Journal of Computational and Applied Mathematics 235 (2011) 2137–2146

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Stability of Pexiderized quadratic functional equation in intuitionistic fuzzy normed space

S.A. Mohiuddine [a,*], H. Şevli [b]

[a] Department of Mathematics, Faculty of Science, Jazan University, Jazan 45142, Saudi Arabia

[b] Department of Mathematics, Istanbul Commerce University, Uskudar, Istanbul, Turkey

ARTICLE INFO

Article history:
Received 15 September 2009

Keywords:
t-norm
t-conorm
Intuitionistic fuzzy normed space
Pexiderized quadratic functional equation
Hyers–Ulam–Rassias stability

ABSTRACT

The object of the present paper is to determine the stability of the Hyers–Ulam–Rassias type theorem concerning the Pexiderized quadratic functional equation in intuitionistic fuzzy normed spaces (IFNS).

1. Introduction and preliminaries

Stability problem of a functional equation was first posed in [1] which was answered in [2] and then generalized in [3,4] for additive mappings and linear mappings respectively. Since then several stability problems for various functional equations have been investigated in [5–8,4]. Recently, fuzzy version is discussed in [9,10]. Quite recently, the stability problem for Jensen functional equation and cubic functional equation is considered in [11,12] respectively in the intuitionistic fuzzy normed spaces; while the idea of intuitionistic fuzzy normed space was introduced in [13] and further studied in [14–17] to deal with some summability problems.

Several results for the Hyers–Ulam–Rassias stability of many functional equations have been proved by several researchers. Our goal is to determine some stability results concerning the Pexiderized quadratic functional equation in intuitionistic fuzzy normed spaces. Here, first we examine the stability for odd and even functions and then we apply our results to a general function.

In this section we recall some notations and basic definitions used in this paper.

Definition 1.1. A binary operation $* : [0, 1] \times [0, 1] \to [0, 1]$ is said to be a *continuous t-norm* if it satisfies the following conditions:

(a) $*$ is associative and commutative, (b) $*$ is continuous, (c) $a * 1 = a$ for all $a \in [0, 1]$, (d) $a * b \le c * d$ whenever $a \le c$ and $b \le d$ for each $a, b, c, d \in [0, 1]$.

Definition 1.2. A binary operation $\diamond : [0, 1] \times [0, 1] \to [0, 1]$ is said to be a *continuous t-conorm* if it satisfies the following conditions:

(a′) $\diamond$ is associative and commutative, (b′) $\diamond$ is continuous, (c′) $a \diamond 0 = a$ for all $a \in [0, 1]$, (d′) $a \diamond b \le c \diamond d$ whenever $a \le c$ and $b \le d$ for each $a, b, c, d \in [0, 1]$.

* Corresponding author.
E-mail addresses: mohiuddine@gmail.com (S.A. Mohiuddine), hsevli@yahoo.com (H. Şevli).

0377-0427/$ – see front matter
doi:10.1016/j.cam.2010.10.010

Using the notions of continuous t-norm and t-conorm, Saadati and Park [13] have recently introduced the concept of intuitionistic fuzzy normed space as follows:

Definition 1.3. The five-tuple $(X, \mu, \nu, *, \diamond)$ is said to be an *intuitionistic fuzzy normed spaces* (for short, IFNS) if X is a vector space, $*$ is a continuous t-norm, $\diamond$ is a continuous t-conorm, and μ, ν are fuzzy sets on $X \times (0, \infty)$ satisfying the following conditions. For every $x, y \in X$ and $s, t > 0$

(i) $\mu(x, t) + \nu(x, t) \leq 1$, (ii) $\mu(x, t) > 0$, (iii) $\mu(x, t) = 1$ if and only if $x = 0$, (iv) $\mu(\alpha x, t) = \mu(x, \frac{t}{|\alpha|})$ for each $\alpha \neq 0$, $(v)\mu(x, t) * \mu(y, s) \leq \mu(x + y, t + s)$, (vi) $\mu(x, \cdot) : (0, \infty) \to [0, 1]$ is continuous, (vii) $\lim_{t\to\infty} \mu(x, t) = 1$ and $\lim_{t\to 0} \mu(x, t) = 0$, (viii) $\nu(x, t) < 1$, (ix) $\nu(x, t) = 0$ if and only if $x = 0$, $(x)\nu(\alpha x, t) = \nu(x, \frac{t}{|\alpha|})$ for each $\alpha \neq 0$, (xi) $\nu(x, t)\diamond\nu(y, s) \geq \nu(x + y, t + s)$, (xii) $\nu(x, \cdot) : (0, \infty) \to [0, 1]$ is continuous, (xiii) $\lim_{t\to\infty} \nu(x, t) = 0$ and $\lim_{t\to 0} \nu(x, t) = 1$.

In this case (μ, ν) is called an *intuitionistic fuzzy norm*.

Example 1.1. Let $(X, \|.\|)$ be a normed space and let $a * b = ab$ and $a\diamond b = \min\{a + b, 1\}$ for all $a, b \in [0, 1]$. For all $x \in X$ and every $t > 0$, consider

$$\mu(x, t) = \begin{cases} 0 & \text{if } t \leq \|x\|; \\ 1 & \text{if } t > \|x\|; \end{cases} \quad \text{and} \quad \nu(x, t) = \begin{cases} 1 & \text{if } t \leq \|x\|; \\ 0 & \text{if } t > \|x\|. \end{cases}$$

Then $(X, \mu, \nu, *, \diamond)$ is an intuitionistic fuzzy normed space.

Example 1.2. Let $(X, \|.\|)$ be a normed space, $a * b = ab$ and $a\diamond b = \min\{a + b, 1\}$ for all $a, b \in [0, 1]$. For all $x \in X$, every $t > 0$ and $k = 1, 2$, consider

$$\mu_k(x, t) = \begin{cases} \dfrac{t}{t + k\|x\|} & \text{if } t > 0 \\ 0 & \text{if } t \leq 0; \end{cases} \quad \text{and} \quad \nu_k(x, t) = \begin{cases} \dfrac{k\|x\|}{t + k\|x\|} & \text{if } t > 0 \\ 1 & \text{if } t \leq 0. \end{cases}$$

Then $(X, \mu, \nu, *, \diamond)$ is an intuitionistic fuzzy normed space.

The concepts of convergence and Cauchy sequences in an intuitionistic fuzzy normed space are studied in [13].

Let $(X, \mu, \nu, *, \diamond)$ be an IFNS. Then, a sequence $x = (x_k)$ is said to be *intuitionistic fuzzy convergent* to $L \in X$ if $\lim \mu(x_k - L, t) = 1$ and $\lim \nu(x_k - L, t) = 0$ for all $t > 0$. In this case we write (μ, ν)-$\lim x = L$ or $x_k \xrightarrow{IF} L$ as $k \to \infty$.

Let $(X, \mu, \nu, *, \diamond)$ be an IFNS. Then, $x = (x_k)$ is said to be *intuitionistic fuzzy Cauchy sequence* if $\lim \mu(x_{k+p} - x_k, t) = 1$ and $\lim \nu(x_{k+p} - x_k, t) = 0$ for all $t > 0$ and $p = 1, 2, \ldots$.

Let $(X, \mu, \nu, *, \diamond)$ be an IFNS. Then $(X, \mu, \nu, *, \diamond)$ is said to be *complete* if every intuitionistic fuzzy Cauchy sequence in $(X, \mu, \nu, *, \diamond)$ is intuitionistic fuzzy convergent in $(X, \mu, \nu, *, \diamond)$.

2. Stability of Pexiderized quadratic functional equation in IFNS

The functional equation

$$f(x + y) + f(x - y) = 2g(x) + 2h(y)$$

is said to be a *Pexiderized quadratic functional equation*. In the case $f = g = h$, it is called the *quadratic functional equation*.

We begin with a stability of Hyers–Ulam–Rassias type theorems in IFNS for the Pexiderized quadratic functional equation.

Theorem 2.1. *Let X be a linear space and let (Z, μ', ν') be an intuitionistic fuzzy normed space. Let $\varphi : X \times X \to Z$ be a function such that*

$$\varphi(2x, 2y) = \alpha\varphi(x, y), \tag{2.1.1}$$

for some real number α with $0 < |\alpha| < 2$. Let (Y, μ, ν) be intuitionistic fuzzy Banach space and f, g and h are odd functions from X to Y such that

$$\left.\begin{aligned} &\mu(f(x + y) + f(x - y) - 2g(x) - 2h(y), t) \geq \mu'(\varphi(x, y), t) \quad \textit{and} \\ &\nu(f(x + y) + f(x - y) - 2g(x) - 2h(y), t) \leq \nu'(\varphi(x, y), t), \end{aligned}\right\} \tag{2.1.2}$$

for all $x, y \in X$ and $t > 0$. Then there is a unique additive mapping $T : X \to X$ such that

$$\left.\begin{aligned}&\mu(f(x) - T(x), t) \geq \mu''\left(x, \frac{2-|\alpha|}{4}t\right) \quad \text{and}\\ &\nu(f(x) - T(x), t) \leq \nu''\left(x, \frac{2-|\alpha|}{4}t\right);\\ &\mu(g(x) + h(x) - T(x), t) \geq \mu''\left(x, \frac{6-3|\alpha|}{14-|\alpha|}t\right) \quad \text{and}\\ &\nu(g(x) + h(x) - T(x), t) \leq \nu''\left(x, \frac{6-3|\alpha|}{14-|\alpha|}t\right),\end{aligned}\right\} \tag{2.1.3}$$

where

$$\left.\begin{aligned}&\mu''(x, t) = \mu'(\varphi(x, x), t/3) * \mu'(\varphi(x, 0), t/3) * \mu'(\varphi(0, x), t/3) \quad \text{and}\\ &\nu''(x, t) = \nu'(\varphi(x, x), t/3) \Diamond \nu'(\varphi(x, 0), t/3) \Diamond \nu'(\varphi(0, x), t/3)\end{aligned}\right\}. \tag{2.1.4}$$

Proof. Replacing x by y and y by x in (2.1.2), we get

$$\left.\begin{aligned}&\mu(f(x+y) - f(x-y) - 2g(y) - 2h(x), t) \geq \mu'(\varphi(y, x), t) \quad \text{and}\\ &\nu(f(x+y) - f(x-y) - 2g(y) - 2h(x), t) \leq \nu'(\varphi(y, x), t)\end{aligned}\right\}. \tag{2.1.5}$$

It follows from (2.1.2) and (2.1.5) that

$$\left.\begin{aligned}&\mu(f(x+y) - g(x) - h(y) - g(y) - h(x), t) \geq \mu'(\varphi(x, y), t) * \mu'(\varphi(y, x), t) \quad \text{and}\\ &\nu(f(x+y) - g(x) - h(y) - g(y) - h(x), t) \leq \nu'(\varphi(x, y), t) \Diamond \nu'(\varphi(y, x), t)\end{aligned}\right\}. \tag{2.1.6}$$

Put $y = 0$ in (2.1.6), we get

$$\left.\begin{aligned}&\mu(f(x) - g(x) - h(x), t) \geq \mu'(\varphi(x, 0), t) * \mu'(\varphi(0, x), t) \quad \text{and}\\ &\nu(f(x) - g(x) - h(x), t) \leq \nu'(\varphi(x, 0), t) \Diamond \nu'(\varphi(0, x), t)\end{aligned}\right\}. \tag{2.1.7}$$

From (2.1.6) and (2.1.7), we conclude that

$$\left.\begin{aligned}&\mu(f(x+y) - f(x) - f(y), 3t) \geq \mu'(\varphi(x, y), t) * \mu'(\varphi(y, x), t)\\ &\quad * \mu'(\varphi(x, 0), t) * \mu'(\varphi(0, x), t) * \mu'(\varphi(y, 0), t) * \mu'(\varphi(0, y), t) \quad \text{and}\\ &\nu(f(x+y) - f(x) - f(y), 3t) \leq \nu'(\varphi(x, y), t) \Diamond \nu'(\varphi(y, x), t)\\ &\quad \Diamond \nu'(\varphi(x, 0), t) \Diamond \nu'(\varphi(0, x), t) \Diamond \nu'(\varphi(y, 0), t) \Diamond \nu'(\varphi(0, y), t)\end{aligned}\right\}. \tag{2.1.8}$$

Then

$$\mu''(2^n x, t) = \mu''\left(x, \frac{t}{\alpha^n}\right) \quad \text{and} \quad \nu''(2^n x, t) = \nu''\left(x, \frac{t}{\alpha^n}\right).$$

If we put $x = y$ in (2.1.8), we get

$$\mu(f(2x) - 2f(x), t) \geq \mu''(x, t) \quad \text{and} \quad \nu(f(2x) - 2f(x), t) \leq \nu''(x, t). \tag{2.1.9}$$

Replacing x by $2^n x$ in (2.1.9) we have

$$\mu\left(\frac{f(2^{n+1}x)}{2^{n+1}} - \frac{f(2^n x)}{2^n}, t\right) = \mu(f(2^{n+1}x) - f(2^n x), 2^n t) \geq \mu''(2^n x, 2^n t) \geq \mu''\left(x, \left(\frac{2}{\alpha}\right)^n t\right)$$

and

$$\nu\left(\frac{f(2^{n+1}x)}{2^{n+1}} - \frac{f(2^n x)}{2^n}, t\right) = \nu(f(2^{n+1}x) - f(2^n x), 2^n t) \leq \nu''(2^n x, 2^n t) \leq \nu''\left(x, \left(\frac{2}{\alpha}\right)^n t\right).$$

Thus

$$\mu\left(\frac{f(2^{n+1}x)}{2^{n+1}} - \frac{f(2^n x)}{2^n}, \left(\frac{\alpha}{2}\right)^n t\right) \geq \mu''(x, t) \quad \text{and} \quad \nu\left(\frac{f(2^{n+1}x)}{2^{n+1}} - \frac{f(2^n x)}{2^n}, \left(\frac{\alpha}{2}\right)^n t\right) \leq \nu''(x, t).$$

Therefore for each $n > m \geq 0$,

$$\left.\begin{aligned} &\mu\left(\frac{f(2^n x)}{2^n} - \frac{f(2^m x)}{2^m}, \sum_{k=m+1}^{n}\left(\frac{\alpha}{2}\right)^{k-1} t\right) = \mu\left(\sum_{k=m+1}^{n} \frac{f(2^k x)}{2^k} - \frac{f(2^{k-1}x)}{2k-1}, \sum_{k=m+1}^{n}\left(\frac{\alpha}{2}\right)^{k-1} t\right) \\ &\quad \geq \prod_{k=m+1}^{n} \mu\left(\frac{f(2^k x)}{2^k} - \frac{f(2^{k-1}x)}{2k-1}, \left(\frac{\alpha}{2}\right)^{k-1} t\right) \geq \mu''(x,t) \quad \text{and} \\ &\nu\left(\frac{f(2^n x)}{2^n} - \frac{f(2^m x)}{2^m}, \sum_{k=m+1}^{n}\left(\frac{\alpha}{2}\right)^{k-1} t\right) \leq \coprod_{k=m+1}^{n} \nu\left(\frac{f(2^k x)}{2^k} - \frac{f(2^{k-1}x)}{2k-1}, \left(\frac{\alpha}{2}\right)^{k-1} t\right) \leq \nu''(x,t) \end{aligned}\right\}, \tag{2.1.10}$$

for all $x \in X$ and $t > 0$ where $\prod_{j=1}^{n} a_j = a_1 * a_2 * \cdots * a_n$, $\coprod_{j=1}^{n} a_j = a_1 \Diamond a_2 \Diamond \cdots \Diamond a_n$. Let $t_0 > 0$ and $\epsilon > 0$ be given. Using the fact that $\lim_{t\to\infty} \mu''(x,t) = 1$ and $\lim_{t\to\infty} \nu''(x,t) = 0$, we can find some $t_1 > t_0$ such that

$$\mu''(x, t_1) > 1 - \epsilon \quad \text{and} \quad \nu''(x, t_1) < \epsilon.$$

The convergence of the series $\sum_{n=1}^{\infty} (\frac{\alpha}{2})^n t_1$ gives some $n_0 \in \mathbb{N}$ such that for each $n > m \geq n_0$,

$$\sum_{k=m+1}^{n}\left(\frac{\alpha}{2}\right)^{k-1} t_1 < t_0.$$

Therefore

$$\mu\left(\frac{f(2^n x)}{2^n} - \frac{f(2^m x)}{2^m}, t_0\right) \geq \mu\left(\frac{f(2^n x)}{2^n} - \frac{f(2^m x)}{2^m}, \sum_{k=m+1}^{n}\left(\frac{\alpha}{2}\right)^{k-1} t_1\right) \geq \mu''(x, t_1) > 1 - \epsilon$$

and

$$\nu\left(\frac{f(2^n x)}{2^n} - \frac{f(2^m x)}{2^m}, t_0\right) \leq \nu\left(\frac{f(2^n x)}{2^n} - \frac{f(2^m x)}{2^m}, \sum_{k=m+1}^{n}\left(\frac{\alpha}{2}\right)^{k-1} t_1\right) \leq \nu''(x, t_1) < \epsilon.$$

Hence, $(\frac{f(2^n x)}{2^n})$ is a Cauchy sequence in (Y, μ, ν). Since (Y, μ, ν) is an intuitionistic fuzzy Banach space, $(\frac{f(2^n x)}{2^n})$ converges to some point $T(x) \in Y$. Define $T : X \to Y$ by $T(x) = (\mu, \nu)\text{-}\lim_{n\to\infty} \frac{f(2^n x)}{2^n}$.

Fix $x, y \in X$ and $t > 0$. It follows from (2.1.8) that

$$\left.\begin{aligned} &\mu\left(\frac{f(2^n(x+y))}{2^n} - \frac{f(2^n x)}{2^n} - \frac{f(2^n y)}{2^n}, \frac{t}{4}\right) = \mu(f(2^n(x+y)) - f(2^n x) - f(2^n y), (2^n t)/4) \\ &\quad \geq \mu'\left(\varphi(x,y), \frac{2^n t}{12\alpha^n}\right) * \mu'\left(\varphi(y,x), \frac{2^n t}{12\alpha^n}\right) * \mu'\left(\varphi(x,0), \frac{2^n t}{12\alpha^n}\right) \\ &\quad * \mu'\left(\varphi(0,x), \frac{2^n t}{12\alpha^n}\right) * \mu'\left(\varphi(y,0), \frac{2^n t}{12\alpha^n}\right) * \mu'\left(\varphi(0,y), \frac{2^n t}{12\alpha^n}\right) \quad \text{and} \\ &\nu\left(\frac{f(2^n(x+y))}{2^n} - \frac{f(2^n x)}{2^n} - \frac{f(2^n y)}{2^n}, \frac{t}{4}\right) = \nu(f(2^n(x+y)) - f(2^n x) - f(2^n y), (2^n t)/4) \\ &\quad \leq \nu'\left(\varphi(x,y), \frac{2^n t}{12\alpha^n}\right) \Diamond \nu'\left(\varphi(y,x), \frac{2^n t}{12\alpha^n}\right) \Diamond \nu'\left(\varphi(x,0), \frac{2^n t}{12\alpha^n}\right) \\ &\quad \Diamond\, \nu'\left(\varphi(0,x), \frac{2^n t}{12\alpha^n}\right) \Diamond \nu'\left(\varphi(y,0), \frac{2^n t}{12\alpha^n}\right) \Diamond \nu'\left(\varphi(0,y), \frac{2^n t}{12\alpha^n}\right) \end{aligned}\right\} \tag{2.1.11}$$

for all n. Moreover,

$$\left.\begin{aligned} &\mu(T(x+y) - T(x) - T(y), t) \geq \mu\left(T(x+y) - \frac{f(2^n(x+y))}{2^n}, \frac{t}{4}\right) * \mu\left(T(x) - \frac{f(2^n x)}{2^n}, \frac{t}{4}\right) \\ &\quad * \mu\left(T(y) - \frac{f(2^n y)}{2^n}, \frac{t}{4}\right) * \mu\left(\frac{f(2^n(x+y))}{2^n} - \frac{f(2^n x)}{2^n} - \frac{f(2^n y)}{2^n}, \frac{t}{4}\right) \quad \text{and} \\ &\nu(T(x+y) - T(x) - T(y), t) \leq \nu\left(T(x+y) - \frac{f(2^n(x+y))}{2^n}, \frac{t}{4}\right) \Diamond \nu\left(T(x) - \frac{f(2^n x)}{2^n}, \frac{t}{4}\right) \\ &\quad \Diamond\, \nu\left(T(y) - \frac{f(2^n y)}{2^n}, \frac{t}{4}\right) \Diamond \nu\left(\frac{f(2^n(x+y))}{2^n} - \frac{f(2^n x)}{2^n} - \frac{f(2^n y)}{2^n}, \frac{t}{4}\right), \end{aligned}\right\} \tag{2.1.12}$$

for all n. Letting $n \to \infty$ in (2.1.11) and (2.1.12), we obtain

$$\mu(T(x+y)-T(x)-T(y),t)=1 \quad \text{and} \quad \nu(T(x+y)-T(x)-T(y),t)=0.$$

Thus $T(x+y)=T(x)+T(y)$. Furthermore, using (2.1.10) with $m=0$, we get

$$\left.\begin{aligned}
\mu(T(x)-f(x),t) &\geq \mu\left(T(x)-\frac{f(2^n x)}{2^n},\frac{t}{2}\right)*\mu\left(\frac{f(2^n x)}{2^n}-f(x),\frac{t}{2}\right)\\
&\geq \mu\left(T(x)-\frac{f(2^n x)}{2^n},\frac{t}{2}\right)*\mu''\left(x,\frac{t}{2\sum_{k=1}^{n}\left(\frac{\alpha}{2}\right)^{k-1}}\right)\geq \mu''\left(x,\frac{t}{2\sum_{k=1}^{\infty}\left(\frac{\alpha}{2}\right)^{k-1}}\right)=\mu''\left(x,\frac{2-\alpha}{4}t\right) \quad \text{and}\\
\nu(T(x)-f(x),t) &\leq \nu\left(T(x)-\frac{f(2^n x)}{2^n},\frac{t}{2}\right)\lozenge\nu\left(\frac{f(2^n x)}{2^n}-f(x),\frac{t}{2}\right)\\
&\leq \nu\left(T(x)-\frac{f(2^n x)}{2^n},\frac{t}{2}\right)\lozenge\nu''\left(x,\frac{t}{2\sum_{k=1}^{n}\left(\frac{\alpha}{2}\right)^{k-1}}\right)\leq \nu''\left(x,\frac{t}{2\sum_{k=1}^{\infty}\left(\frac{\alpha}{2}\right)^{k-1}}\right)=\nu''\left(x,\frac{2-\alpha}{4}t\right)
\end{aligned}\right\}. \tag{2.1.13}$$

It follows from (2.1.7) and (2.1.13) that

$$\begin{aligned}
\mu\left(g(x)+h(x)-T(x),\frac{14-\alpha}{12}t\right) &\geq \mu(f(x)-T(x),t)*\mu\left(g(x)+h(x)-f(x),\frac{2-\alpha}{12}t\right)\\
&\geq \mu''\left(x,\frac{2-\alpha}{4}t\right)*\mu'\left(\varphi(x,0),\frac{2-\alpha}{12}t\right)*\mu'\left(\varphi(0,x),\frac{2-\alpha}{12}t\right)\\
&\geq \mu''\left(x,\frac{2-\alpha}{4}t\right)
\end{aligned}$$

and similarly

$$\nu\left(g(x)+h(x)-T(x),\frac{14-\alpha}{12}t\right)\leq \nu''\left(x,\frac{2-\alpha}{4}t\right).$$

Thus we obtained (2.1.3). To prove the uniqueness of T, assume that T' be another additive mapping from X into Y, which satisfies (2.1.2). Since for each $n\in\mathbb{N}$, $T(2^n x)=4^n T(x)$ and $T'(2^n x)=4^n T'(x)$. Then

$$\begin{aligned}
\mu(T(x)-T'(x),t) &= \mu(T(2^n x)-T'(2^n x),4^n t)\\
&\geq \mu\left(T'(2^n x)-f(2^n x),\frac{4^n t}{2}\right)*\mu\left(f(2^n x)-T(2^n x),\frac{4^n t}{2}\right)\\
&\geq \mu''\left(2^n x,\frac{(2-|\alpha|)4^n t}{8}\right)=\mu''\left(x,\frac{(2-|\alpha|)2^n t}{8}\right) \quad \text{and}\\
\nu(T(x)-T'(x),t) &= \nu(T(2^n x)-T'(2^n x),4^n t)\\
&\leq \nu\left(T'(2^n x)-f(2^n x),\frac{4^n t}{2}\right)\lozenge\nu\left(f(2^n x)-T(2^n x),\frac{4^n t}{2}\right)\leq \nu''\left(x,\frac{(2-|\alpha|)2^n t}{8}\right)
\end{aligned}$$

for all $x\in X$, $t>0$ and $n\in\mathbb{N}$. Also, we have

$$\lim_{n\to\infty}\mu''\left(x,\frac{(2-|\alpha|)2^n t}{8}\right)=1 \quad \text{and} \quad \lim_{n\to\infty}\nu''\left(x,\frac{(2-|\alpha|)2^n t}{8}\right)=0.$$

Therefore,

$$\mu(T(x)-T'(x),t)=1 \quad \text{and} \quad \nu(T(x)-T'(x),t)=0,$$

for all $x\in X$ and $t>0$. Hence $T(x)=T'(x)$ for all $x\in X$. □

Theorem 2.2. *Suppose that (2.1.1) holds with $0<|\alpha|<4$. Let (Y,μ,ν) be intuitionistic fuzzy Banach space and f, g and h are even functions from X to Y such that $f(0)=g(0)=h(0)=0$ and*

$$\left.\begin{aligned}
&\mu(f(x+y)+f(x-y)-2g(x)-2h(y),t)\geq \mu'(\varphi(x,y),t) \quad \textit{and}\\
&\nu(f(x+y)+f(x-y)-2g(x)-2h(y),t)\leq \nu'(\varphi(x,y),t)
\end{aligned}\right\}, \tag{2.2.1}$$

for all $x, y \in X$ and $t > 0$. Then there is a unique quadratic mapping $Q : X \to Y$ such that

$$\mu(Q(x) - f(x), t) \geq \mu''\left(x, \frac{(4 - |\alpha|)}{16}t\right), \qquad \mu(Q(x) - g(x), t) \geq \mu''\left(x, \frac{(12 - 3|\alpha|)}{52 - |\alpha|}t\right),$$
$$\mu(Q(x) - h(x), t) \geq \mu''\left(x, \frac{(12 - 3|\alpha|)}{52 - |\alpha|}t\right) \quad \textit{and}$$
$$\nu(Q(x) - f(x), t) \leq \nu''\left(x, \frac{(4 - |\alpha|)}{16}t\right), \qquad \nu(Q(x) - g(x), t) \leq \nu''\left(x, \frac{(12 - 3|\alpha|)}{52 - |\alpha|}t\right),$$
$$\nu(Q(x) - h(x), t) \leq \nu''\left(x, \frac{(12 - 3|\alpha|)}{52 - |\alpha|}t\right),$$

where $\mu''(x, t)$ and $\nu''(x, t)$ are defined by (2.1.4).

Proof. Replacing x by y and y by x in (2.2.1), we get

$$\left.\begin{array}{l}\mu(f(x+y) + f(x-y) - 2g(y) - 2h(x), t) \geq \mu'(\varphi(y, x), t) \quad \text{and} \\ \nu(f(x+y) + f(x-y) - 2g(y) - 2h(x), t) \leq \nu'(\varphi(y, x), t)\end{array}\right\}. \tag{2.2.2}$$

Put $y = x$ in (2.2.1). Then for all $x \in X$ and $t > 0$

$$\mu(f(2x) - 2g(x) - 2h(x), t) \geq \mu'(\varphi(x, x), t) \quad \text{and}$$
$$\nu(f(2x) - 2g(x) - 2h(x), t) \leq \nu'(\varphi(x, x), t).$$

Put $x = 0$ in (2.2.1), we get

$$\mu(2f(y) - 2h(y), t) \geq \mu'(\varphi(0, y), t) \quad \text{and} \quad \nu(2f(y) - 2h(y), t) \leq \nu'(\varphi(0, y), t), \tag{2.2.3}$$

for all $x \in X$ and $t > 0$. For $y = 0$, (2.2.1) becomes

$$\mu(2f(x) - 2g(x), t) \geq \mu'(\varphi(x, 0), t) \quad \text{and} \quad \nu(2f(x) - 2g(x), t) \leq \nu'(\varphi(x, 0), t). \tag{2.2.4}$$

Combining (2.2.2)–(2.2.4), we get

$$\left.\begin{array}{l}\mu(f(x+y) - f(x-y) - 2f(x) - 2f(y), t) \\ \quad \geq \mu'(\varphi(x, y), t/3) * \mu'(\varphi(x, 0), t/3) * \mu'(\varphi(0, y), t/3) \quad \text{and} \\ \nu(f(x+y) - f(x-y) - 2f(x) - 2f(y), t) \\ \quad \leq \nu'(\varphi(x, y), t/3) \diamond \nu'(\varphi(x, 0), t/3) \diamond \nu'(\varphi(0, y), t/3)\end{array}\right\}. \tag{2.2.5}$$

Setting $y = x$ in (2.2.5), we have

$$\mu(2f(x) - 4f(x), t) \geq \mu''(x, t) \quad \text{and} \quad \nu(2f(x) - 4f(x), t) \leq \nu''(x, t), \tag{2.2.6}$$

where $\mu''(x, t)$ and $\nu''(x, t)$ are defined in (2.1.4). By (2.1.1),

$$\mu''(2^n x, t) = \mu''\left(x, \frac{t}{\alpha^n}\right) \quad \text{and} \quad \nu''(2^n x, t) = \nu''\left(x, \frac{t}{\alpha^n}\right), \tag{2.2.7}$$

for every $x \in X$ and for each $n \geq 0$. It follows from (2.2.6) and (2.2.7) that

$$\left.\begin{array}{l}\mu(f(2^{n+1}x) - 4f(2^n x), t) \geq \mu''\left(x, \dfrac{t}{\alpha^n}\right) \quad \text{and} \\ \nu(f(2^{n+1}x) - 4f(2^n x), t) \leq \nu''\left(x, \dfrac{t}{\alpha^n}\right)\end{array}\right\}. \tag{2.2.8}$$

From (2.2.8), we obtain

$$\mu\left(\frac{f(2^{n+1}x)}{4^{n+1}} - \frac{f(2^n x)}{4^n}, t\right) = \mu(f(2^{n+1}x) - 4f(2^n x), 4^{n+1}t) \geq \mu''\left(x, \frac{4^{n+1}t}{\alpha^n}\right) \quad \text{and}$$
$$\nu\left(\frac{f(2^{n+1}x)}{4^{n+1}} - \frac{f(2^n x)}{4^n}, t\right) = \nu(f(2^{n+1}x) - 4f(2^n x), 4^{n+1}t) \leq \nu''\left(x, \frac{4^{n+1}t}{\alpha^n}\right)$$

or equivalently,

$$\mu\left(\frac{f(2^{n+1}x)}{4^{n+1}} - \frac{f(2^n x)}{4^n}, \frac{\alpha^n t}{4^{n+1}}\right) \geq \mu''(x, t) \quad \text{and} \quad \nu\left(\frac{f(2^{n+1}x)}{4^{n+1}} - \frac{f(2^n x)}{4^n}, \frac{\alpha^n t}{4^{n+1}}\right) \leq \nu''(x, t),$$

for all $x \in X$ and $t > 0$. Therefore, for all $x \in X$, $t > 0$ and for each $n > m \geq 0$

$$\left.\begin{aligned} &\mu\left(\frac{f(2^n x)}{4^n} - \frac{f(2^m x)}{4^m}, \sum_{k=m+1}^{n} \frac{\alpha^{k-1}t}{4^k}\right) = \mu\left(\sum_{k=m+1}^{n}\left(\frac{f(2^k x)}{4^k} - \frac{f(2^{k-1}x)}{4^{k-1}}\right), \sum_{k=m+1}^{n}\frac{\alpha^{k-1}t}{4^k}\right) \\ &\quad \geq \prod_{k=m+1}^{n} \mu\left(\frac{f(2^k x)}{4^k} - \frac{f(2^{k-1}x)}{4^{k-1}}, \frac{\alpha^{k-1}t}{4^k}\right) \geq \mu''(x,t) \quad \text{and} \\ &\nu\left(\frac{f(2^n x)}{4^n} - \frac{f(2^m x)}{4^m}, \sum_{k=m+1}^{n} \frac{\alpha^{k-1}t}{4^k}\right) = \nu\left(\sum_{k=m+1}^{n}\left(\frac{f(2^k x)}{4^k} - \frac{f(2^{k-1}x)}{4^{k-1}}\right), \sum_{k=m+1}^{n}\frac{\alpha^{k-1}t}{4^k}\right) \\ &\quad \leq \coprod_{k=m+1}^{n} \nu\left(\frac{f(2^k x)}{4^k} - \frac{f(2^{k-1}x)}{4^{k-1}}, \frac{\alpha^{k-1}t}{4^k}\right) \leq \nu''(x,t) \end{aligned}\right\}, \tag{2.2.9}$$

where $\prod$ and $\coprod$ are defined in the proof of Theorem 2.1. Given $\epsilon > 0$ and $t_0 > 0$. Since $\lim_{t\to\infty}\mu''(x,t) = 1$ and $\lim_{t\to\infty}\nu''(x,t) = 0$ there is some $t_1 > t_0$ such that $\mu''(x,t_1) > 1-\epsilon$ and $\nu''(x,t_1) < \epsilon$. By the convergence of the series $\sum_{k=1}^{\infty}\frac{\alpha^{k-1}}{4^k}t_1$, we can find some n_0 such that $\sum_{k=m+1}^{n}\frac{\alpha^{k-1}}{4^k}t_1 < t_0$ for each $n > m \geq n_0$. It follows that

$$\mu\left(\frac{f(2^n x)}{4^n} - \frac{f(2^m x)}{4^m}, t_0\right) \geq \mu\left(\frac{f(2^n x)}{4^n} - \frac{f(2^m x)}{4^m}, \sum_{k=m+1}^{n}\frac{\alpha^{k-1}}{4^k}t_1\right) \geq \mu''(x,t_0) > 1-\epsilon$$

and

$$\nu\left(\frac{f(2^n x)}{4^n} - \frac{f(2^m x)}{4^m}, t_0\right) \leq \nu\left(\frac{f(2^n x)}{4^n} - \frac{f(2^m x)}{4^m}, \sum_{k=m+1}^{n}\frac{\alpha^{k-1}}{4^k}t_1\right) \leq \nu''(x,t_0) < \epsilon,$$

for every $x \in X$, $t > 0$ and each $n > m > n_0$. This shows that $(\frac{f(2^n x)}{4^n})$ is a Cauchy sequence in the intuitionistic fuzzy Banach space (Y,μ,ν), therefore it is convergence to some $Q(x) \in Y$. So we can define a mapping $Q : X \to Y$ by $Q(x) = (\mu,\nu)\text{-}\lim_{n\to\infty}\frac{f(2^n x)}{4^n}$. Fix $x, y \in X$ and $t > 0$. It follows from (2.2.5) that

$$\left.\begin{aligned} &\mu\left(\frac{f(2^n(x+y))}{4^n} + \frac{f(2^n(x-y))}{4^n} - 2\frac{f(2^n x)}{4^n} - 2\frac{f(2^n y)}{4^n}, \frac{t}{5}\right) \\ &\quad = \mu\left(f(2^n(x+y)) + f(2^n(x-y)) - 2f(2^n x) - 2f(2^n y), \frac{4^n t}{5}\right) \\ &\quad \geq \mu'\left(\varphi(x,y), \frac{4^n t}{15\alpha^n}\right) * \mu'\left(\varphi(x,0), \frac{4^n t}{15\alpha^n}\right) * \mu'\left(\varphi(0,y), \frac{4^n t}{15\alpha^n}\right) \quad \text{and} \\ &\nu\left(\frac{f(2^n(x+y))}{4^n} + \frac{f(2^n(x-y))}{4^n} - 2\frac{f(2^n x)}{4^n} - 2\frac{f(2^n y)}{4^n}, \frac{t}{5}\right) \\ &\quad = \nu\left(f(2^n(x+y)) + f(2^n(x-y)) - 2f(2^n x) - 2f(2^n y), \frac{4^n t}{5}\right) \\ &\quad \leq \nu'\left(\varphi(x,y), \frac{4^n t}{15\alpha^n}\right) \diamond \nu'\left(\varphi(x,0), \frac{4^n t}{15\alpha^n}\right) \diamond \nu'\left(\varphi(0,y), \frac{4^n t}{15\alpha^n}\right) \end{aligned}\right\} \tag{2.2.10}$$

for all n. Furthermore,

$$\left.\begin{aligned} &\mu(Q(x+y) + Q(x-y) - 2Q(x) - 2Q(y), t) \geq \mu\left(Q(x+y) - \frac{f(2^n(x+y))}{4^n}, \frac{t}{5}\right) \\ &\quad * \mu\left(Q(x-y) - \frac{f(2^n(x-y))}{4^n}, \frac{t}{5}\right) * \mu\left(2Q(x) - 2\frac{f(2^n x)}{4^n}, \frac{t}{5}\right) * \mu\left(2Q(y) - 2\frac{f(2^n y)}{4^n}, \frac{t}{5}\right) \\ &\quad * \mu\left(\frac{f(2^n(x+y))}{4^n} + \frac{f(2^n(x-y))}{4^n} - 2\frac{f(2^n x)}{4^n} - 2\frac{f(2^n y)}{4^n}, \frac{t}{5}\right) \quad \text{and} \\ &\nu(Q(x+y) + Q(x-y) - 2Q(x) - 2Q(y), t) \leq \nu\left(Q(x+y) - \frac{f(2^n(x+y))}{4^n}, \frac{t}{5}\right) \\ &\quad \diamond \nu\left(Q(x-y) - \frac{f(2^n(x-y))}{4^n}, \frac{t}{5}\right) \diamond \nu\left(2Q(x) - 2\frac{f(2^n x)}{4^n}, \frac{t}{5}\right) \diamond \nu\left(2Q(y) - 2\frac{f(2^n y)}{4^n}, \frac{t}{5}\right) \\ &\quad \diamond \nu\left(\frac{f(2^n(x+y))}{4^n} + \frac{f(2^n(x-y))}{4^n} - 2\frac{f(2^n x)}{4^n} - 2\frac{f(2^n y)}{4^n}, \frac{t}{5}\right) \end{aligned}\right\}. \tag{2.2.11}$$

Letting $n \to \infty$ in (2.2.10) and (2.2.11), we get

$$\mu(Q(x+y)+Q(x-y)-2Q(x)-2Q(y),t)=1 \quad \text{and}$$
$$\nu(Q(x+y)+Q(x-y)-2Q(x)-2Q(y),t)=0,$$

for all $x, y \in X$ and $t > 0$. Thus, $Q(x+y)+Q(x-y)=2Q(x)+2Q(y)$. Using (2.2.9) with $m=0$, we get

$$\left.\begin{aligned}
\mu(Q(x)-f(x),t) &\geq \mu\left(Q(x)-\frac{f(2^n x)}{4^n},\frac{t}{2}\right)*\mu\left(\frac{f(2^n x)}{4^n}-f(x),\frac{t}{2}\right)\\
&\geq \mu\left(Q(x)-\frac{f(2^n x)}{4^n},\frac{t}{2}\right)*\mu''\left(x,\frac{4t}{2\sum_{k=1}^{n}\left(\frac{\alpha}{4}\right)^{k-1}}\right)\\
&\geq \mu''\left(x,\frac{4t}{2\sum_{k=0}^{\infty}\left(\frac{\alpha}{4}\right)^{k}}\right)=\mu''\left(x,\frac{4-\alpha}{16}t\right) \quad \text{and}\\
\nu(Q(x)-f(x),t) &\leq \nu\left(Q(x)-\frac{f(2^n x)}{4^n},\frac{t}{2}\right)\lozenge\nu\left(\frac{f(2^n x)}{4^n}-f(x),\frac{t}{2}\right)\\
&\leq \nu\left(Q(x)-\frac{f(2^n x)}{4^n},\frac{t}{2}\right)\lozenge\nu''\left(x,\frac{4t}{2\sum_{k=1}^{n}\left(\frac{\alpha}{4}\right)^{k-1}}\right)\\
&\leq \nu''\left(x,\frac{4t}{2\sum_{k=0}^{\infty}\left(\frac{\alpha}{4}\right)^{k}}\right)=\nu''\left(x,\frac{4-\alpha}{16}t\right)
\end{aligned}\right\}, \tag{2.2.12}$$

for sufficiently large n. From (2.2.4) and (2.2.12), we conclude that

$$\begin{aligned}
\mu\left(Q(x)-g(x),\frac{52-\alpha}{48}t\right) &\geq \mu(Q(x)-f(x),t)*\mu\left(f(x)-g(x),\frac{4-\alpha}{48}t\right)\\
&\geq \mu''\left(x,\frac{4-\alpha}{16}t\right)*\mu'\left(\varphi(x,0),\frac{4-\alpha}{48}t\right)\geq \mu''\left(x,\frac{4-\alpha}{16}t\right) \quad \text{and}\\
\nu\left(Q(x)-g(x),\frac{52-\alpha}{48}t\right) &\leq \nu(Q(x)-f(x),t)\lozenge\nu\left(f(x)-g(x),\frac{4-\alpha}{48}t\right)\\
&\leq \nu''\left(x,\frac{4-\alpha}{16}t\right)\lozenge\nu'\left(\varphi(x,0),\frac{4-\alpha}{48}t\right)\leq \nu''\left(x,\frac{4-\alpha}{16}t\right).
\end{aligned}$$

Thus

$$\mu(Q(x)-g(x),t)\geq \mu''\left(x,\frac{12-3\alpha}{52-\alpha}t\right) \quad \text{and} \quad \nu(Q(x)-g(x),t)\leq \nu''\left(x,\frac{12-3\alpha}{52-\alpha}t\right).$$

Similarly, we show that the above inequality also holds for h. The uniqueness assertion can be done on the same lines as in Theorem 2.1. □

Theorem 2.3. *Suppose that (2.1.1) holds with $|\alpha| < 2$. Let f be a mapping from X to intuitionistic fuzzy Banach space (Y, μ, ν) such that $f(0) = 0$ and*

$$\left.\begin{aligned}
&\mu(f(x+y)+f(x-y)-2f(x)-2f(y),t)\geq \mu'(\varphi(x,y),t) \quad \text{and}\\
&\nu(f(x+y)+f(x-y)-2f(x)-2f(y),t)\leq \nu'(\varphi(x,y),t)
\end{aligned}\right\}, \tag{2.3.1}$$

for all $x, y \in X$ and all $t > 0$. Then there are unique mappings $T, Q : X \to Y$ such that T is additive, Q is quadratic and

$$\mu(f(x)-T(x)-Q(x),t)\geq M\left(x,\left\{\left(\frac{2-\alpha}{8}\right)*\left(\frac{4-\alpha}{32}\right)\right\}t\right) \quad \text{and}$$
$$\nu(f(x)-T(x)-Q(x),t)\leq N\left(x,\left\{\left(\frac{2-\alpha}{8}\right)\lozenge\left(\frac{4-\alpha}{32}\right)\right\}t\right),$$

for all $x \in X$ and all $t > 0$, where

$$\begin{aligned} M(x,t) &= \mu'(\varphi(x,x),t/3) * \mu'(\varphi(-x,-x),t/3) * \mu'(\varphi(x,0),t/3) \\ &\quad * \mu'(\varphi(0,x),t/3) * \mu'(\varphi(-x,0),t/3) * \mu'(\varphi(0,-x),t/3) \quad \textit{and} \\ N(x,t) &= \nu'(\varphi(x,x),t/3) \lozenge \nu'(\varphi(-x,-x),t/3) \lozenge \nu'(\varphi(x,0),t/3) \\ &\quad \lozenge\, \nu'(\varphi(0,x),t/3) \lozenge \nu'(\varphi(-x,0),t/3) \lozenge \nu'(\varphi(0,-x),t/3). \end{aligned}$$

Proof. Passing to the odd part f° and even part f^e of f, we deduce from (2.3.1) that

$$\begin{aligned} &\mu(f^\circ(x+y)+f^\circ(x-y)-2f^\circ(x)-2f^\circ(y),t) \ge \mu'(\varphi(x,y),t) * \mu'(\varphi(-x,-y),t), \quad \text{and} \\ &\nu(f^\circ(x+y)+f^\circ(x-y)-2f^\circ(x)-2f^\circ(y),t) \le \nu'(\varphi(x,y),t) \lozenge \nu'(\varphi(-x,-y),t). \end{aligned}$$

On the other hand,

$$\begin{aligned} &\mu(f^e(x+y)+f^e(x-y)-2f^e(x)-2f^e(y),t) \ge \mu'(\varphi(x,y),t) * \mu'(\varphi(-x,-y),t) \quad \text{and} \\ &\nu(f^e(x+y)+f^e(x-y)-2f^e(x)-2f^e(y),t) \le \nu'(\varphi(x,y),t) \lozenge \nu'(\varphi(-x,-y),t). \end{aligned}$$

Using the proofs of Theorems 2.1 and 2.2, we get a unique additive mapping T and a unique quadratic mapping Q satisfying

$$\mu(f^\circ(x)-T(x),t) \ge M\left(x, \frac{2-|\alpha|}{4}t\right) \quad \text{and} \quad \nu(f^\circ(x)-T(x),t) \le N\left(x, \frac{2-|\alpha|}{4}t\right).$$

Also,

$$\mu(f^e(x)-Q(x),t) \ge M\left(x, \frac{4-|\alpha|}{16}t\right) \quad \text{and} \quad \nu\left(f^e(x)-Q(x),t) \le N(x, \frac{4-|\alpha|}{16}t\right).$$

Therefore

$$\begin{aligned} \mu(f(x)-T(x)-Q(x),t) &\ge \mu\left(f^\circ - T(x), \frac{t}{2}\right) * \mu\left(f^e - Q(x), \frac{t}{2}\right) \\ &\ge M\left(x, \frac{2-|\alpha|}{8}t\right) * M\left(x, \frac{4-|\alpha|}{32}t\right) = M\left(x, \left\{\left(\frac{2-\alpha}{8}\right) * \left(\frac{4-\alpha}{32}\right)\right\} t\right) \quad \text{and} \\ \nu(f(x)-T(x)-Q(x),t) &\le \nu\left(f^\circ - T(x), \frac{t}{2}\right) \lozenge \nu\left(f^e - Q(x), \frac{t}{2}\right) \\ &\le N\left(x, \frac{2-|\alpha|}{8}t\right) \lozenge N\left(x, \frac{4-|\alpha|}{32}t\right) = N\left(x, \left\{\left(\frac{2-\alpha}{8}\right) \lozenge \left(\frac{4-\alpha}{32}\right)\right\} t\right), \end{aligned}$$

for all $x \in X$ and $t > 0$. □

The following example provides an illustration.

Example 2.1. Let $(X, \langle ., . \rangle)$ be an inner product space, Y be a normed space and Z be the real line $\mathbb{R}$. Let (μ, ν) and (μ', ν') be the intuitionistic fuzzy norms on Y and $\mathbb{R}$, defined by Example 1.2 with $k = 1$, respectively. Suppose that the intuitionistic fuzzy norm (μ, ν), makes Y into an intuitionistic fuzzy Banach space. Fix $x_\circ, y_\circ, z_\circ \in Y$ and $a \in X$, we define

$$\begin{aligned} &f(x) = \langle x, a\rangle x_\circ + \|x\|^2 y_\circ + \sqrt{\|x\|} z_\circ, \\ &g(x) = \langle x, a\rangle x_\circ + \|x\|^2 y_\circ, \\ &h(x) = \|x\|^2 y_\circ + \sqrt{\|x\|} z_\circ, \\ &\varphi(x,y) = (\sqrt{\|x+y\|} + \sqrt{\|x-y\|} - 2\sqrt{\|y\|})\|z_\circ\|, \end{aligned}$$

for each $x, y \in X$. One can easily verified that

$$f(x+y)+f(x-y)-2g(x)-2h(y) = (\sqrt{\|x+y\|} + \sqrt{\|x-y\|} - 2\sqrt{\|y\|}) z_\circ,$$

for all $x, y \in X$. Therefore

$$\begin{aligned} &\mu(f(x+y)+f(x-y)-2g(x)-2h(y),t) = \mu'(\varphi(x,y),t) \quad \text{and} \\ &\nu(f(x+y)+f(x-y)-2g(x)-2h(y),t) = \nu'(\varphi(x,y),t), \end{aligned}$$

for each $x, y \in X$ and $t \in \mathbb{R}$. Moreover, $\varphi(2x, 2y) = \sqrt{2}\varphi(x,y)$ for each $x, y \in X$. Therefore, conditions of Theorems 2.1 and 2.2 for f, g, h and $|\alpha| = \sqrt{2} < 2$ are satisfied. It follows that odd and even parts of f can be approximated by linear and quadratic functions, respectively. In fact f°, the odd part of f and $f^\circ(x) = \langle x, a\rangle x_\circ$ is linear. The even part of f is f^e, and $f^e(x) = \|x\|^2 y_\circ + \sqrt{\|x\|} z_\circ$ contains a quadratic $Q(x) = \|x\|^2 y_\circ$. Also

$$\mu(f^e(x) - Q(x), t) = \mu'(\sqrt{\|x\|}\,\|z_\circ\|, t) \geq \mu''\left(x, \frac{4-\sqrt{2}}{16}t\right) \quad \text{and}$$

$$\nu(f^e(x) - Q(x), t) = \nu'(\sqrt{\|x\|}\,\|z_\circ\|, t) \leq \nu''\left(x, \frac{4-\sqrt{2}}{16}t\right).$$

3. Conclusion

This work indeed presents a relationship between three different disciplines, namely, the theory of fuzzy spaces, the Hyers–Ulam–Rassias stability and the theory of functional equations. We established the Hyers–Ulam–Rassias stability of a Pexiderized quadratic functional equation in intuitionistic fuzzy normed spaces.

References

[1] S.M. Ulam, Problems in Modern Mathematics (Chapter VI, Some Questions in Analysis: Section 1, Stability), Science ed., John Wiley & Sons, New York, 1960.
[2] D.H. Hyers, On the stability of the linear functional equation, Proc. Natl. Acad. Sci. USA 27 (1941) 222–224.
[3] T. Aoki, On the stability of the linear transformation in Banach spaces, J. Math. Soc. Japan 2 (1950) 64–66.
[4] T.M. Rassias, On the stability of functional equations and a problem of Ulam, Acta. Appl. Math. 62 (2000) 123–130.
[5] K.-W. Jun, H.-M. Kim, On the Hyers–Ulam–Rassias stability of a general cubic functional equation, Math. Inequal Appl. 6 (1) (2003) 87–95.
[6] K.-W. Jun, H.-M. Kim, On the Hyers–Ulam stability of a generalized quadratic and additive functional equation, Bull. Korean Math. Soc. 42 (1) (2005) 133–148.
[7] Y.-S. Lee, S.-Y. Chung, Stability of the Jensen type functional equation, Banach J. Math. Anal. 1 (1) (2007) 91–100.
[8] A. Najati, Hyers–Ulam–Rassias stability of a cubic functional equation, Bull. Korean Math. Soc. 44 (4) (2007) 825–840.
[9] A.K. Mirmostafaee, M.S. Moslehian, Fuzzy almost quadratic functions, Results Math. 52 (2008) 161–177.
[10] A.K. Mirmostafaee, M.S. Moslehian, Fuzzy versions of Hyers–Ulam–Rassias theorem, Fuzzy Sets Syst. 159 (2008) 720–729.
[11] S.A. Mohiuddine, Stability of Jensen functional equation in intuitionistic fuzzy normed space, Chaos Solitons Fractals 42 (2009) 2989–2996.
[12] M. Mursaleen, S.A. Mohiuddine, On stability of a cubic functional equation in intuitionistic fuzzy normed spaces, Chaos Solitons Fractals 42 (2009) 2997–3005.
[13] R. Saadati, J.H. Park, On the intuitionistic fuzzy topological spaces, Chaos Solitons Fractals 27 (2006) 331–344.
[14] S.A. Mohiuddine, Q.M. Danish Lohani, On generalized statistical convergence in intuitionistic fuzzy normed space, Chaos Solitons Fractals 42 (2009) 1731–1737.
[15] M. Mursaleen, S.A. Mohiuddine, Statistical convergence of double sequences in intuitionistic fuzzy normed spaces, Chaos Solitons Fractals 41 (2009) 2414–2421.
[16] M. Mursaleen, S.A. Mohiuddine, Nonlinear operators between intuitionistic fuzzy normed spaces and Fréchet differentiation, Chaos Solitons Fractals 42 (2009) 1010–1015.
[17] M. Mursaleen, S.A. Mohiuddine, On lacunary statistical convergence with respect to the intuitionistic fuzzy normed space, J. Comput. Appl. Math. 233 (2009) 142–149.