Contents lists available at SciVerse ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Inscribing a regular octahedron into polytopes

Arseniy Akopyan [a,b,*], Roman Karasev [c,b]

[a] *Institute for Information Transmission Problems RAS, Bolshoy Karetny per. 19, Moscow, 127994, Russia*

[b] *B. N. Delone International Laboratory "Discrete and Computational Geometry", P. G Demidov Yaroslavl State University, Sovetskaya st. 14, Yaroslavl',
150000, Russia*

[c] *Department of Mathematics, Moscow Institute of Physics and Technology, Institutskiy per. 9, Dolgoprudny, 141700, Russia*

ARTICLE INFO

ABSTRACT

We prove that any simple polytope (and some non-simple polytopes) in $\mathbb{R}^3$ admits a regular
octahedron inscribed into its surface.

© 2012 Elsevier B.V. All rights reserved.

1. Introduction

The famous theorem of Shnirel'man [10] (see also [5]) asserts that for every smooth simple closed curve γ in the plane
there exists a square Q such that all four vertices of Q lie on γ.

In the thesis of Makeev [6] the following theorem was proved (reproved and generalized for higher prime power
dimensions in [2,4]):

Theorem 1.1. *Let H be a smooth surface embedded into $\mathbb{R}^3$ and homeomorphic to the sphere S^2. Let C be some $\mathbb{Z}_3$-symmetric
octahedron. Then there exists an octahedron $C' \subset \mathbb{R}^3$ similar to C with all its vertices lying on H.*

Remark 1.2. Here by $\mathbb{Z}_3$-*symmetric* we mean the following: The group $\mathbb{Z}_3 \subset SO(3)$ acts on C by cyclically permuting three
pairs of its opposite vertices so that this action is extended to an action by isometries on the whole $\mathbb{R}^3$.

The word *similar* here means equivalent up to a similarity transform with positive determinant, that is a composition of
a proper rigid motion and a positive homothety. By *inscribing* a polytope P into a surface H we will always mean finding its
similar copy such that all vertices of P lie on H.

It is known (see [5,7,8] for example) that squares in the plane can be also inscribed into any polygonal simple curve; the
approximation by smooth curves and going to the limit works well in this case. Informally, the key feature here is that if
you look at the square from some direction in the plane then you do not see one of its vertices. The problem of inscribing a
square into any continuous simple closed curve in the plane remains unsolved, the approximation by smooth curves seems
to be insufficient in this case, see [7] for further information about this.

In $\mathbb{R}^3$ the situation is different even for regular octahedra: One can see all the vertices of an octahedron from some
directions. Thus we have to be careful when going to the limit and this is the main content of this paper. Note also that in the

* Corresponding author at: Institute for Information Transmission Problems RAS, Bolshoy Karetny per. 19, Moscow, 127994, Russia.
E-mail addresses: akopjan@gmail.com (A. Akopyan), r_n_karasev@mail.ru (R. Karasev).
URL: http://www.rkarasev.ru/en/ (R. Karasev).

plane there exists a direct proof [9] of the Shnirel'man theorem for polygonal curves, while in this paper we cannot avoid using the smooth case and going to the limit.

The main result of this paper is:

Theorem 1.3. *Suppose P is a simple convex polytope in $\mathbb{R}^3$. Then there exists a regular octahedron inscribed into the surface ∂P.*

Remark 1.4. The case of non-simple polytopes is considered in Section 4. In this case we are able to establish the existence of an inscribed octahedron under certain additional assumptions on its solid angles (see Theorem 4.4).

Remark 1.5. Unlike the results for inscribing into smooth surfaces [2,3], here we only consider inscribing of a regular octahedron, leaving out the case of inscribing an octahedron similar to a given $\mathbb{Z}_3$-symmetric octahedron. This case seems to be out of reach of the methods presented here.

2. Approximation of ∂P by smooth surfaces

We are going to use the following way to approximate a polytope by smooth bodies:

Definition 2.1. Denote by P_ε the union of all ε-balls that are contained in P

$$P_\varepsilon = \bigcup_{B_\varepsilon(x) \subseteq P} B_\varepsilon.$$

The body P_ε has a smooth boundary and, by the known results, admits an inscribed regular octahedron. Moreover, in [2, 3] it is proved that there is a nontrivial $\mathbb{Z}_3$-equivariant 1-homology class of such octahedra in the configuration space of all octahedra, for which we take the space of similarity transforms with positive determinant S_3, which is topologically equal to $\mathbb{R}^+ \times \mathbb{R}^3 \times SO(3)$. This space has an action of the symmetry group of the octahedron, and out of this group we are going to use the $\mathbb{Z}_3$ that cyclically permutes the three axes of the octahedron.

In a certain "generic" case the set $\mathcal{l}$ of inscribed octahedra is indeed a smooth 1-manifold and itself represents a nonzero class $[\mathcal{l}] \in H_1(S_3/\mathbb{Z}_3; \mathbb{F}_3)$. To handle degenerate cases (see [2,3]) it is useful to replace $[\mathcal{l}]$ with its Poincaré dual class from some relative cohomology, which is actually convenient to treat as a member of the compact support cohomology $\zeta \in H_c^9(S_3/\mathbb{Z}_3; \mathbb{F}_3)$ (see [1] as a reference for compact support cohomology). Using the technique of compact support cohomology, we restate the results of [2,3] as follows: There exists a nonzero obstruction in cohomology $\zeta \in H_c^9(S_3/\mathbb{Z}_3; \mathbb{F}_3)$ such that the set of inscribed octahedra $\mathcal{l}$ has ζ supported in its arbitrarily small neighborhood even in the degenerate cases, when $\mathcal{l}$ itself does not look like a 1-cycle geometrically.

From here on, we argue in terms of the 1-dimensional homology cycle of $\mathcal{l}$ informally, having in mind that everything becomes rigorous when passing to codimension 1 compact support cohomology.

Now let ε tend to zero. If the diameters of the inscribed octahedra of P_ε do not tend to zero, then we obtain an inscribed octahedron for P by standard compactness considerations. Hence for the rest of the proof we assume the contrary: Let the maximum diameter of inscribed octahedra for P_ε be at most $\delta(\varepsilon)$ and $\lim_{\varepsilon \to +0} \delta(\varepsilon) = 0$. Denote the set of inscribed octahedra for P_ε by $\mathcal{l}_\varepsilon \subset S_3$.

Since $\delta(\varepsilon)$ tends to zero, the octahedra from $\mathcal{l}_\varepsilon$ tend (say, in Hausdorff metric) to points at the boundary of P. The first observation is that they obviously cannot tend to a relative interior point of a facet. Moreover, the detailed analysis near an edge shows that for small enough ε the octahedra in $\mathcal{l}_\varepsilon$ cannot tend to an interior point of an edge. Indeed, if we project ∂P along an edge then we obtain a plane angle A, the smoothening ∂P_ε being projected to a smoothened plane angle A_ε. Let C be an octahedron inscribed into ∂P_ε; it projects to a quadrilateral or hexagon C' inscribed into ∂A_ε. Now it remains to note that C' is centrally symmetric and we cannot see one of its vertices from any direction (of course, the invisible vertex depends on the direction), while we can see the entire ∂A_ε from some directions. This is a contradiction and we conclude:

Lemma 2.2. *If there is no octahedron inscribed into ∂P then all octahedra in $\mathcal{l}_\varepsilon$ tend to the vertices of P in Hausdorff metric. For small enough ε the family $\mathcal{l}_\varepsilon$ becomes a disjoint union of the sets $\mathcal{l}_\varepsilon(v)$, corresponding to different vertices $v \in P$.*

The decomposition of $\mathcal{l}_\varepsilon$ into the sets $\mathcal{l}_\varepsilon(v)$ corresponds to decomposition of the 1-homology class of inscribed octahedra into a smooth body into a sum of 1-homology classes. Hence we have proved:

Lemma 2.3. *There exists a vertex v, such that for arbitrarily small ε the octahedra $\mathcal{l}_\varepsilon(v)$ (inscribed into ∂P_ε near v) carry a nontrivial $\mathbb{Z}_3$-equivariant 1-homology.*

Thus we have to study the situation near the vertices of P. More precisely, we have to consider solid angles $A(v)$ of corresponding vertices of P and their smoothenings $A_\varepsilon(v)$ (not depending on $\varepsilon > 0$ essentially because of the possibility to apply a homothety).

We want to describe the $\mathbb{Z}_3$-equivariant 1-homology of the set of octahedra inscribed into $\partial A_\varepsilon(v)$. The configuration space S_3 of all octahedra is homotopy equivalent to $SO(3)$ and it has trivial homology modulo 3 in dimensions 1 and 2;
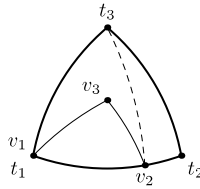
Fig. 1.

the $\mathbb{Z}_3$-equivariant 1-homology modulo 3 (the homology of $S_3/\mathbb{Z}_3$) is therefore equal to $H_1(\mathbb{Z}_3; \mathbb{F}_3) = \mathbb{F}_3$. We choose the generator of $H_1(S_3/\mathbb{Z}_3; \mathbb{F}_3)$ to be equal to the 1-homology of octahedra inscribed into a generic smooth convex body.

The homology class of octahedra inscribed into $\partial A_\varepsilon(v)$ is well defined in the case when there are no arbitrarily large octahedra inscribed into ∂A_ε (note that we actually work in compact support cohomology and therefore must ensure that everything remains bounded); in this case it corresponds to the well defined 1-homology class. Using an appropriate homothety we conclude that fixed ε and arbitrarily large octahedron is the same as arbitrarily small ε and a fixed size octahedron. By compactness considerations we conclude the following:

Lemma 2.4. *The surface $\partial A_\varepsilon(v)$ has no well defined homology class of inscribed octahedra only if there exists an octahedron inscribed into $\partial A(v)$ without smoothening.*

Definition 2.5. Call a solid angle A *special* if its boundary admits an inscribed octahedron.

Denote the configuration space of all congruence classes ($SO(3)$-orbits) of solid angles (with $n \geq 3$ facets) by $\mathcal{A}^n$ and denote the subset of special angles by $\mathcal{A}_S^n$, this is a closed subset of $\mathcal{A}^n$. Define the function

$$\varphi : \mathcal{A}^n \setminus \mathcal{A}_S^n \to \mathbb{F}_3$$

by assigning to a solid angle A the 1-homology of octahedra (divided by the generator of $H_1(S_3/\mathbb{Z}_3; \mathbb{F}_3)$) inscribed into the smoothened solid angle A_ε.

Lemma 2.6. *The function φ is locally constant on $\mathcal{A}^n \setminus \mathcal{A}_S^n$.*

Proof. Fix some $\varepsilon > 0$. Under deformations of a solid angle A that avoid arbitrarily large inscribed octahedra for its smoothening A_ε, the smoothenings A_ε are deformed continuously and the sizes of their inscribed octahedra remain bounded. Hence, by the continuity of the 1-homology (or compact support cohomology), the class of octahedra inscribed into A_ε is preserved. □

Now in the inscribing problem we have the following options:
(a) P has a special angle. In this case it already has an inscribed octahedron.
(b) If (the sum is over all vertices in P)

$$\sum_{v \in P} \varphi(A(v)) \neq 1 \tag{2.1}$$

then ∂P admits an inscribed octahedron. This follows from additivity of the homology classes.

Note that for every solid angle A close enough to a halfspace $\varphi(A) = 0$, since its smoothening does not admit an inscribed octahedron.

3. Proof of Theorem 1.3

If the polytope P is simple then we deal with $\mathcal{A}_S^3 \subset \mathcal{A}^3$. Taking into account the observations in the previous section, we see that to prove Theorem 1.3 it is enough to prove the following lemma (because in this case the left hand part of (2.1) vanishes):

Lemma 3.1. *The set $\mathcal{A}^3 \setminus \mathcal{A}_S^3$ is arcwise connected.*

The proof of Lemma 3.1 will follow from the description of all solid angles that admit an octahedron inscribed into the boundary:

Lemma 3.2. *A solid angle A admits an octahedron inscribed into its boundary if and only if it is possible to place its corresponding spherical triangle $v_1 v_2 v_3$ inside of the regular spherical triangle $t_1 t_2 t_3$ with $|t_1 t_2| = \pi/3$ in the following way (up to relabeling the vertices v_1, v_2, v_3): The vertices v_1 and t_1 coincide, the vertex v_2 lies on the segment $t_1 t_2$, and v_3 lies inside the triangle $t_1 v_2 t_3$. (see Fig. 1)*

Proof. Let us check how an octahedron C could be inscribed into A. There are two alternatives:
Case 1: Some three vertices of C are on one facet of A, two are on the other facet, and one is on the third facet.
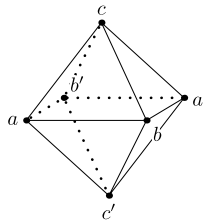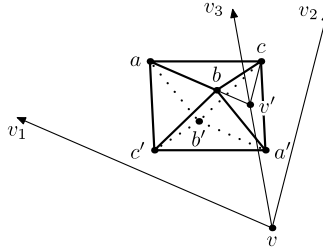*Case* 2: Every facet of A contains two vertices of C.
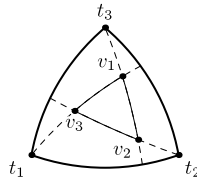
Fig. 2.

Fig. 3.

Fig. 4.

There are other degenerate cases, but we may ignore them since they all are limit cases of these two.

Denote the vertices of the octahedron by a, b, c, a', b', and c' (see Fig. 2).

Consider the first case. Suppose one of the facets of the angle A contains the facet $a'b'c'$, the second facet contains the edge ab, and the third facet contains the vertex c.

Denote the vertex of the angle A by v. It is easy to see that the common edge of the first and the second facets of A is parallel to the edge ab of C, because both facets are parallel to ab. Without loss of generality we may assume that the point b is closer to v than a. Let v_1 be the vertex of the spherical triangle that corresponds to the common edge of the first and the second facet of A, v_2 correspond to the first and the third facet, and v_3 is the remaining vertex (see Fig. 3).

Let t_1, t_2, and t_3 be the vertices of the regular spherical triangle that corresponds to the vectors $\overrightarrow{ba}$, $\overrightarrow{c'b'}$, and $\overrightarrow{a'c}$.

Denote by v' the point where the edge v_3 and the plane abc intersect (Fig. 3). Obviously, $v'c$ is parallel to the edge v_2. Since $v'c$ does not intersect the interior of the triangle abc, it follows that the vector $\overrightarrow{v'c}$ lies "between" the vectors $\overrightarrow{ba}$ and $\overrightarrow{bc}$. Therefore on the sphere the vertex v_2 lies on the segment $[t_1, t_2]$.

Consider the plane α of the third facet that contains the point c of the octahedron. It contains the line cv' and does not intersect the octahedron in the interior. This means that α lies "between" the planes (in terms of normals to oriented planes) $v'ca'$ and $v'ca$, the latter coinciding with the plane abc. The line v_2t_3 (on the sphere) corresponds to the plane $v'ca'$ and the line t_1v_2 corresponds to the plane $v'ca$. Therefore the third facet corresponds to a line "between" v_2t_1 and v_2t_3.

It is clear that the second facet corresponds to the line passing through t_1 that lies "between" $[t_1, t_2]$ and $[t_1, t_3]$. Therefore the point v_3 lies inside the triangle $t_1v_2t_3$.

To prove the lemma in the opposite direction, we note that for any triangle from the statement of the lemma it is possible to construct an inscribed octahedron by the way depicted in Fig. 3.

Consider the second case. Without loss of generality we may assume that the first facet of A contains the edge ab of the octahedron, the second facet contains the edge ca', and the third facet contains the edge $b'c'$. This means that extensions of the sides of the spherical triangle $\triangle v_1v_2v_3$ pass through the vertices of $\triangle t_1t_2t_3$ (see Fig. 4). If the fact that $\triangle v_1v_2v_3$ is inside $\triangle t_1t_2t_3$ is not clear to the reader, please see the proof of Lemma 4.1 for a rigorous explanation.

Let us show that $\triangle v_1v_2v_3$ can be placed in $\triangle t_1t_2t_3$ in the proper way. Since the area of $\triangle v_1v_2v_3$ is less than the area of $\triangle t_1t_2t_3$ it follows that one of the angles of $\triangle v_1v_2v_3$ is less than $\angle t_2t_1t_3$ (note that $\triangle t_1t_2t_3$ is regular). Without loss of generality we may assume that this angle is $\angle v_2$ and the triangle $v_1v_2v_3$ is placed in $t_1t_2t_3$ in the way shown in Fig. 4 (the points t_1 and v_3 are on the same side of the line v_1v_2).
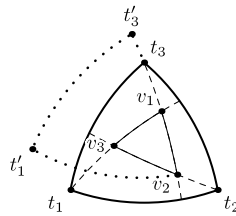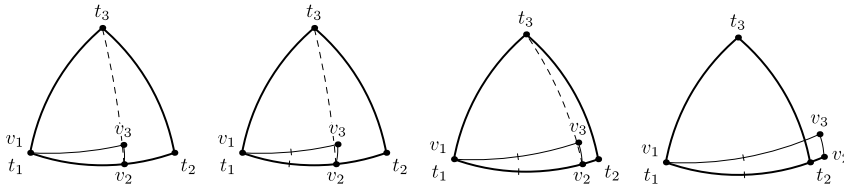
Fig. 5.

Fig. 6.

Choose a point t'_1 so that $|t'_1 v_2| = |t_1 t_2|$ and $\angle t'_1 v_2 t_3 = \angle t_1 t_2 t_3$; and choose a point t'_3 on the ray $[v_2, t_3)$ so that $|t'_3 v_2| = |t_1 t_2|$ (Fig. 5). Note that $\angle t_3 v_2 t_1 > \angle t_3 t_2 t_1$. Therefore the segment $[t'_1, v_2]$ intersects the segment $[t_1, v_3]$, thus giving the inclusion $\triangle v_1 v_2 v_3 \subset \triangle t'_1 v_1 v_2$. We obtain that $\triangle v_1 v_2 v_3$ is positioned in the proper way inside $\triangle v_2 t'_1 t'_3$, which is congruent to $\triangle t_1 t_2 t_3$.   □

Lemma 3.3. *In Lemma* 3.2 *we may assume that* $|v_1 v_2| \geq |v_1 v_3| \geq |v_2 v_3|$.

Proof of Lemma 3.3. If $|v_1 v_3| > |v_1 v_2|$ then reflect $\triangle v_1 v_2 v_3$ with respect to the bisector of $\angle v_2 v_1 v_3$ and the triangle $v_1 v'_2 v'_3$, which lies in $\triangle t_1 t_2 t_3$ in a proper way, because $v'_2 \in [v_1, v_3]$ and $\triangle v_1 v_2 v_3 \subset v_1 v'_3 t_3$.

If $|v_2 v_3| > |v_1 v_3|$ then reflect $\triangle v_1 v_2 v_3$ with respect to the perpendicular bisector of the segment $[v_1, v_2]$. Denote by v'_3 the image of v_3. We have

$$\angle v'_3 v_2 v_1 = \angle v_3 v_1 v_2 < \angle t_3 v_1 v_2 < \angle t_3 v_2 v_1.$$

Since $|v_2 v_3| > |v_1 v_3|$, we have

$$\angle v'_3 v_1 v_2 = \angle v_3 v_2 v_1 < \angle v_3 v_1 v_2 < \angle t_3 v_1 v_2.$$

Therefore the "rays" $[v_1, v'_3)$ and $[v_2, v'_3)$ are directed into the interior of $\triangle t_1 v_2 t_3$ and the point v'_3 lies inside this triangle.

Using this two kinds of operations we can rearrange the side lengths of the $\triangle v_1 v_2 v_3$ in the required order.   □

Corollary 3.4. *If all facet angles of a solid angle* $A \in \mathcal{A}^3$ *are less than* $\pi/6$ *then* $A \in \mathcal{A}^3_S$. *If one facet angle of a solid angle* $A \in \mathcal{A}^3$ *is greater than* $\pi/3$ *then* $A \in \mathcal{A}^3 \setminus \mathcal{A}^3_S$.

Remark 3.5. It is *not true* that if all facet angles of a solid angle $A \in \mathcal{A}^3$ are at most $\pi/3$ then $A \in \mathcal{A}^3_S$. Indeed, we start from the spherical triangle $t_1 t_2 t_3$ and decrease slightly its side $t_2 t_3$. The resulting triangle will not fit into $\triangle t_1 t_2 t_3$, so by Lemma 3.2 it cannot be in $\mathcal{A}^3_S$.

Now we make the final step:

Proof of Lemma 3.1. Consider the triangle T corresponding to a solid angle $A \in \mathcal{A}^3 \setminus \mathcal{A}^3_S$. Let $T = \triangle v_1 v_2 v_3$ and $|v_1 v_2| \geq |v_1 v_3| \geq |v_2 v_3|$. Let T_0 be the regular triangle $t_1 t_2 t_3$ with side length $\pi/3$.

We are going to show how to increase the sides of the triangle T and obtain a triangle with a side greater than $\pi/3$ (the set of triangles of this kind is obviously arcwise connected and by Corollary 3.4 belongs to the set $\mathcal{A}^3 \setminus \mathcal{A}^3_S$).

Suppose all sides are less than $\pi/3$. Let us try to place the triangle T into the right triangle T_0 in the way prescribed by Lemma 3.3. The only way that makes the position not proper is that v_3 is outside $\triangle v_1 v_2 t_3$, which is possible only if the segment $v_1 v_3$ goes outside the segment $v_2 t_3$. From Corollary 3.4 it follows that $|v_1 v_2| \geq \pi/6$ and therefore $\angle v_1 v_2 t_3$ is acute. Now we increase the length of the side $v_1 v_3$ up to $|v_1 v_2|$ preserving the angle $\angle v_2 v_1 v_3$. The position of $\triangle v_1 v_2 v_3$ remains not proper (as required by Lemma 3.3).

Now we start to increase the length of the sides $v_1 v_2$ and $v_1 v_3$ in such a way that they remain equal during the process. The angle $\angle v_1 v_2 v_3$ will increase while the angle $\angle v_1 v_2 t_3$ will decrease during this process. Hence the point v_3 will remain outside $\triangle v_1 v_2 t_3$. Finally we obtain an isosceles triangle with two sides greater than $\pi/3$ (see Fig. 6).   □
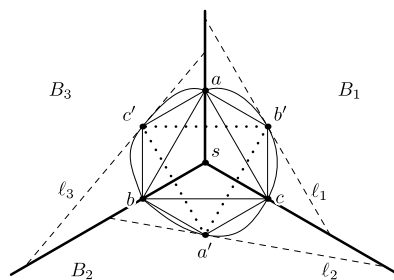
Fig. 7.

4. The case of non-simple polytopes

In this case we have a weaker analogue of Lemma 3.2. We again associate a solid angle A with its spheric convex polygon and denote by T_0 the regular spherical triangle with side length $\pi/3$.

Lemma 4.1. *If $A \in \mathcal{A}_S$ then some congruence takes the spherical polygon of A inside T_0.*

Proof. The assumption $A \in \mathcal{A}_S$ means that the vertex v of A is outside C and we "see" all vertices of C from v. Here we *see* a vertex $c \in C$ if the segment $[vc]$ does not intersect the interior of C. We also note that it is sufficient to prove the lemma in the "generic" case and then go to the limit using compactness of the group of rotations. So we are free to perturb anything.

Consider a regular tetrahedron Θ formed by some four alternating facets of C. Observe that v is not in the interior of Θ, otherwise some vertex of C would be "behind" C when looking from v. Look at Θ from v, there are three alternatives:

Case I: We see some vertex of Θ and this vertex is not "behind" Θ. Denote by B the solid angle of this vertex, its spherical triangle is congruent to T_0. Let us make a small perturbation of A keeping C inscribed into X and making the intersection $\partial A \cap \partial B$ transversal.

Let ∂B consist of three flat angles B_1, B_2, and B_3. Every intersection $X_i = A \cap B_i$ is a convex set containing the vertex s of B_i and having three vertices x_1, x_2, and x_3 of the octahedron C on its boundary. Consider the facet B_1, in this case x_1, x_2 and x_3 is the triple of verticies a, b' and c (Fig. 7).

Note that a and c are on the sides of B_1 and the segments $[a, b']$ and $[b', c]$ are parallel to the sides of B_1, so $sab'c$ is a parallelogram. There exists a support line ℓ_1 to X_1 passing through b' in the plane of B_1. The points a, c, and s are on the one side of ℓ_1 and the line ℓ_1 separates X_1 from infinity, except for the case when ℓ_1 is parallel to a side of B_1. But the latter situation is degenerate and can be excluded by a small perturbation. A similar statement holds for facets B_2 and B_3.

Thus $\partial A \cap \partial B$ is bounded and after the translation that identifies the vertices of A and B the whole solid angle A will get inside B. This is true after an arbitrarily small perturbation of A, so it was true for the original A by the continuity.

Case II: We see some vertex of Θ and this vertex is "behind" Θ. In this case we replace Θ with another tetrahedron Θ' symmetric to Θ with respect to the center of C. The considered case cannot happen to both Θ and Θ' at the same time.

Case III: We see Θ as a quadrangle from v. In this case we do not see the vertex of C that corresponds to the farthest in the pair of edges of Θ that intersect as we see them from v, because we see every face of C adjacent to v from inside. $\square$

Now we make a definition:

Definition 4.2. Denote by $\mathcal{A}_0 \subset \mathcal{A}$ the set of solid angles that cannot be put into T_0 by a congruence.

Remark 4.3. A careful analysis of Lemmas 3.2 and 3.3 shows that $\mathcal{A}^3 \cap \mathcal{A}_0 \neq \mathcal{A}^3 \setminus \mathcal{A}_S^3$. It is sufficient to take $T = T_0$ and shrink one of its sides to the midpoint of that side slightly.

Theorem 4.4. *Suppose P is a convex polytope in $\mathbb{R}^3$ such that all its solid angles either in $\mathcal{A}^3$ or in $\mathcal{A}_0$. Then there exists a regular octahedron inscribed into ∂P.*

Proof. It remains to show that any $A \in \mathcal{A}_0$ can be deformed to a halfspace staying inside $\mathcal{A}_0$. We can make a strong monotonic (monotonic with respect to inclusion) deformation of A to a halfspace, and obviously A will remain in $\mathcal{A}_0$ under such a deformation. $\square$

Acknowledgments

References

[1] A. Borel, J. Moore, Homology theory for locally compact spaces, Michigan Mathematical Journal 7 (1960) 137–159.

[2] R.N. Karasev, Inscribing a regular crosspolytope, 2009. Arxiv preprint arXiv:0905.2671.

[3] R.N. Karasev, Equipartition of a measure by $(Z_p)^k$-invariant fans, Discrete & Computational Geometry 43 (2) (2010) 477–481.

[4] R.N. Karasev, Knaster's problem for $(Z_2)^k$-symmetric subsets of the sphere S^{2^k-1}, Discrete & Computational Geometry 44 (2) (2010) 429–438.

[5] V. Klee, S. Wagon, Old and New Unsolved Problems in Plane Geometry and Number Theory, in: The Dolciani Mathematical Expositions, vol. 11, Mathematical Association of America, Washington, DC, 1991, p. xv, 333.

[6] V.V. Makeev, Universally inscribed and outscribed polytopes, Doctor of Sciences thesis, Saint-Petersburg State University, 2003.

[7] B. Matschke, On the square peg problem and some relatives, 2009. Arxiv preprint http://arxiv.org/abs/1001.0186.

[8] I. Pak, Lectures on discrete and polyhedral geometry, Preliminary version available at author's web page, 2010.

[9] I. Pak, The discrete square peg problem, 2008. Arxiv preprint http://arxiv.org/abs/0804.0657.

[10] L.G. Shnirel'man, On certain geometrical properties of closed curves, Uspekhi Matematicheskikh Nauk (10) (1944) 34–44.