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A T_X-Approach to Some Results on Cuts and Metrics

Victor Chepoi*

Laboratoire de Biomathématiques, Faculté de Medecine, Université d'Aix Marseille II, 27 Bd Jean Moulin, F-13385 Marseille Cedex 5, France

We give simple algorithmic proofs of some theorems of Papernov (1976) and Karzanov (1985, 1990) on the packing of metrics by cuts.

1. INTRODUCTION

Let us commence by recalling the multicommodity flow problem and its dual, the problem of packing metrics by cuts. A pair $S = \{A, B\}$ of nonempty disjoint subsets of a finite set V is called a *cut* if $B = V - A$. Consider a network $N = (G, H, c, q)$ consisting of a supply graph $G = (V, E)$ endowed with a capacity function $c: E \to \mathbb{R}^+ \cup \{0\}$, a demand graph $H = (X, F)$ with $X \subseteq V$, and a demand function $q: F \to \mathbb{R}^+ \cup \{0\}$. Denote the edges of H by $s_1t_1, \dots, s_mt_m$. For a cut $S = \{A, B\}$ of V let $E(S)$ denote the set of edges of G with one end in A and the other in B, and let $c(S) = \sum_{e \in E(S)} c(e)$ be the *capacity* of the cut S.

The well-known *multicommodity flow problem* is to find flows $f_1, \dots, f_m$, where each f_i is a flow from s_i to t_i of value q_i, such that for each $e \in E$ the total flow through e does not exceed $c(e)$, or to establish that no such flows exist. By linear programming duality, a multicommodity flow exists if and only if

$$\sum q_i d_l(s_i, t_i) \le \sum_{e \in E} c(e) l(e), \qquad i = 1, \dots, m,$$

for any nonnegative real-valued length function l on E, and $d_l(s_i, t_i)$ denotes the distance between s_i and t_i in the graph G whose edges are

* On leave from the Universitatea de stat din Moldova, Chişinău.

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weighted by l [8, 15]. If a multicommodity flow exists then the following condition of Ford–Fulkerson type is verified:

$$c(S) \geq \sum_{i=1}^{m} q_i \qquad \text{for any cut } S = (A, B) \text{ with } s_i \in A, t_i \in B. \tag{1}$$

For what commodity graphs H is this necessary condition also sufficient? The answer was given by the following result of Papernov [17]:

if H is the complete graph K_4 with four vertices or the circuit C_5 with five vertices or a union of two stars and (1) *holds, then the multicommodity flow problem has a solution.*

This result generalizes many earlier known theorems on multicommodity flows established in [11, 12, 16, 18, 19].

Let $G = (V, E)$ be a complete graph the edges $e \in E$ of which have nonnegative real-valued lengths $l(e)$. Suppose that $d_l(x, y)$ denotes the distance between vertices x and y with respect to l; in other words d_l is the *metric closure* of l. Then (V, d_l) is a finite metric space. A sequence $u = x_0, x_1, \ldots, x_n, x_{n+1} = v$ of points of V is called a *shortest path* between the points u and v if $d_l(u, v) = \sum_{i=1}^{n} d_l(x_i, x_{i+1})$. We will say that l satisfies the *parity condition* if $l(u, v) + l(v, w) + l(w, u)$ is an even integer for any $u, v, w \in V$. Evidently, the parity condition is preserved while passing to d_l, and, moreover, all distances of (V, d_l) are integers because $d_l(u, v) + d_l(v, v) + d_l(v, u) = 2d_l(u, v)$ is an even integer.

Now we recall a cut packing problem which is dual to the multicommodity flow problem. Given a graph $H = (X, F)$ with $X \subseteq V$, a family $\{d_1, \ldots, d_m\}$ of metrics on V is called an *H-packing* for l (or (V, d_l)) [13, 14] if

$$d_l(x, y) \geq d_1(x, y) + \cdots + d_m(x, y) \qquad \text{for all } x, y \in V \tag{2}$$

and

$$d_l(s, t) = d_1(s, t) + \cdots + d_m(s, t) \qquad \text{for all } st \in F. \tag{3}$$

If $d_1, \ldots, d_m$ is an H-packing of l, and $u = x_0, x_1, \ldots, x_n, x_{n+1} = v$ is a shortest path between u and v with $uv \in F$, then necessarily

$$d_l(x_i, x_{i+1}) = d_1(x_i, x_{i+1}) + \cdots + d_m(x_i, x_{i+1})$$

for any $i = 0, \ldots, n$. If equality (3) holds, then we say that the metric d_l admits an additive decomposition $d_l = d_1 + \cdots + d_m$. The simplest building stones are the *cut* (*pseudo*-) *metrics* associated to cuts of the set V: for a cut $S = (A, B)$ of V define

$$\delta_S(x, y) = \begin{cases} 0 & \text{if } x, y \in A \text{ or } x, y \in B, \\ 1 & \text{otherwise, i.e., if } S \text{ separates } x \text{ and } y. \end{cases}$$

More generally, a metric d on V is called *Hamming* if for some $\lambda > 0$ and some cut metric δ_S we have $d = \lambda \delta_S$. An example of a metric not decomposable into a sum of cut metrics (or Hamming metrics) gives the standard graph-metric d' of the complete bipartite graph $K_{2,3}$. A 2, 3-*metric* d' on V is defined as follows: take a partition of V into five blocks, and consider each of them as a vertex of $K_{2,3}$. Put $d'(x, y) = 0$, if x and y belong to a common block, otherwise let $d'(x, y)$ be the distance in $K_{2,3}$ between the blocks containing x and y. Finally, if $d = \lambda d'$ for some positive λ, we will say that d is a *Hamming* 2, 3-*metric*.

Combining linear programming arguments with the result of Papernov one can obtain the following theorem (see [13]).

THEOREM A. *If H is K_4 or C_5 or a union of two stars, then there exists an H-packing for l consisting of Hamming metrics.*

As is noted in [13, 20], Theorem A implies the Papernov theorem. Karzanov [13] presented a stronger, "half-integral" version of this result.

THEOREM B. *If H is as in Theorem* A *and l satisfies the parity condition, then there exists an H-packing for l consisting of cut metrics.*

Karzanov's proof yields an $O(|V|^3)$ algorithm for finding an H-packing for l. A shorter (but nonconstructive) proof of Theorem B was given by Schrijver [20].

Let d be a metric on V. An *extremal graph* (*antipodal graph* in the terminology of [16]) of d is a graph $H = (X, F)$ with $X \subseteq V$ such that for any distinct $x, y \in V$ there is an edge $st \in F$ such that

$$d(s, x) + d(x, y) + d(y, t) = d(s, t);$$

see [13, 14]. A basic property of extremal graphs is that any shortest path between two points x, y of V can be extended to a shortest path between s, t of X with $st \in F$. As is shown in [1] from Theorems A and B one can derive the following result.

THEOREM C. *Let d be a metric on V whose extremal graph H is either K_4, or C_5, or a union of two stars. Then*

(i) *d is decomposable into a sum of Hamming metrics*;

(ii) *if, in addition, d satisfies the parity condition, then d is decomposable into a sum of cut metrics.*

In [14] Karzanov, continuing this line of research, established the following results.

THEOREM D. *If the length function l on V satisfies the parity condition and $H = (X, F)$ is a graph with $X \subseteq V$ and $|X| = 5$, then there exists an H-packing for l consisting of cut metrics and* $2, 3$*-metrics*.

THEOREM E. *Let d be a metric on V whose extremal graph H has five vertices. If d satisfies the parity condition, then d is decomposable into a sum of cut metrics and* $2, 3$*-metrics*.

In this note we present alternative algorithmic proofs of Theorems A–E. If the metric closure d_l of l is given, then one can find the corresponding H-packings in optimal $O(|V|^2)$ time.

2. TIGHT EXTENSIONS OF METRIC SPACES

Let $X := (X, d)$ be a metric space. The closed *ball* of center x and radius r will be $B(x, r)$. A metric space X is called *hyperconvex* if for any collection of closed balls in X, $B(x_i, r_i)$, $i \in I$, satisfying the condition that $d(x_i, x_j) \leq r_i + r_j$ for all $i, j \in I$, the intersection $\bigcap_{i \in I} B(x_i, r_i)$ is nonempty, i.e., the family of balls of X has the Helly property.

The notion of hyperconvex spaces has been introduced by Aronszajn and Panitchpakdi [1], who proved that a hyperconvex space is injective, i.e., is a retract of any metric space in which it is isometrically embedded (for additional information consult [2, 10]). To be more precise, here are the basic notions: a metric space (X, d) is *isometrically embedded* into a metric space (Y, d') if there is a map $h: X \to Y$ such that $d'(h(x), h(y)) = d(x, y)$ for all $x, y \in X$. In this case we say that X is a *subspace* of Y and that Y is an *extension* of X. Now, a *retraction* $h: Y \to X$ from a metric space (Y, d') to a subspace X is an idempotent $(h(x) = x$ for any $x \in X)$ nonexpansive $(d'(h(x), h(y)) \leq d'(x, y)$ for any $x, y \in Y)$ mapping; its image X is called a *retract* of Y. A metric space (X, d) is *injective* if X is a retract of every metric space in which X embeds isometrically.

THEOREM 1 [1]. *A metric space (X, d) is injective if and only if it is hyperconvex*.

Let $\mathbb{R}^X$ denote the set of all functions which map X into $\mathbb{R}$, endowed with the L_∞-metric

$$d(f, g) = \sup_{x \in X} |f(x) - g(x)|$$

for all elements f and g of $\mathbb{R}^X$. The resulting metric space $(\mathbb{R}^X, d)$ is a basic example of an injective space.

Isbell [9], Dress [6], and Chrobak and Larmore [5] independently established that every metric space (X, d) has a smallest containing injective space, which is compact if X is compact (in a more general framework a similar result was presented in [10]). Such a space is called the *injective envelope* by Isbell, the *convex hull* by Chrobak and Larmore, and the *tight extension* (notation T_X) by Dress. We will follow the terminology of [6], where a systematic treatment of this construction and its applications were given (for applications see also [5]). Although we need only a few elementary facts, mainly concerning the structure of T_X of small metric spaces, let us review some essential features of tight extensions.

An extension (Y, d) of a metric space X is called a *tight extension*, if for any map $\rho: Y \times Y \to \mathbb{R}$ satisfying the conditions

(i) $\rho(x, y) = \rho(y, x) \geq 0$ for all $x, y \in Y$;

(ii) $\rho(x, z) + \rho(z, y) \geq \rho(x, y)$ for all $x, y, z \in Y$;

(iii) $\rho(x_1, x_2) = d(x_1, x_2)$ for all $x_1, x_2 \in X$ and $\rho(y_1, y_2) \leq d(y_1, y_2)$ for all $y_1, y_2 \in Y$;

one has necessarily $\rho(y_1, y_2) = d(y_1, y_2)$ for all $y_1, y_2 \in Y$.

It has been shown in [6] that an extension (Y, d) of a metric space X is tight if and only if

$$d(y_1, y_2) = \sup\{d(x_1, x_2) - d(x_1, y_1) - d(x_2, y_2) : x_1, x_2 \in X\}$$

holds for all $y_1, y_2 \in Y$.

In case X is compact, one can find a uniquely determined smallest subset F_X of X, such that any tight extension of X is a tight extension of F_X. The following result shows that F_X coincides with the vertex set of the extremal graph of a metric space defined in the previous section.

THEOREM 2 [6]. *Let (Y, d) be a compact metric space and let X be a closed subspace of Y. Then the following conditions are equivalent*:

(i) *Y is a tight extension of X*;

(ii) *X contains the set F_X of all $x \in Y$ for which there exists some $y \in Y$ with $d(y, x) + d(x, z) > d(y, z)$ for all $z \in Y - \{x\}$.*

In particular, for any $y_1, y_2 \in Y$ there exist $x_1, x_2 \in F_X$ such that

$$d(x_1, x_2) = d(x_1, y_1) + d(y_2, y_2) + d(y_2, x_2)$$

and for any $y \in Y$ and $x \in F_X$ there is some $z \in F_X$ with $d(z, x) = d(z, y) + d(y, x)$.

For a metric space (X, d) let T_X denote the set of all $f \in \mathbb{R}^X$ satisfying

$$f(x) = \sup\{d(x, y) - f(y) : y \in X\}$$

for all $x \in X$. There is a canonical map, h_X, of the space (X, d) into T_X, which is given by $x \to h_x$, where the function h_x is defined by the formula

$$h_x(y) = d(x, y) \qquad \text{for all } y \in X.$$

From Theorem 3 of [6] it follows that T_X endowed with the L_∞-metric is a tight extension of X and the map h_X is an isometric embedding of X into T_X. It has been shown in [5, 6, 9] that T_X is the universal tight extension of X, i.e., it contains, up to canonical isometries, every tight extension of X, and it has no proper tight extension itself. On the other hand, from the proof of Theorem 2.1 of [9] it follows that T_X is the smallest injective extension of X, i.e., T_X is the injective hull of X.

Now, suppose that X is finite, say $|X| = n$. Then T_X can be isometrically embedded in $\mathbb{R}^n$ with the L_∞-metric and it consists of the finite union of a number of convex polyhedra of dimensions between 1 and $[n/2]$ [5, 6]. For our purposes we need the precise structure of T_X for small metric spaces ($n \leq 5$) only. T_X of metric spaces with at most four points has been described in [4–6] and T_X of metric spaces with five points was established in [4, 6]. Before we present these results, notice that in all these cases T_X is a union of a number of line segments, rectangles, or half-squares endowed with the rectilinear distance (due to the well-known fact that there is an isometry from the l_1-plane to the l_∞-plane).

For a cut $S = (A, B)$ of a metric space (X, d) define

$$\alpha_{A,B} = \tfrac{1}{2} \cdot \min_{\substack{a, a' \in A \\ b, b' \in B}} \big(\max\{d(a,b) + d(a',b'), d(a,b') + d(a',b), d(a,a') + d(b,b')\} - d(a,a') - d(b,b')\big).$$

According to [4], $\alpha_{A,B}$ is called the *isolation index* of the cut $S = (A, B)$. If $S = \{\{x\}, X - \{x\}\}$ we simply write α_x instead of $\alpha_{\{x\}, X-\{x\}}$. If d satisfies the parity condition, then all isolation indices of cuts are integers; cf. [3]. Indeed, for a cut $S = (A, B)$ and points $a, a' \in A$ and $b, b' \in B$,

$$\begin{aligned} d(a,b) + d(a',b') - d(a,a') - d(b,b') \\ = (d(a,a') + d(a,b') + d(a',b')) \\ + (d(a',b) + d(a',b') + d(b,b')) \\ - 2(d(a,a') + d(a',b') + d(b,b')), \end{aligned}$$

is an even integer. Hence all numbers over which this minimum is taken for $\alpha_{A,B}$ are integers, whence the isolation index of any cut is an integer. Now we are ready to describe T_X for $|X| \le 5$, actually reproducing the results from [4–6].

If $|X| = 2$, T_X is a line segment, with two points of X at the ends.

If $|X| = 3$, say $X = \{x, y, z\}$, T_X consists of three line segments joined at a point, with the points of X at the ends of the arms. The lengths of these segments are α_x, α_y, and α_z, respectively (see Fig. 1). The metric d defined on X can be expressed in the form

$$d = \alpha_x \delta_{\{x\},\{y,z\}} + \alpha_y \delta_{\{y\},\{x,z\}} + \alpha_z \delta_{\{z\},\{x,y\}}.$$

In consequence, T_X isometrically embeds in the l_1-plane.

If $|X| = 4$, say $X = \{u, v, x, y\}$, T_X consists of a rectangle with the rectilinear metric, together with a line segment attached by one end to each corner. The points of X are the outer ends of these segments, whose lengths are α_u, α_v, α_x, and α_y, respectively. If

$$\max\{d(u,v) + d(x,y), d(u,x) + d(v,y), d(u,y) + d(v,x)\} = d(u,v) + d(x,y),$$

then the sides of the rectangle are given by the isolation indices $\alpha_{\{u,x\},\{y,v\}}$ and $\alpha_{\{u,y\},\{v,x\}}$ (see Fig. 2); for details consult [4–6]. Again, d decomposes into a sum of Hamming metrics

$$d = \alpha_u \delta_{\{u\},\{v,x,y\}} + \alpha_v \delta_{\{v\},\{u,x,y\}} + \alpha_x \delta_{\{x\},\{u,v,y\}} + \alpha_y \delta_{\{y\},\{u,v,x\}} + \alpha_{\{u,x\},\{v,y\}} \delta_{\{u,x\},\{v,y\}} + \alpha_{\{u,y\},\{v,x\}} \delta_{\{u,y\},\{v,x\}},$$

and T_X embeds in the l_1-plane.

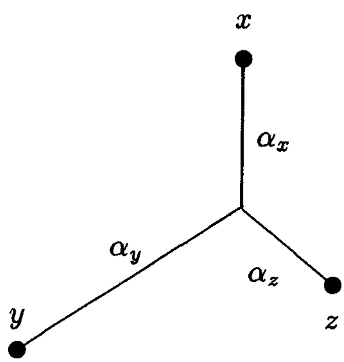

FIG. 1. T_X of three points $X = \{x, y, z\}$.

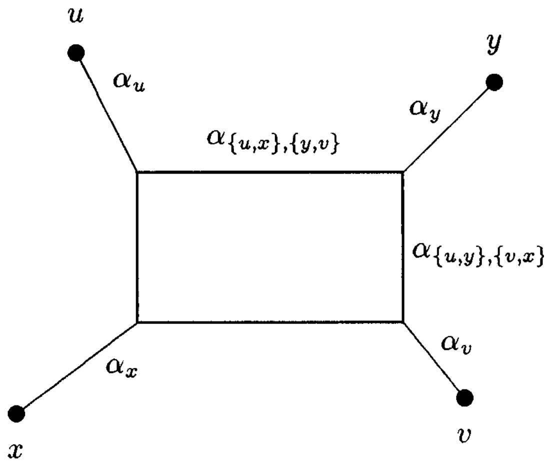

FIG. 2. T_X of four points $X = \{u, v, x, y\}$.

Finally, if X has cardinality five, there are three "generic" types of metrics defined on X. The corresponding spaces T_X taken from [4, 6] are shown in Figs. 3–5.

TYPE I. For $X = \{x_0, x_1, x_2, x_3, x_4\}$ put

$$d = \sum_{i=0}^{4} \gamma_i \delta_{\{x_i\}, X-\{x_i\}} + \sum_{i=0}^{4} \beta_i \delta_{\{x_i, x_{i+1}\}, X-\{x_i, x_{i+1}\}}$$

(indices modulo 5), where $\gamma_i = \alpha_{x_i}$ and $\beta_i = \alpha_{\{x_i, x_{i+1}\}, X-\{x_i, x_{i+1}\}}$. As is shown in Fig. 3, T_X consists of five rectangles glued together to form a "star" and five line segments attached by one end to each corner of the star. In this case T_X isometrically embeds in $\mathbb{R}^3$ endowed with the l_1-metric.

TYPE II. For $X = \{z_1, z_2, y_1, y_2, y_3\}$ let

$$d = \sum_{i=1}^{2} \gamma_i \delta_{\{z_i\}, X-\{z_i\}} + \sum_{i=1}^{3} \eta_i \delta_{\{y_i\}, X-\{y_i\}} + \beta_1 \delta_{\{y_1, z_1\}, X-\{y_2, z_1\}}$$
$$+ \beta_2 \delta_{\{y_1, z_2\}, X-\{y_1, z_2\}} + \beta_3 \delta_{\{y_2, z_2\}, X-\{y_2, z_2\}} + \beta_4 \delta_{\{y_2, z_1\}, X-\{y_2, z_1\}} + \alpha d',$$

where γ_1, γ_2, η_1, η_2, η_3, β_0, β_1, β_2, β_3, and β_4 are the isolation indices of the respective cuts and d' is the 2, 3-metric defined by

$$d'(z_1, z_2) = d'(y_i, y_j) = 2 \qquad (1 \le i < j \le 3)$$
$$d'(z_i, y_j) = 1 \qquad (i = 1, 2;\ j = 1, 2, 3).$$

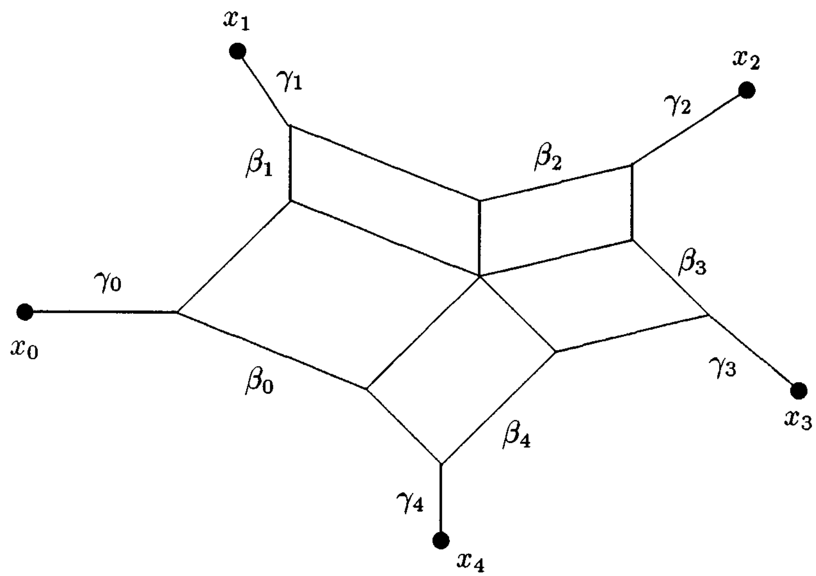

FIG. 3. T_X of five points $X = \{x_0, x_1, x_2, x_3, x_4\}$: type I.

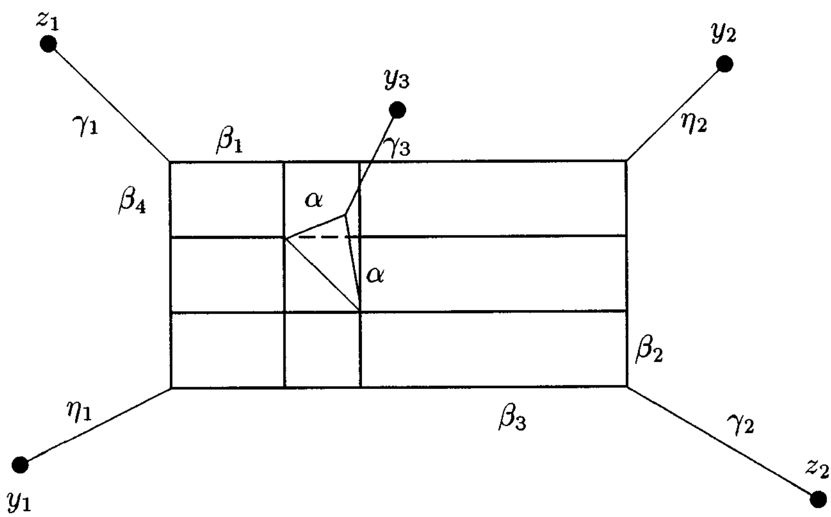

FIG. 4. T_X of five points $X = \{z_1, z_2, y_1, y_2, y_3\}$: type II.

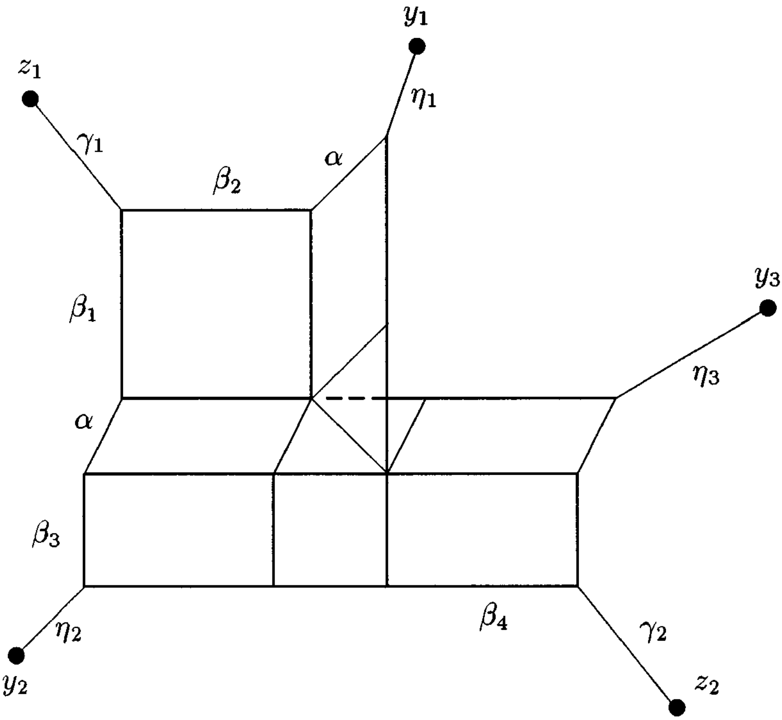

FIG. 5. T_X of five points $X = \{z_1, z_2, y_1, y_2, y_3\}$: type III.

TYPE III. The labels and parameters are as in type II, but now

$$d = \sum_{i=1}^{2} \gamma_i \delta_{\{z_i\}, X-\{z_i\}} + \sum_{i=1}^{3} \eta_i \delta_{\{y_i\}, X-\{y_i\}} + \beta_1 \delta_{\{y_1, z_1\}, X-\{y_1, z_1\}}$$
$$+ \beta_2 \delta_{\{y_2, z_1\}, X-\{y_2, z_1\}} + \beta_3 \delta_{\{y_2, z_2\}, X-\{y_2, z_2\}} + \beta_4 \delta_{\{y_3, z_2\}, X-\{y_3, z_2\}} + \alpha d'.$$

Elementary cells of T_X, $|X| \leq 5$ will be called the pendant line segments, the full rectangles, or the triplets of identical triangles glued together along their common diagonal to form a solid $K_{2,3}$ (for them we will use the short-name $K_{2,3}$-*cell*). We will end this section by stating some useful properties of the space T_X. A straightforward verification shows that every elementary cell is a gated set of T_X. Recall that according to [7] a subset M of a metric space (T_x, d) is *gated*, if for any point $y \notin M$ there exists a (unique) point $g_y \in M$ (the *gate* for y in M) such that $d(y, z) = d(y, g_y) + d(g_y, z)$ for all $z \in M$. This shows how given a point $x \in T_X$ and a radius $r > 0$ to construct the ball $B(x, r)$. First, we find the gate g_x of x

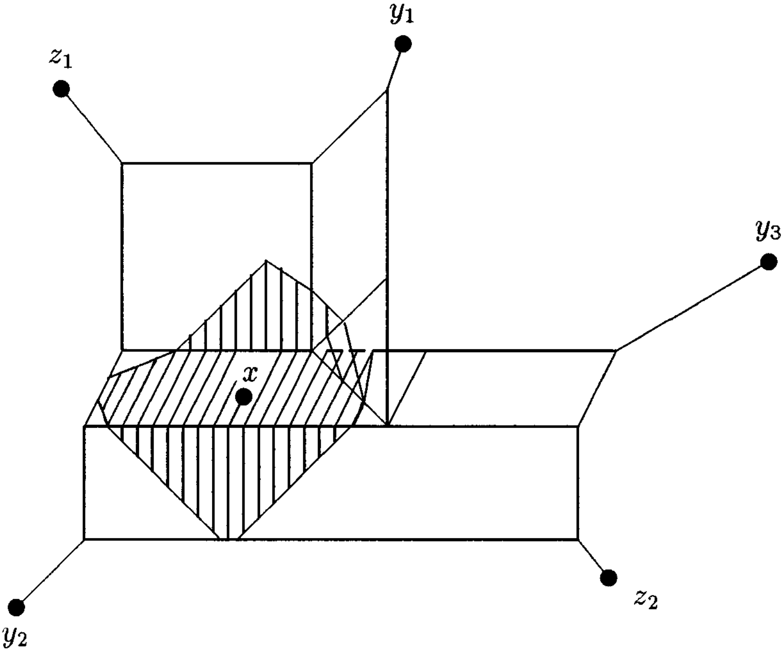

FIG. 6. The ball $B(x, r)$.

on each elementary cell C of T_X. Then $B(x, r) \cap C$ coincides with $B(g_x, r - d(x, g_x)) \cap C$. The latter intersection can be easily found, because on all rectangular cells and triangles of $K_{2,3}$-cells the metric d is of l_1-type. Therefore, we can perform the whole construction of $B(x, r)$ in constant time $O(1)$; for an illustration see Fig. 6.

Due to the specific form of balls, we can solve the following problem in only $O(m)$ time: find the intersection B of m balls $B(x_1, r_1), \ldots, B(x_m, r_m)$. Indeed, it suffices to compute this intersection B_R inside each rectangular cell R or each triangle of a $K_{2,3}$-cell R. To find B_R we first compute the intersection of balls of radii $r_i - d(x_i, g_{x_i})$ centered at g_{x_i} in the whole plane of R, and then intersect the obtained figure with R.

If d satisfies the parity condition, then the lengths of edges of elementary cells of T_X are integers, because each of them is an isolation index of a certain cut of X. Therefore, one can identify every rectangular cell R of T_X with a rectangle R' of $\mathbb{R}^2$ whose all edges are axis-parallel and all corners are vertices of the grid $\mathbb{Z}^2$. It will be convenient to call *integer points* all points of R whose images in R' belong to $\mathbb{Z}^2$. Similarly, we can define the integer points of triangles of $K_{2,3}$-cells. The gates of an integer point on elementary cells of T_X are integer, too. Now, suppose that

$x_1, \ldots, x_m$ are integer points of T_X and $r_1, \ldots, r_m \in \mathbb{Z}^+$. One can easily show that in this case the set $B = \bigcap_{i=1}^m B(x_i, r_i)$ contains integer points (actually, the boundary segments of every nonempty set of the type B_R have such points). Therefore, with B in hands we can find at least one its integer point in only constant time.

2. PROOFS OF THEOREMS A–E

Let $G = (V, E)$ be a complete graph the edges $e \in E$ of which have nonnegative lengths $l(e)$, and let $H = (X, F)$ be a graph with $X \subseteq V$. From now on $d := d_l$ will denote the metric closure of l.

Karzanov [13] outlined a simple way to reduce the case when H is a union of two stars S_1 and S_2 to that when H is K_4. Let S_1 contain the edges pp_i, $i = 1, \ldots, r$ and S_2 contains the edges qq_j, $j = 1, \ldots, t$. Put

$$\delta_1 = \max\{d(p, x) : x \in V\}, \qquad \delta_2 = \max\{d(q, x) : x \in V\},$$

and $\delta = \delta_1 + \delta_2$. Add two new points p' and q' to V and denote by V' the resulting set. Let H' be the complete graph K_4 with the vertices p, q, p', q'. Extend d to a metric d' on V' letting $d'(p', x) = \delta - d(p, x)$, $d'(q', x) = \delta - d(q, x)$, for any $x \in V$, and $d'(p', q') = 2\delta - d(p, q)$. Then $d'(p, p') = d'(q, q') = \delta$, and, moreover, if a sequence $p, \ldots, p_i$ (respectively, $q, \ldots, q_j$) is a shortest path of (V, d), then $p, \ldots, p_i, p'$ (respectively, $q, \ldots, q_j, q'$) is a shortest path of (V', d'). In particular, if H is the extremal graph of (V, d), then H' will be the extremal graph of the new metric space (V', d'). Finally, if d satisfies the parity condition, then d' satisfies it as well. Now, assume that there exists an H'-packing $d_1, \ldots, d_m$ of d' consisting of Hamming metrics (respectively, cut metrics, if d' fulfills the parity condition). Since p, p_i, p' and q, q_j, q' are shortest paths of (V', d'), as we already noted

$$d'(p, p_i) = \sum_{k=1}^{m} d_k(p, p_i) \qquad \text{and} \qquad d'(q, q_j) = \sum_{k=1}^{m} d_k(q, q_j).$$

Taking the restriction of each d_k, $k = 1, \ldots, m$, on V we will get the required H-packing of d consisting of Hamming (respectively, cut) metrics. Thus, it suffices to establish the validity of Theorems A, B, and C only for $H = K_4$ and $H = C_5$. Therefore, in all cases to be considered the graph $H = (X, F)$ has at most five vertices.

Let Y be the union of the sets V and T_X glued together along their common subspace X. Define the distance $d(x, y)$ between two points of Y as the length of the shortest path joining them. Since on X the metric closure of the length function l and the injective metric of T_X coincide, we conclude that both sets V and T_X endowed with their own metrics are isometric subspaces of the metric space (Y, d). From the definition of T_X and the results of Section 2 it follows that there is a retraction from Y to T_X. We construct a retraction map step by step, starting with the identity map h acting on T_X. At each step we extend h to a larger subset of V, finding an image in T_X of a new point from $V - X$. Namely, let $V = \{v_1, \dots, v_n\}$ and suppose that h has been defined on a subset $V' = \{v_1, v_2, \dots, v_{k-1}\}$ of V containing X. Set $w_j = h(v_j)$, $j = 1, \dots, k-1$. Pick a point $v_k \in V - V'$, and for any point $v_j \in V'$ put $r_j = d(v_j, v_k)$. By the triangle inequality and because h is non-expansive on V', we conclude that

$$d(w_i, w_j) \leq d(v_i, v_j) \leq r_i + r_j$$

for any $i, j \in \{1, \dots, k-1\}$. Consider the balls $B(w_j, r_j)$, $j \leq k-1$. Since T_X is hyperconvex, these balls intersect. Take as $w_k := h(v_k)$ any point of $\bigcap_{j=1}^{k-1} B(w_j, r_j)$. Evidently, this iterative procedure provides a non-expansive map h from V to T_X. Therefore, an H-packing of d restricted to the set $W = \{w_1, \dots, w_n\}$ can be easily transformed into an H-packing of d on the initial set V.

The properties of T_X stated at the end of Section 2 point the way how to construct the balls $B(w_j, r_j)$, $j = 1, \dots, k-1$, and to select a new point $w_k \in \bigcap_{j=1}^{k-1} B(w_j, r_j)$, $k = 1, \dots, n$, in total $O(n^2)$ time. In addition, if d obeys the parity condition, then within the same time bounds we can select all w_k, $k = 1, \dots, n$ among integer points of T_X.

Pick a point $w_i \in W$ in a rectangular cell R. Consider two segments which pass through w_i and are translates of the edges of R. If such a segment intersects an edge of a rectangular cell R' incident to R, then extend it in the same way to a maximal chain whose endpoints belong to the boundary of T_X. Transform T_X into a grid Γ by taking all such chains, analogous chains formed by the edges of the rectangular cells, and the points of W located on the pendant edges of T_X; for an illustration see Fig. 7. To construct Γ we have to sort the coordinates of the points of W inside each rectangular cell or pendant edge of T_X. By a strip of T_X we mean an area of T_X comprised between two consecutive nonintersecting chains and which does not intersect the $K_{2,3}$-cell. This notion extends in an evident fashion to the case of pendant edges of T_X. Suppose now that T_X has m strips $\mathsf{S}_1, \dots, \mathsf{S}_m$, whose widths are the numbers $\lambda_1, \dots, \lambda_m$. Notice here that if W consists of integer points only, then the widths of all strips must be integral. Each strip S_i defines a cut $S_i = (A_i, B_i)$ of W (and of the initial set V, of course).

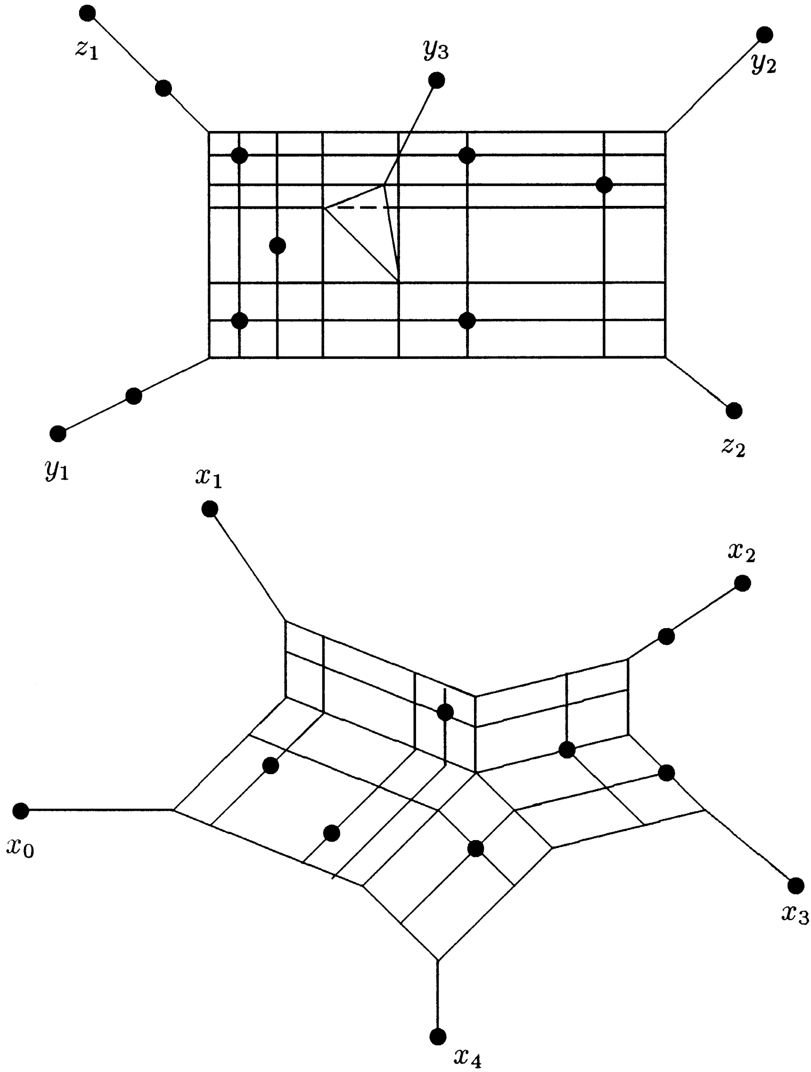

FIG. 7. Two examples of the grid Γ.

Let d_0 be a metric on W obtained by summing up the Hamming metrics $\lambda_i \delta_{S_i}$, $i = 1, \dots, m$, i.e.,

$$d_0 = \lambda_1 \delta_{S_1} + \cdots + \lambda_m \delta_{S_m}.$$

If T_X does not contain a $K_{2,3}$-cell, then one can easily show that d and d_0 coincide, giving us the desired H-packing of d on W (and V) consisting of

Hamming metrics. If, in addition, d satisfies the parity condition, then each λ_i, $i = 1, \ldots, m$, is an integer, i.e., we will have an H-packing of d consisting of cut metrics. This settles the case $H = K_4$ in Theorems A, B, and C. If $H = (X, F)$ is the extremal graph of the metric d on V, then the mapping h will be an isometry. If $H = C_5$, then T_X of Type II or III cannot occur, because in these cases the vertices y_1, y_2, and y_3 will be pairwise adjacent in H. This concludes the proof of Theorem C.

Now, suppose that T_X contains a $K_{2,3}$-cell C consisting of three congruent triangles T_1, T_2, T_3. For each point x of X, let R_x be the union of the pendant edge of T_X containing x and of the rectangular cell sharing a common vertex with this edge. Replace each point of W by its gate in C. We prefer to use the same symbol w_i for the gate of w_i in C. The unique common vertex of R_x and C will be the gate of every point of $W \cap R_x$. For convenience, we will denote it also by x. Then each distance between two points w_i and w_j of W decreases by the value $d_0(w_i, w_j)$. Therefore, it suffices to find an H-packing of $d - d_0$ defined on the new set W.

We are ready, finally, to complete the proof of Theorems A and B. Let $H = C_5$, and suppose, without loss of generality, that the vertices y_2 and y_3 are nonadjacent in H. Identify the triangles T_2 and T_3 as is shown in Fig. 8. This mapping is nonexpansive. Namely, it preserves the distances between points from the same triangle T_i, $i = 1, 2, 3$, or from a point in T_1 and another one in $T_2 \cup T_3$. All other distances decrease. Transform the rectangle $R = T_1 \cup T_2$ into a rectilinear grid by taking all vertical and horizontal lines passing through the images of points of W. Again, the strips $\mathcal{S}_{m+1}, \ldots, \mathcal{S}_{m+p}$ define the cuts $S_{m+1} = (A_{m+1}, B_{m+1}), \ldots, S_{m+p} = (A_{m+p}, B_{m+p})$ of W (and V). If $\lambda_{m+1}, \ldots, \lambda_{m+p}$ are the widths of these strips, then

$$\lambda_{m+1}\delta_{S_{m+1}} + \cdots + \lambda_{m+p}\delta_{S_{m+p}}$$

is an H-packing of $d - d_0$ consisting of Hamming metrics (or cut metrics, if d fulfills the parity condition). The cuts which take part in this decomposition can be found in total $O(n^2)$ time in a straightforward way. This finishes the proof of Theorems A and B.

Finally, suppose that we are in the conditions of Theorems D or E. To construct the required 2, 3-metrics we identify the triangles T_1, T_2 and T_3. Consider a rectilinear grid within the resulting triangle (recall, it represents a half-square) by taking all vertical and horizontal lines passing through the images of points of W as is sketched in Fig. 9. We copy the obtained grid in all three triangles of C. Then T_1, T_2, T_3 are subdivided into a collection of rectangles and half-squares, latter being arranged along the common edge of these triangles. The triplets $C_{m+1}, \ldots, C_{m+p}$ of identical half-squares of sizes $\lambda_{m+1}, \ldots, \lambda_{m+p}$ taken from distinct triangles

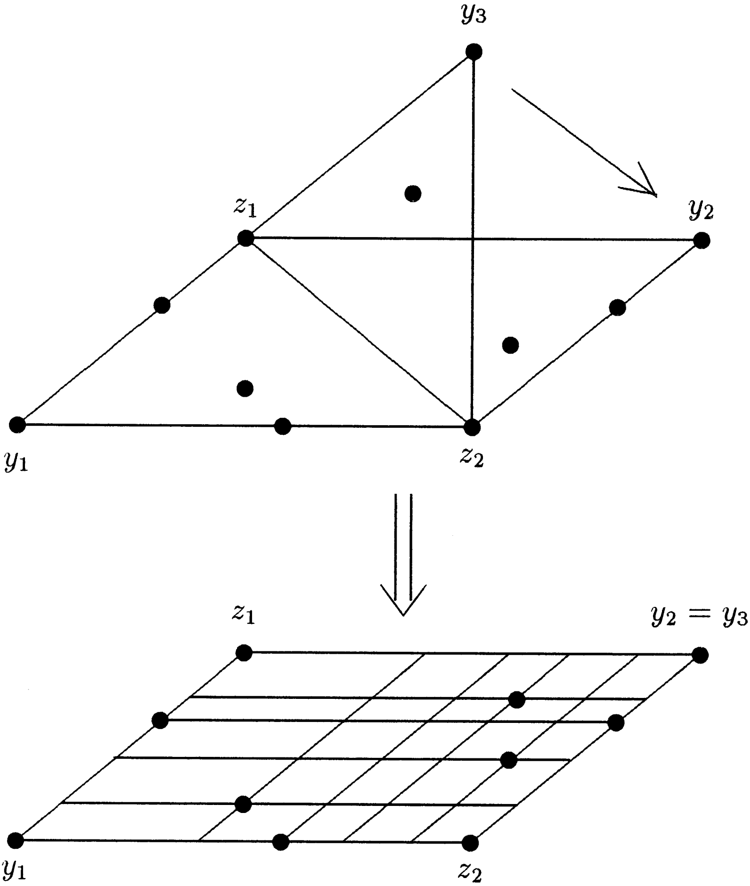

FIG. 8. An illustration to the proof of Theorems A and B.

define a collection of 2, 3-metrics $d'_{m+1}, \ldots, d'_{m+p}$ on W. Namely, the gate in C_{m+j} of every point of W is a vertex of the bounding $K_{2,3}$-graph, this giving us the blocks of the 2, 3-metric d'_{m+j}, $j = 1, \ldots, p$. One can easily show that

$$\lambda_{m+1} d'_{m+1} + \cdots + \lambda_{m+p} d'_{m+p}$$

represents a decomposition of $d - d_0$ into a sum of Hamming 2, 3-metrics. If d satisfies the parity condition, then our preceding discussion yields that $\lambda_{m+1}, \ldots, \lambda_{m+p}$ are integers, concluding the proof of Theorems D and E. Again the H-packing of $d - d_0$ can be computed in $O(n^2)$ total time. We conclude with the following variant of Theorems A and D.

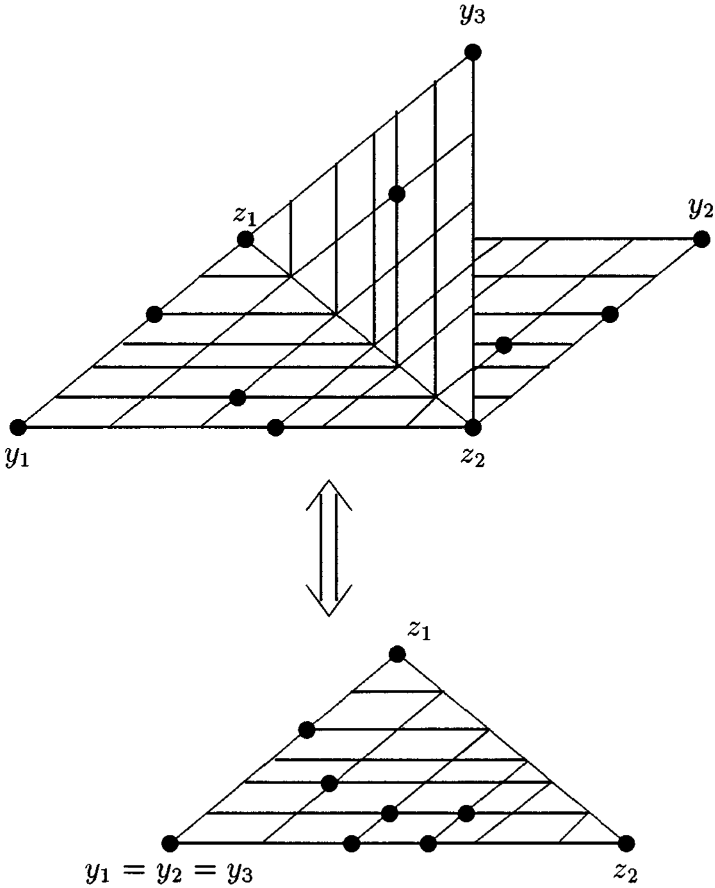

FIG. 9. Construction of the grid in Theorems D and E.

THEOREM D′. *If the graph $H = (X, F)$ has at most five vertices, then there exists an H-packing for l consisting of Hamming metrics and Hamming* 2, 3-*metrics*.

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REFERENCES

1. N. Aronszajn and P. Panitchpakdi, Extensions of uniformly continuous transformations and hyperconvex metric spaces, *Pacific J. Math.* **6** (1956), 405–439.
2. J.-B. Baillon, Nonexpansive mappings and hyperconvex spaces, *in* "Contemporary Mathematics," Vol. 72, pp. 11–19, Amer. Math. Soc., Providence, RI, 1988.
3. H.-J. Bandelt and V. Chepoi, Embedding into the rectilinear grid, submitted for publication.
4. H.-J. Bandelt and A. W. M. Dress, A canonical decomposition theory for metrics on a finite set, *Adv. Math.* **92** (1992), 47–105.
5. M. Chrobak and L. L. Larmore, Generosity helps or an 11-competitive algorithm for three servers, *J. Algorithms* **16** (1994), 234–263.
6. A. W. M. Dress, Trees, tight extensions of metric spaces, and the cohomological dimension of certain groups: a note on combinatorial properties of metric spaces, *Adv. Math.* **5** (1984), 321–402.
7. A. W. M. Dress and R. Scharlau, Gated sets in metric spaces, *Aequationes Math.* **34** (1987), 112–120.
8. M. Iri, On an extension of the maximum-flow minimum-cut theorem to multicommodity flows, *J. Oper. Res. Soc. Japan* **13** (1970/1971), 129–135.
9. J. Isbell, Six theorems about metric spaces, *Comment. Math. Helv.* **39** (1964), 65–74.
10. E. Jawhari, D. Misane, and M. Pouzet, Retracts: Graphs and ordered sets from the metric point of view, *in* "Contemporary Mathematics," Vol. 57, pp. 175–225, Amer. Math. Soc., Providence, RI, 1986.
11. L. R. Ford and D. R. Fulkerson, Maximal flow through a network, *Canad. J. Math.* **8** (1956), 399–404.
12. T. C. Hu, Multicommodity network flows, *Oper. Res.* **11** (1963), 344–360.
13. A. V. Karzanov, Metrics and undirected cuts, *Math. Programm.* **32** (1985), 183–198.
14. A. V. Karzanov, Sums of cuts and bipartite metrics, *Europ. J. Combin.* **11** (1990), 473–484.
15. O. Kenji and K. Osamu, On feasibility conditions on multicommodity flows in networks, *IEEE Trans. Circuit Theory* **CT-18** (1971), 425–429.
16. M. V. Lomonosov, Combinatorial approaches to multiflow problems, *Discrete Appl. Math.* **11** (1985), 1–94.
17. B. A. Papernov, On existence of multicommodity flows, *in* "Studies Discrete Optimization," (A. A. Fridman, Ed.), Nauka, Moscow, 1976. [in Russian]
18. B. Rothschild and A. Whinston, On two commodity network flows, *Oper. Res.* **14** (1966), 377–387.
19. P. D. Seymour, Four-terminus flows, *Networks* **10** (1980), 79–86.
20. A. Schrijver, Short proofs on multicommodity flows and cuts, *J. Combin. Theory Ser. B* **53** (1991), 32–39.