that classical relevantists have strong justifications on some points and specifically the following:

(1) Angell's and McCall's connexive systems worked out in the Sixties are seriously defective (suffice it to remark that in them $(p \wedge p) \to p$ is not a theorem simply because

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[15] See for instance [7].

$(p \wedge q) \rightarrow p$ is not such). Such faults have been carefully identified by Montgomery and Routley [13].

(2) The idea of requiring that the antecedent and the consequent should have something in common in order to grant an interconnection between them is a sound principle, which however classical relevantists have forced into the rigid syntactical criterion of variable-sharing.

(3) (Related to 2) Any appropriate logic of implication should be non-Scotian, i.e. should avoid such formulas as $\bot \rightarrow \neg\bot$ or $(q \wedge \neg q) \rightarrow p$ in which the clauses lack a common element either from a semantic or syntactical viewpoint.

It is difficult to imagine a formula which expresses a minimal condition for the idea of a relevance relation between the clauses of a conditional better than Aristotle's Thesis does. The interest which classical relevantists have always shown for it may be understood only because it has a strong relevantist flavor, even if no positive condition which classical relevantists put on relevance is able to grasp. However, as we saw, on the one hand it appears that only in a paraconsistent framework Aristotle's Thesis may be subjoined to relevant systems—and, on the other hand, being a non-classical thesis it cannot be selected as a relevant formula in the KWS-construction.

Which is then the sense in which Aristotle's Thesis expresses an intuition about relevance? A suggestion which may be developed is that the required characterization should be found not by looking to subsystems of classical logics but to extensions of it, and mainly to modal extension of the standard calculus. Let us recall that the four modal status described in the Aristotelian square of oppositions are formalized as proposition-forming monadic operators symbolized by $\Box$, $\Box\neg$, $\Diamond$, $\Diamond\neg$. An equivalent description of the square granted by the $\Box - \Diamond$ interchange is of course given by the wffs $\Box$, $\neg\Diamond$, $\Diamond$, $\neg\Box$. Note in particular that $\Box A$ and $\Box\neg A$ are contraries, which means that a basic thesis for the logic of such operators is $\neg(\Box A \wedge \Box\neg A)$, i.e. $\Box A \supset \Diamond A$, where $\Box A \supset \Diamond A$ is the well-known deontic axiom **D**.

If we require that an Aristotelian modal logic should be represented by a normal modal system, i.e. should contain at least **K** $(\Box(A \supset B) \supset (\Box A \supset \Box B))$ and closed under Necessitation ($\vdash A$ only if $\vdash\Box A$) then no logical system representing Aristotelian logic could be weaker than the minimal deontic logic **KD**.

Now the modal status of a proposition is a function of the meaning of the proposition itself. If we require that the two clauses of a true conditional have some connection of meaning, then, they cannot have incompatible meaning, so they cannot have incompatible modal status. A minimal condition for relevance, which in the new context we could call *consequential relevance*, is then as follows:

(CR) *The antecedent and the consequent of a true conditional cannot have incompatible modal status.*

As is well known, the incompatible modal status are located at the extremities of the two diagonals of the square and are represented by the two couples $\{\Box, \neg\Box\}$ and $\{\Diamond, \neg\Diamond\}$. So we may ask that a conditional $A \rightarrow B$ is true whenever

(1) A strictly implies B ($A \supset B$ is necessarily true).
(2) It is false that A and B have incompatible modal status, or in other words we have:
   (a) $\neg(\Box A \wedge \neg\Box B)$, i.e., $\Box A \supset \Box B$.
   (b) $\neg(\Diamond A \wedge \neg\Diamond B)$, i.e., $\Diamond A \supset \Diamond B$.
   (c) $\neg(\neg\Box A \wedge \Box B)$, i.e., $\neg\Box A \supset \neg\Box B$.
   (d) $\neg(\neg\Diamond A \wedge \Diamond B)$, i.e., $\neg\Diamond A \supset \neg\Diamond B$.

Since (a) and (b) follow from $\Box(A \supset B)$ by the normality of **K**, a simple definition of the arrow is the following:

$$(\text{Def} \rightarrow) \qquad A \rightarrow B =_{\text{Df}} A - 3B \wedge (\Box B \supset \Box A) \wedge (\Diamond B \supset \Diamond A).$$

And it is easy to check that (Def $\rightarrow$) is equivalent to the following:

$$(\text{Def} \rightarrow') \qquad A \rightarrow B =_{\text{Df}} A - 3B \wedge (\Box A \equiv \Box B) \wedge (\Diamond A \equiv \Diamond B) \wedge (\neg\Box A \equiv \neg\Box B) \wedge (\neg\Diamond A \equiv \neg\Diamond B).$$

(Def $\rightarrow'$) asserts, as a matter of fact, that A implies B and that the modal status of A and B is coincident.

Thus a synthetic way to express the same idea is by introducing a variable Ξ for modal status and write

$$(\text{Def} \rightarrow'') \qquad A \rightarrow B =_{\text{Df}} \text{ for every } \Xi,\ A - 3B \wedge (\Xi A \equiv \Xi B).$$

So the basic idea that the modal status of the clauses cannot be incompatible leads to an apparently stronger claim: *antecedent and consequent of a true conditional must have the same modal status*. The two assertions are indeed equivalent. So modal logic allows a simple rendering of the idea that the clauses should have something in common: where this something in common is not a variable, as classical relevant logic claims, but exactly the modal status.

Given the minimal normal system **K** the definition yields the following theorems:

$$(\bot \rightarrow p) \supset (p \rightarrow \bot),$$
$$(p \rightarrow \top) \supset (\top \rightarrow p).$$ [16]

In other words, if a contradiction (or a tautology) implies consequentially a wff A, it is equivalent to it. Such a peculiarity is enough to grant the non-Scotian character of this kind of logic.

Now let us recall that any Aristotelian logic should be as strong as deontic logic, or in other words it should include the deontic law $\Box A \supset \Diamond A$. Let us now suppose by Reductio $A \rightarrow \neg A$. By the definition (Def $\rightarrow$) this means to suppose by Reductio

$$(1) \quad \Box(A \supset \neg A) \wedge (\Box\neg A \supset \Box A) \wedge (\Diamond\neg A \supset \Diamond A).$$

But $\Box(A \supset \neg A)$ equals $\Box\neg A$, and *via* the conditional $(\Box\neg A \supset \Box A)$ this implies $\Box A$. So our supposition implies $\Box\neg A \wedge \Box A$, which is the negation of $\neg(\Box A \wedge \Box\neg A)$, i.e. the

[16] Suffice it to say that $\bot \rightarrow p$ implies the equivalence between $\Diamond p$ and $\Diamond\bot$. But $\neg\Diamond\bot$ is a **K**-theorem, so $\neg\Diamond p$ is such. This means that $\Box\neg p$ is a theorem, so by definition $\top \rightarrow \neg p$ and $p \rightarrow \bot$ are such.

negation of **D**. Then in presence of **D** the given supposition leads to an inconsistency, which means to establish in **KD** the negation of the supposition, i.e. $\neg(A \to \neg A)$.[17]

Aristotle's Thesis is then a theorem of **KD** if we define the arrow in the mentioned sense.

On the other hand, let us presuppose as a background system the simple **K** and add as an axiom Aristotle's Thesis $\neg(A \to \neg A)$, where $\to$ is defined as before. In other words we subjoin to **K** the axiom

$$(2)\quad \neg(\Box(A \supset \neg A) \wedge (\Box\neg A \supset \Box A) \wedge (\Diamond\neg A \supset \Diamond A)).$$

Now a substitution instance of (2) is

$$(3)\quad \neg(\Box(\top \supset \neg\top) \wedge (\Box\neg\top \supset \Box\top) \wedge (\Diamond\neg\top \supset \Diamond\top))$$

or in other words

$$(4)\quad \Diamond\top \vee (\Box\neg\top \wedge \neg\Box\top) \vee (\neg\Box\top \wedge \neg\Diamond\top).$$

Now since $\neg\Box\top$ is the negation of the **K**-thesis $\Box\top$, from (4) we have $\Diamond\top$ as a theorem. But $\Diamond\top$ this is equivalent, as is well known, to the axiom **D**, i.e. to $\Box A \supset \Diamond A$.

Thus subjoining to **K** Aristotle's Thesis along with the mentioned definition of the arrow is the same as extending **K** with the deontic axiom **D**. Of course, since having **AT** as a theorem is the same as having Boethius' Thesis $(A \to B) \supset \neg(A \to \neg B)$,[18] the same result holds when subjoining Boethius' Thesis to **K** in place of Aristotle's Thesis.

It may be of some interest to remark that **K** is sometimes called in the literature "Aristotle's law". In a sense it describes a feature of implication which is embodied in Modus Ponens: If A logically implies B (so $A \supset B$ is logically necessary) and A is necessary, B is also necessary. So axioms **K** and **AT** (and the equivalent **D** are both basic expressions of features of Aristotelian logic.[19]

It may be proved that $\to$ is transitive and contrapositive,[20] so we may endorse the proof given in note 3 which shows the equivalence between Boethius' Thesis $(A \to B) \supset \neg(A \to \neg B)$ and Aristotle's Thesis. Boethius' Thesis is then also equivalent to **D**.

The preceding remarks are simply the background for more complex results. In fact we may build one or more systems in which the arrow $\to$ for consequential implication is a primitive symbol. Such systems may be endowed with the definition of modal operators by introducing the definition

$$(\text{Def } \Box)\qquad \Box A =_{\text{Df}} \top \to A.$$

The minimal system of consequential implication containing Aristotle's Thesis is called **CI** in [20] and is proved to be definitionally equivalent to **KD**. This paper shows that for every normal modal system we may find a consequential system which is its consequential

[17] This informal proof presupposes conditionalization, but might be reconstructed without this rule.

[18] This result is granted by the fact that $\to$ may be proved to be reflexive, transitive and contrapositive (see footnote 3).

[19] See [15] for a careful investigation of the historical roots of connexive and consequential implication.

[20] For such proofs we may make use of the tableaux procedures for normal system (see [17]).

translation. The central system of consequential implication, named **CI.0**, is **CI** $+ (A \to B) \supset (A \supset B)$ and is analyzed in [17].

As reminded at the beginning, Aristotle's Thesis is the characteristic axiom of connexive logics. This does not mean that every thesis belonging to original connexive systems is valid once the arrow is interpreted as consequential implication. The "Factor Law" $(A \to B) \supset (A \wedge R \to B \wedge R)$ for instance is connexively valid but not consequentially valid, even if it is consistent with consequential systems. It may be shown that the addition of such a formula to a modal-consequential system as **CI.0** yields the collapse of modalities.[21] The same could be said of Strong Boethius' Thesis, $(A \to B) \to \neg(A \to \neg B)$, which in consequential systems turns out to be equivalent to $A \to B \equiv \neg(A \to \neg B)$.[22] In such a way the arrow collapses to the dual relation of cotenability. Thanks to the relation $\top \to A \equiv \Box A$ this equivalence yields the collapse of possibility to necessity $\Box A \equiv \Diamond A$, and in presence of $(A \to B) \supset (A \supset B)$ it yields the collapse $\Box B \equiv B$ and the equivalence $A \to B \equiv (A \equiv B)$.

We mention here that the arrow $\to$ is not the only relation modally definable which satisfies Aristotle's Thesis. Other two implicative relations can be defined as follows:

$$\text{(Def} \Rightarrow) \qquad A \Rightarrow B =_{\text{Df}} \Box(A \supset B) \wedge (\Diamond B \supset \Diamond A),$$

$$\text{(Def} >) \qquad A > B =_{\text{Df}} \Box({}^{*}A \supset B) \wedge (\Diamond B \supset \Diamond {}^{*}A),$$

where * is a "circumstantial operator" in Åqvist's sense.[23]

In such weaker implicative relations, however, it is more difficult to characterize the element which is in common between the two clauses and which allows to treat consequential relations as special relations of relevance.

A final remark concerns the open question of the relation between the consequential operators and the relations axiomatized in relevant logics. We have tried to stress two different points: (1) if $\to$ stands for relevant implication, the addition of Aristotle's Thesis yields systems of implication which are not necessarily trivial, (2) If $\to$ is translated into a particular modal statement Aristotle's Thesis is not inconsistent with classical calculus but is a translation of a simple modal thesis.

This couple of remarks suggests that any system of relevant logic which allows us for a definition of a normal system of modal logic can also include Aristotle's Thesis, provided it is modally defined in the sense outlined in the preceding sections. For instance, we already know that the system **E** of entailment allows the definition of the box as $\Box A =_{\text{Df}} (A \to A) \to A$ and that the modal fragment of **E** is exactly **S4**. Thus, in defining an operator $\to$ in the consequentialist fashion Aristotle's Thesis $\neg(A \to \neg A)$ becomes a thesis of **E** since it is equivalent to $\Diamond\top$.

Of course the arrow which is the primitive axiomatized in **E** is not coincident with the consequentialist arrow, but the latter is construed in terms of the former and receives a sense in terms of it. The study of the interrelations between such implicative relations may

[21] See [22]. The Factor law is an axiom of McCall's system **CFL** (see [10, p. 442 ff]), which has received a modal analysis by Meyer in [12].

[22] See [19].

[23] See [5]. For the logic of $\Rightarrow$ see [21] and for the logic of $>$ see [18].

be an interesting exercise for classical or non-classical relevantists, since it may allow to grasp the differences and the kinships between different intuitions about relevance.

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