Let $G$ be a semi-simple algebraic group over an algebraically closed field $K$ and $g$ be the Lie algebra of $G$. It has been shown by Springer [6] that, provided the characteristic of $K$ is either 0 or a 'good prime' for $G$, there is a bijection between the conjugacy classes of unipotent elements in $G$ and the classes of nilpotent elements of $g$ under the adjoint action of $G$. We shall give a way of describing these conjugacy classes when the characteristic of $K$ satisfies the more restrictive condition of being either 0 or a prime $p > 4h + 3$ where $h$ is the height of the highest root of $g$.

Let $P_J$ be a parabolic subgroup of $G$ with Levi decomposition $P_J = U_J L_J$ where $U_J$ is the unipotent radical of $P_J$ and $L_J$ is a Levi subgroup of $P_J$. It has been shown by Richardson [5] that, provided $G$ has only finitely many unipotent conjugacy classes, $P_J$ has a dense orbit on the Lie algebra $g_J$ of $U_J$ under the adjoint action and a dense orbit on $U_J$ under conjugation. Since these dense orbits are unique every parabolic subgroup $P_J$ gives rise to a well-determined class of nilpotent elements (the class containing the elements in the dense orbit) and a well-determined class of unipotent elements. It is not known whether $G$ always has only finitely many unipotent classes, but this is certainly true when the characteristic of $K$ is either 0 or a good prime for $G$. [4].

A parabolic subgroup $P_J$ of $G$ is called distinguished if $\dim U_J / U_J' = -\dim L_J$, where $U_J'$ is the commutator subgroup of $U_J$. It is clear that Borel subgroups are distinguished. In a simple group $G$ of type $A_t$ the Borel subgroups are the only distinguished parabolic subgroups, but in groups of other types there are others. For example in a simple group of type $C_l$ there is a bijection between conjugacy classes of distinguished parabolic subgroups of $G$ and partitions of $l$ into distinct parts. In a group of type $B_l$ the classes of distinguished parabolic subgroups correspond to partitions of $2l + 1$ into distinct odd parts, and in a group of type $D_l$ they correspond to partitions of $2l$ into distinct odd parts. The classes of distinguished parabolic subgroups in the exceptional groups can also be determined in a straightforward manner. The number of such classes...
is 2 for $G_2$, 4 for $F_4$, 3 for $E_6$, 6 for $E_7$, and 11 for $E_8$. The significance of the distinguished parabolic subgroups lies in the following result.

**Theorem 1.** Suppose the characteristic of $K$ is either 0 or $p > 4h + 3$. Let $P_J$ be a distinguished parabolic subgroup of $G$ and $n$ be an element in the dense orbit of $P_J$ on $g_J$. Then $n$ is a nilpotent element of $g$ which commutes with no non-zero semi-simple element of $g$. Moreover any nilpotent element $n$ of $g$ which commutes with no non-zero semi-simple element arises in this way from some distinguished parabolic subgroup $P_J$, which is determined by $n$ to within conjugacy in $G$.

This result shows that there is a bijection between conjugacy classes of distinguished parabolic subgroups in $G$ and classes of nilpotent elements of $g$ which do not commute with non-zero semi-simple elements. We observe that the unipotent elements of $G$ related by Springer’s correspondence to the nilpotent elements of $g$ of this special type are those for which the connected component of the centralizer does not contain non-identity semi-simple elements of $G$.

We may extend the above result to arbitrary nilpotent elements of $g$ by introducing the concept of a regular subgroup of $G$. A subgroup $R$ of $G$ is called a regular subgroup if $R = L_J'$ for some parabolic subgroup $P_J$ of $G$. This is the commutator subgroup, i.e. the semi-simple part, of the Levi subgroup $L_J$ of $P_J$. The conjugacy classes of regular subgroups of a given group $G$ are easy to obtain. Two regular subgroups are conjugate in $G$ if and only if their root systems are equivalent under the action of the Weyl group of $G$. Given a regular subgroup $R$ of $G$ and a distinguished parabolic subgroup $P_R$ of $R$ we may again define a class of nilpotent elements in $g$ via the dense orbit of $P_R$ on the Lie algebra of its unipotent radical.

**Theorem 2.** Suppose the characteristic of $K$ is either 0 or $p > 4h + 3$. Let $R$ be a regular subgroup of $G$ and $P_R$ be a distinguished parabolic subgroup of $R$. Let $n$ be an element in the dense orbit of $P_R$ on the Lie algebra of its unipotent radical. Then $n$ lies in a nilpotent class which is determined by the pair $(R, P_R)$ of subgroups of $G$. Every nilpotent class in $g$ is obtained from some pair $(R, P_R)$ in this way, and the pair $(R, P_R)$ is determined by the given nilpotent class to within conjugacy in $G$.

We obtain a similar result for unipotent classes of $G$, using the Springer correspondence.

**Theorem 3.** Suppose the characteristic of $K$ is either 0 or $p > 4h + 3$. There is a bijection between the unipotent conjugacy classes in $G$ and conjugacy classes of pairs $(R, P_R)$ where $R$ is a regular subgroup of $G$ and $P_R$ a distinguished parabolic subgroup of $R$. The class corresponding to $(R, P_R)$ contains the elements in the dense orbit of $P_R$ on its unipotent radical.
As an illustration of these results we take $G = SL_n(K)$. Every regular subgroup of $G$ is conjugate to a subgroup $R$ containing matrices of form

$$\begin{pmatrix}
M_1 & 0 \\
M_2 \\
0
\end{pmatrix}$$

where $M_1 \in SL_\lambda(K)$ and $\lambda = (\lambda_1 > \lambda_2 > \ldots)$ is a partition of $\lambda$. The only distinguished parabolic subgroups of $R$ are those conjugate to the subgroup $P_R$ consisting of matrices of the above form where each $M_1$ is upper triangular. The unipotent radical of $P_R$ is the subgroup in which each $M_1$ is upper unitriangular. The dense orbit of $P_R$ on its unipotent radical contains all matrices for which each super-diagonal entry in each $M_1$ is non-zero. For example

$$J(\lambda) = \begin{pmatrix}
J(\lambda_1) & 0 \\
0 & J(\lambda_2) \\
\vdots & \vdots
\end{pmatrix}$$

lies in this dense orbit, where

$$J(\lambda_4) = \begin{pmatrix}
1 & 1 & 0 \\
0 & 1 & \\
\vdots & \vdots & 1
\end{pmatrix}$$

Thus we recover the standard description of unipotent classes in $SL_n(K)$ by Jordan block matrices.

The restriction on the characteristic of $K$ has been imposed in order to be able to use the Jacobson-Morozov theorem in the characteristic $p$ situation. It is likely that a much weaker restriction on the characteristic will be sufficient to give the same result. However the restriction cannot be removed altogether since it is known that in the orthogonal and symplectic groups of characteristic 2 the unipotent classes cannot be parametrized in the above manner. [7]

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Mathematics Institute
University of Warwick
Coventry
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