THE CLASSIFICATION OF UNIPOTENT AND NILPOTENT ELEMENTS

BY

P. BALA AND R. W. CARTER

(Communicated by Prof. T. A. SPRINGER at the meeting of October 27, 1973)

Let G be a semi-simple algebraic group over an algebraically closed field K and $\mathfrak{g}$ be the Lie algebra of G. It has been shown by SPRINGER [6] that, provided the characteristic of K is either 0 or a 'good prime' for G, there is a bijection between the conjugacy classes of unipotent elements in G and the classes of nilpotent elements of $\mathfrak{g}$ under the adjoint action of G. We shall give a way of describing these conjugacy classes when the characteristic of K satisfies the more restrictive condition of being either 0 or a prime $p \geqslant 4h+3$ where h is the height of the highest root of $\mathfrak{g}$.

Let P_J be a parabolic subgroup of G with Levi decomposition $P_J = U_J L_J$ where U_J is the unipotent radical of P_J and L_J is a Levi subgroup of P_J. It has been shown by RICHARDSON [5] that, provided G has only finitely many unipotent conjugacy classes, P_J has a dense orbit on the Lie algebra $\mathfrak{u}_J$ of U_J under the adjoint action and a dense orbit on U_J under conjugation. Since these dense orbits are unique every parabolic subgroup P_J gives rise to a well-determined class of nilpotent elements (the class containing the elements in the dense orbit) and a well-determined class of unipotent elements. It is not known whether G always has only finitely many unipotent classes, but this is certainly true when the characteristic of K is either 0 or a good prime for G. [4].

A parabolic subgroup P_J of G is called distinguished if $\dim U_J/U_J' = \dim L_J$, where U_J' is the commutator subgroup of U_J. It is clear that Borel subgroups are distinguished. In a simple group G of type A_l the Borel subgroups are the only distinguished parabolic subgroups, but in groups of other types there are others. For example in a simple group of type C_l there is a bijection between conjugacy classes of distinguished parabolic subgroups of G and partitions of l into distinct parts. In a group of type B_l the classes of distinguished parabolic subgroups correspond to partitions of $2l+1$ into distinct odd parts, and in a group of type D_l they correspond to partitions of $2l$ into distinct odd parts. The classes of distinguished parabolic subgroups in the exceptional groups can also be determined in a straightforward manner. The number of such classes is 2 for G_2, 4 for F_4, 3 for E_6, 6 for E_7, and 11 for E_8. The significance of the distingushed parabolic subgroups lies in the following result.

THEOREM 1. Suppose the characteristic of K is either 0 or $p \geqslant 4h+3$. Let P_J be a distinguished parabolic subgroup of G and n be an element in the dense orbit of P_J on $\underline{u}_J$. Then n is a nilpotent element of $\underline{g}$ which commutes with no non-zero semi-simple element of $\underline{g}$. Moreover any nilpotent element n of $\underline{g}$ which commutes with no non-zero semi-simple element arises in this way from some distinguished parabolic subgroup P_J, which is determined by n to within conjugacy in G.

This result shows that there is a bijection between conjugacy classes of distinguished parabolic subgroups in G and classes of nilpotent elements of $\underline{g}$ which do not commute with non-zero semi-simple elements. We observe that the unipotent elements of G related by Springer's correspondence to the nilpotent elements of $\underline{g}$ of this special type are those for which the connected component of the centralizer does not contain non-identity semi-simple elements of G.

We may extend the above result to arbitrary nilpotent elements of $\underline{g}$ by introducing the concept of a regular subgroup of G. A subgroup R of G is called a regular subgroup if $R = L_J'$ for some parabolic subgroup P_J of G. This is the commutator subgroup, i.e. the semi-simple part, of the Levi subgroup L_J of P_J. The conjugacy classes of regular subgroups of a given group G are easy to obtain. Two regular subgroups are conjugate in G if and only if their root systems are equivalent under the action of the Weyl group of G. Given a regular subgroup R of G and a distinguished parabolic subgroup P_R of R we may again define a class of nilpotent elements in $\underline{g}$ via the dense orbit of P_R on the Lie algebra of its unipotent radical.

THEOREM 2. Suppose the characteristic of K is either 0 or $p \geqslant 4h+3$. Let R be a regular subgroup of G and P_R be a distinguished parabolic subgroup of R. Let n be an element in the dense orbit of P_R on the Lie algebra of its unipotent radical. Then n lies in a nilpotent class which is determined by the pair (R, P_R) of subgroups of G. Every nilpotent class in $\underline{g}$ is obtained from some pair (R, P_R) in this way, and the pair (R, P_R) is determined by the given nilpotent class to within conjugacy in G.

We obtain a similar result for unipotent classes of G, using the Springer correspondence.

THEOREM 3. Suppose the characteristic of K is either 0 or $p \geqslant 4h+3$. There is a bijection between the unipotent conjugacy classes in G and conjugacy classes of pairs (R, P_R) where R is a regular subgroup of G and P_R a distinguished parabolic subgroup of R. The class corresponding to (R, P_R) contains the elements in the dense orbit of P_R on its unipotent radical.

As an illustration of these results we take $G = SL_n(K)$. Every regular subgroup of G is conjugate to a subgroup R containing matrices of form

$$\begin{pmatrix} M_1 & & 0 \\ & M_2 & \\ & & \ddots \\ 0 & & & \ddots \end{pmatrix}$$

where $M_i \in SL_{\lambda_i}(K)$ and $\lambda = (\lambda_1 \geqslant \lambda_2 \geqslant \ldots)$ is a partition of n. The only distinguished parabolic subgroups of R are those conjugate to the subgroup P_R consisting of matrices of the above form where each M_i is upper triangular. The unipotent radical of P_R is the subgroup in which each M_i is upper unitriangular. The dense orbit of P_R on its unipotent radical contains all matrices for which each super-diagonal entry in each M_i is non-zero. For example

$$J(\lambda) = \begin{pmatrix} J(\lambda_1) & 0 & & \\ 0 & J(\lambda_2) & & \\ & & \ddots & \\ & & & \ddots \end{pmatrix}$$

lies in this dense orbit, where

$$J(\lambda_i) = \begin{pmatrix} 1 & 1 & 0 & \\ & \ddots & \ddots & \\ & & \ddots & \ddots \\ 0 & & \ddots & 1 \\ & & & 1 \end{pmatrix}$$

Thus we recover the standard description of unipotent classes in $SL_n(K)$ by Jordan block matrices.

The restriction on the characteristic of K has been imposed in order to be able to use the Jacobson-Morozov theorem in the characteristic p situation. It is likely that a much weaker restriction on the characteristic will be sufficient to give the same result. However the restriction cannot be removed altogether since it is known that in the orthogonal and symplectic groups of characteristic 2 the unipotent classes cannot be parametrized in the above manner. [7]

Proofs of the above results will appear in a forthcoming paper. The first author wishes to thank the Science Research Council for support while the work was being carried out.

Mathematics Institute
University of Warwick
Coventry

REFERENCES

1. BOREL, A., R. CARTER, C. W. CURTIS, N. IWAHORI, T. A. SPRINGER and R. STEINBERG, Seminar on algebraic groups and related finite groups. Lecture Notes in Mathematics, Springer **131** (1970).
2. DYNKIN, E. B., Semisimple subalgebras of semisimple Lie algebras. Amer. Math. Soc. Translations (2) **6** 111–244 (1957).
3. ELKINGTON, G. B., Centralizers of unipotent elements in semisimple algebraic groups, J. Algebra **23**, 137–163 (1972).
4. RICHARDSON, R. W., Conjugacy classes in Lie algebras and algebraic groups. Ann. Math. **86**, 1–15 (1967).
5. ———, Conjugacy classes in parabolic subgroups of semisimple algebraic groups. Bull. London Math. Soc. To appear.
6. SPRINGER, T. A., The unipotent variety of a semisimple group, Proceedings of the Colloquium in Algebraic Geometry (Tata Institute), 373–391 (1969).
7. WALL, G. E., On the conjugacy classes in the unitary, symplectic and orthogonal groups, J. Australian Math. Soc. **3**, 1–62 (1963).