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Effective Dini's Theorem on Effectively Compact Metric Spaces

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Abstract

We show that if a computable sequence of real-valued functions on an effectively compact metric space converges pointwise monotonically to a computable function, then the sequence converges effectively uniformly to the function. This is an effectivized version of Dini's Theorem.

Keywords: Dini's Theorem, effectively compact metric space, computable function, effective uniform convergence

1 Introduction

If a sequence of real-valued continuous functions on a compact space converges pointwise monotonically to a continuous function, then the sequence converges uniformly to the function. It is called *Dini's theorem* and one of the fundamental theorems in functional analysis and general topology.

From the viewpoint of computability, the question arises: whether we can effectivize Dini's theorem, in other words, whether there is a theorem which is a Dini's theorem with some topological concepts replaced by their computational counterparts. In this paper, we show a positive answer to this question in the case of metric spaces, more precisely, the theorem that if a computable sequence of real-valued functions on an effectively compact metric space con-

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verges pointwise monotonically to a computable function, then the sequence converges effectively uniformly to the function.

Meanwhile, if a computable sequence of real numbers converges monotonically to a computable real number, then the sequence converges effectively to the real number. It is called the *monotonic convergence theorem* [6]. The main theorem in this paper is not only an effectivization of Dini's thorem but also an extension of the monotonic convergence theorem to real-valued continuous functions on effectively compact metric spaces. The monotonic convergence theorem is considered a special case of our theorem on $C(\{0\})$, the space of all real-valued continuous functions on a singlton.

2 Preliminaries

For a relation $S \subset X_1 \times \cdots \times X_m \times Y_1 \times \cdots \times Y_n$ and a tuple $(x_1, \ldots, x_m) \in X_1 \times \cdots \times X_m$, the set $\{(y_1, \ldots, y_n) \in Y_1 \times \cdots \times Y_n \mid (x_1, \ldots, x_m, y_1, \ldots, y_n) \in S\}$ is denoted by $S(x_1, \ldots, x_m)$.

We fix some standard tuple functions and corresponding projection functions on $\mathbb{N}$. $\langle \text{-}, \ldots, \text{-} \rangle$ denotes the n-tuple function. $(\text{-})^n_k$ denotes the corresponding kth projection function. We often identify an n-tuple sequence $(x_{k_1,\ldots,k_n})$ with its serialization $(x_{(k)^n_1,\ldots,(k)^n_n})_{k \in \mathbb{N}}$.

We use the terminology and the notation on computability of real numbers and of real functions that Pour-El and Richards have used in [6].

The following four definitions are introduced in [9] by Yasugi, Mori, and Tsujii.

Definition 2.1 Let (M, d) be a metric space. A set $\mathcal{S} \subset M^\omega$ is a *computability structure* on (M, d) if the following three conditions hold.

- (i) If $(x_n), (y_n) \in \mathcal{S}$, then $(d(x_n, y_{n'}))_{n,n'}$ forms a computable double sequence of real numbers.
- (ii) If $(x_n) \in \mathcal{S}$ and $\sigma : \mathbb{N} \to \mathbb{N}$ is a recursive function, then $(x_{\sigma(n)}) \in \mathcal{S}$.
- (iii) If $(x_{n,k}) \in \mathcal{S}$, $(x'_n) \in M^\omega$, and $(x_{n,k})$ converges to (x'_n) effectively in n and k as $k \to \infty$, then $(x'_n) \in \mathcal{S}$.

An element of $\mathcal{S}$ is called a *computable sequence* in M.

Definition 2.2 A metric space with a computability structure $(M, d, \mathcal{S})$ is *effectively totally bounded* if there exists a computable sequence $(e_l) \in \mathcal{S}$ and a recursive function $\gamma : \mathbb{N} \to \mathbb{N}$ such that

$$M = \overline{\{e_l \mid l \in \mathbb{N}\}} \quad \text{and} \quad (\forall i \in \mathbb{N})\; M = \bigcup_{l=0}^{\gamma(i)} B(e_l, 1/2^i).$$

$(M, d, \mathcal{S})$ is *effectively compact* if it is effectively totally bounded and d is a complete metric.

Definition 2.3 Let $(M, d, \mathcal{S})$ be a metric space with a computability structure. A subset $K \subset M$ is an *effectively compact subset* of M if $(K, d|_K, \mathcal{S} \cap K^\omega)$ is an effectively compact metric space.

In other words, K is an effectively compact subset of $(M, d, \mathcal{S})$ iff it is a compact subset of (M, d) and there exists a computable sequence $(e_l) \in \mathcal{S}$ and a recursive function $\gamma : \mathbb{N} \to \mathbb{N}$ such that

$$K = \overline{\{e_l \mid l \in \mathbb{N}\}} \quad \text{and} \quad (\forall i \in \mathbb{N})\; K \subset \bigcup_{l=0}^{\gamma(i)} B(e_l, 1/2^i).$$

Definition 2.4 Let $(M, d, \mathcal{S})$ be an effectively compact metric space. A sequence of functions (f_n), $f_n : M \to \mathbb{R}$, is *computable* if the following two conditions hold.

- (i) (*Sequential computability*) For any computable sequence (x_k) in M, $(f_n(x_k))_{n,k}$ forms a computable sequence of real numbers.
- (ii) (*Effective uniform continuity*) There exists a recursive function $\alpha : \mathbb{N}^2 \to \mathbb{N}$ such that for any $n, j \in \mathbb{N}$ and any $x, y \in M$,

$$d(x, y) < 1/2^{\alpha(n,j)} \implies |f_n(x) - f_n(y)| < 1/2^j.$$

A function $f : M \to \mathbb{R}$ is a *computable function* if $(f)_{n \in \mathbb{N}}$, the sequence all of whose elements are equal to f, is a computable sequence of functions.

The recursive function α in Definition 2.4 (ii) is called an *effective modulus of continuity* of (f_n).

3 Effective Dini's Theorem

First, we show two propositions with no assumptions on computability.

Proposition 3.1 *Let (M, d) be a metric space. Let $K \subset M$ be a nonempty compact subset with $(e_l) \in M^\omega$ and $\gamma : \mathbb{N} \to \mathbb{N}$ such that $K = \overline{\{e_l \mid l \in \mathbb{N}\}}$ and $(\forall i)\; K \subset \bigcup_{l=0}^{\gamma(i)} B(e_l, 1/2^i)$. Then, for any finite sequence of open balls $(B(x_k, r_k))_{k=0,\dots,m}$, the following holds:*

$$K \subset \bigcup_{k=0}^{m} B(x_k, r_k) \iff (\exists i)(\forall l \le \gamma(i))(\exists k \le m)\; d(x_k, e_l) + \frac{1}{2^i} < r_k.$$

Proof. ($\Longleftarrow$) Suppose $y \in K$. Then, $(\exists l \leq \gamma(i))\ d(y, e_l) < 1/2^i$. It follows that

$$(\exists i)(\exists l \leq \gamma(i))(\exists k \leq m)\ \left[d(y, e_l) < \frac{1}{2^i}\ \wedge\ d(x_k, e_l) + \frac{1}{2^i} < r_k\right].$$

This implies $(\exists k \leq m)\ d(y, x_k) < r_k$, i.e., $y \in \bigcup_{k=0}^{m} B(x_k, r_k)$. This shows $K \subset \bigcup_{k=0}^{m} B(x_k, r_k)$.

($\Longrightarrow$) We will prove the contraposition. The negation of the conclusion is:

$$(\forall i)(\exists l \leq \gamma(i))(\forall k \leq m) \quad d(x_k, e_l) + \frac{1}{2^i} \geq r_k.$$

By choosing an l for each i, we obtain a function $\gamma' : \mathbb{N} \to \mathbb{N}$ such that

$$(\forall i)(\forall k \leq m) \quad d(x_k, e_{\gamma'(i)}) + \frac{1}{2^i} \geq r_k.$$

Since K is sequentially compact, there exists a subsequence $(e_{\gamma'(\theta(i))})$ that converges to a point in K. Let y be the limit. A simple manipulation of limits yields $(\forall k \leq m)\ d(x_k, y) \geq r_k$, i.e., $y \notin \bigcup_{k=0}^{m} B(x_k, r_k)$. This shows $K \not\subset \bigcup_{k=0}^{m} B(x_k, r_k)$. □

Proposition 3.2 *Let (M, d) be a separable metric space with a dense, at most countable subset $\{e_l \mid l \in \mathbb{N}\}$. Let a be a real number such that $a \geq 1$. Let (ε_i) be a sequence of positive real numbers converging to 0. Then,*

$$B(x, r) = \bigcup_{\substack{l,i \in \mathbb{N},\\ d(x,e_l)+a\varepsilon_i<r}} B(e_l, \varepsilon_i).$$

Proof. Comparison of the distance between the centers with the difference between the radii yields $B(x, r) \supset B(e_l, \varepsilon_i)$ if $d(x, e_l) + a\varepsilon_i < r$. This implies $B(x, r) \supset \bigcup_{l,i\in\mathbb{N},\ d(x,e_l)+a\varepsilon_i<r} B(e_l, \varepsilon_i)$.

Suppose $y \in B(x, r)$. There exists an ε_i such that $d(x, y) < r - (a + 1)\varepsilon_i$. Furthermore, there exists an e_l such that $d(y, e_l) < \varepsilon_i$. From these two inequalities as well as $d(x, e_l) \leq d(x, y) + d(y, e_l)$, we obtain $d(x, e_l) + a\varepsilon_i < r$. Therefore $y \in \bigcup_{l,i\in\mathbb{N},\ d(x,e_l)+a\varepsilon_i<r} B(e_l, \varepsilon_i)$. This shows $B(x, r) \subset \bigcup_{l,i\in\mathbb{N},\ d(x,e_l)+a\varepsilon_i<r} B(e_l, \varepsilon_i)$. □

Next, we show two lemmata. Lemma 3.3 is on a consequence of effective compactness. Lemma 3.4 is on another characterization of computable real-valued functions.

Lemma 3.3 *Let $(M, d, \mathcal{S})$ be a metric space with a computability structure. For any effectively compact subset K of M and any computable double sequence*

$(B(x_{n,k}, r_{n,k}))$ of open balls in M, there exists a recursive partial function $\alpha : \mathbb{N} \rightharpoonup \mathbb{N}$ such that $\alpha(n)$ is defined and $K \subset \bigcup_{k=0}^{\alpha(n)} B(x_{n,k}, r_{n,k})$ holds if $K \subset \bigcup_{k=0}^{\infty} B(x_{n,k}, r_{n,k})$, and $\alpha(n)$ is undefined otherwise.

Proof. We choose a computable sequence (e_l) and a recursive function $\gamma : \mathbb{N} \to \mathbb{N}$ such that $K = \overline{\{e_l \mid l \in \mathbb{N}\}}$ and $(\forall i)\ K \subset \bigcup_{l=0}^{\gamma(i)} B(e_l, 1/2^i)$. By applying Proposition 3.1 to each finite sequence $(B(x_{n,k}, r_{n,k}))_{k=0,\ldots,m}$ for $n = 0, 1, 2, \ldots$, we obtain

$$K \subset \bigcup_{k=0}^{m} B(x_{n,k}, r_{n,k}) \iff (\exists i)(\forall l \leq \gamma(i))(\exists k \leq m)\ d(x_{n,k}, e_l) + \frac{1}{2^i} < r_{n,k}.$$

It is clear that the right-hand side is recursively enumerable in n and m. Therefore, there exists a primitive recursive predicate φ on $\mathbb{N}^3$ such that $(\exists i')\varphi(n, m, i')$ iff $K \subset \bigcup_{k=0}^{m} B(x_{n,k}, r_{n,k})$. Hence, we can construct α by $\alpha(n) \simeq (\min\{m' \mid \varphi(n, (m')^2_1, (m')^2_2)\})^2_1$. □

Lemma 3.3 corresponds to "$\delta'_{\mathrm{range}} \leq \delta_{\mathrm{Haine-Borel}}|^{\mathcal{K}^*}$" shown by Brattka and Presser in [1]. The proof here is essentially the same as that in [1] with some correction for a minor error.

Lemma 3.4 *Let $(M, d, \mathcal{S})$ be an effectively compact metric space. Let $(e_l) \in \mathcal{S}$ be dense in M. For any sequence (f_n) of real-valued functions on M, the following two conditions are equivalent.*

(i) *(f_n) is a computable sequence of functions.*

(ii) *There exists a recursively enumerable set $S \subset \mathbb{N} \times \mathbb{Q} \times \mathbb{Q}^+ \times \mathbb{N} \times \mathbb{N}$ such that*

$$(\forall n \in \mathbb{N})(\forall c \in \mathbb{Q})(\forall r \in \mathbb{Q}^+) \quad f_n^{-1}((c - r,\ c + r)) = \bigcup_{(l,i) \in S(n,c,r)} B(e_l, 1/2^i).$$

Proof. [(i)⇒(ii)] Let $\alpha : \mathbb{N}^2 \to \mathbb{N}$ be an effective modulus of continuity of (f_n). Let $S \subset \mathbb{N} \times \mathbb{Q} \times \mathbb{Q}^+ \times \mathbb{N} \times \mathbb{N}$ be the set defined by

$$\begin{aligned} &(n, c, r, l, i) \in S \\ &\iff (\exists j \in \mathbb{N})(i = \alpha(n, j)\ \wedge\ c - r < f_n(e_l) - 1/2^j\ \wedge\ f_n(e_l) + 1/2^j < c + r). \end{aligned}$$

Since α is a recursive function and $(f_n(e_l))$ is a computable double sequence of real numbers, it follows immediately from the definition of S that S is a recursively enumerable set.

Let $(l,i) \in S(n,c,r)$. Then we have, for some $j \in \mathbb{N}$,

$$\begin{aligned} f_n(B(e_l, 1/2^i)) &= f_n(B(e_l, 1/2^{\alpha(n,j)})) \\ &\subset (f_n(e_l) - 1/2^j,\ f_n(e_l) + 1/2^j) \\ &\subset (c-r,\ c+r). \end{aligned}$$

Therefore, $B(e_l, 1/2^i) \subset f_n^{-1}((c-r,\ c+r))$. This shows $f_n^{-1}((c-r,\ c+r)) \supseteq \bigcup_{(l,i)\in S(n,c,r)} B(e_l, 1/2^i)$.

Suppose $x \in f_n^{-1}((c-r,\ c+r))$, i.e., $|f_n(x) - c| < r$. Then, for some $j \in \mathbb{N}$, it holds that $2/2^j \le r - |f_n(x) - c|$. For such a j, there exists an e_l such that $d(x, e_l) < 1/2^{\alpha(n,j)}$. Hence $|f_n(e_l) - f_n(x)| < 1/2^j$. By using $f_n(e_l) - 1/2^j < f_n(x)$, we obtain

$$f_n(e_l) + 1/2^j < f_n(x) + 2/2^j \le f_n(x) + r - |f_n(x) - c| \le c + r.$$

Analogously, by using $f_n(e_l) + 1/2^j > f_n(x)$, we obtain $f_n(e_l) - 1/2^j > c - r$. The conjunction of the obtained two inequalities implies $(n, c, r, l, \alpha(n,j)) \in S$. Therefore, $x \in B(e_l, 1/2^i)$ for some $(l,i) \in S(n,c,r)$. This shows $f_n^{-1}((c-r,\ c+r)) \subset \bigcup_{(l,i)\in S(n,c,r)} B(e_l, 1/2^i)$.

[(ii)⇒(i)] (*Sequential computability*) Suppose $(x_k) \in \mathcal{S}$. From (ii), we obtain

$$|f_n(x_k) - c| < 1/2^j \iff (\exists l)(\exists i)[(n, c, 1/2^j, l, i) \in S \ \wedge\ d(x_k, e_l) < 1/2^i].$$

Hence $\{(n,k,j,c) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N}\times\mathbb{Q} \mid |f_n(x_k) - c| < 1/2^j\}$ is a recursively enumerable set. Meanwhile, $(\forall n)(\forall k)(\forall j)(\exists c \in \mathbb{Q})\ |f_n(x_k) - c| < 1/2^j$ holds since $\mathbb{Q}$ is dense in $\mathbb{R}$. Therefore, there exists a computable triple sequence of rational numbers $(c_{n,k,j})$ such that $(\forall n)(\forall k)(\forall j)\ |f_n(x_k) - c_{n,k,j}| < 1/2^j$, i.e., $(f_n(x_k))_{n,k}$ is a computable double sequence of real numbers.

(*Effective uniform continuity*) Using Proposition 3.2, we obtain that

$$B(e_l, 1/2^i) = \bigcup_{\substack{l',i'\in\mathbb{N},\\ d(e_l,e_{l'})+2/2^{i'}<1/2^i}} B(e_{l'}, 1/2^{i'}).$$

It is clear that $d(e_l, e_{l'}) + 2/2^{i'} < 1/2^i$ is a recursively enumerable predicate of l, i, l', i'. It is also clear that $(\forall l)(\forall i)(\exists l')(\exists i')\ d(e_l, e_{l'}) + 2/2^{i'} < 1/2^i$ holds. Hence there exist recursive functions $\sigma, \rho : \mathbb{N}^3 \to \mathbb{N}$ such that for any $l, i \in \mathbb{N}$,

$$\begin{aligned} &\{(l,i,l',i') \in \mathbb{N}\times\mathbb{N}\times\mathbb{N}\times\mathbb{N} \mid d(e_l, e_{l'}) + 2/2^{i'} < 1/2^i\} \\ &\qquad = \{(l, i, \sigma(l,i,k), \rho(l,i,k)) \mid k \in \mathbb{N}\}. \end{aligned}$$

Therefore, for any $l, i \in \mathbb{N}$,

$$B(e_l, 1/2^i) = \bigcup_{k=0}^{\infty} B(e_{\sigma(l,i,k)}, 1/2^{\rho(l,i,k)}) \tag{1}$$

and

$$(\forall k) \quad d(e_l, e_{\sigma(l,i,k)}) + 2/2^{\rho(l,i,k)} < 1/2^i. \tag{2}$$

Since S is a recursively enumerable set, so is $\{(n, j, l, i, c) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} \times \mathbb{N} \times \mathbb{Q} \mid (n, c, 1/2^{j+1}, l, i) \in S\}$. On the other hand, since $M = \bigcup_{c \in \mathbb{Q}} f_n^{-1}((c - 1/2^{j+1},\ c + 1/2^{j+1}))$, it holds that

$$M = \bigcup_{c \in \mathbb{Q}} \bigcup_{(l,i) \in S(n,c,1/2^{j+1})} B(e_l, 1/2^i).$$

Hence $(\forall n)(\forall j)(\exists l)(\exists i)(\exists c \in \mathbb{Q})\ (n, c, 1/2^{j+1}, l, i) \in S$. Therefore, there exist recursive functions $\sigma', \rho' : \mathbb{N}^3 \to \mathbb{N}$ and a computable triple sequence of rational numbers $(c_{n,j,k})$ such that for any $n, j \in \mathbb{N}$,

$$\begin{aligned}&\{(n, j, l, i, c) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} \times \mathbb{N} \times \mathbb{Q} \mid (n, c, 1/2^{j+1}, l, i) \in S\} \\ &\qquad = \{(n, j, \sigma'(n, j, k), \rho'(n, j, k), c_{n,j,k}) \mid k \in \mathbb{N}\}.\end{aligned}$$

Thus, for any $n, j \in \mathbb{N}$,

$$M = \bigcup_{k=0}^{\infty} B(e_{\sigma'(n,j,k)}, 1/2^{\rho'(n,j,k)}) \tag{3}$$

and

$$(\forall k) \quad f_n(B(e_{\sigma'(n,j,k)}, 1/2^{\rho'(n,j,k)})) \subset (c_{n,j,k} - 1/2^{j+1},\ c_{n,j,k} + 1/2^{j+1}). \tag{4}$$

From (1) and (3), we obtain

$$M = \bigcup_{k=0}^{\infty} B(e_{\sigma(\sigma'(n,j,(k)_1^2),\rho'(n,j,(k)_1^2),(k)_2^2)}, 1/2^{\rho(\sigma'(n,j,(k)_1^2),\rho'(n,j,(k)_1^2),(k)_2^2)}).$$

Application of Lemma 3.3 yields that there exists a recursive function $\gamma : \mathbb{N}^2 \to \mathbb{N}$ such that

$$M = \bigcup_{k=0}^{\gamma(n,j)} B(e_{\sigma(\sigma'(n,j,(k)_1^2),\rho'(n,j,(k)_1^2),(k)_2^2)}, 1/2^{\rho(\sigma'(n,j,(k)_1^2),\rho'(n,j,(k)_1^2),(k)_2^2)}).$$

Let $\alpha : \mathbb{N}^2 \to \mathbb{N}$ be a recursive function defined by

$$\alpha(n,j) = \max_{k \le \gamma(n,j)} \rho(\sigma'(n,j,(k)^2_1), \rho'(n,j,(k)^2_1), (k)^2_2).$$

Suppose points $x, y \in M$ satisfy $d(x,y) < 1/2^{\alpha(n,j)}$. Then there exists some $k \le \gamma(n,j)$ such that

$$d(x, e_{\sigma(\sigma'(n,j,(k)^2_1),\rho'(n,j,(k)^2_1),(k)^2_2)}) < 1/2^{\rho(\sigma'(n,j,(k)^2_1),\rho'(n,j,(k)^2_1),(k)^2_2)}.$$

With such a k,

$$\begin{aligned}
&d(x, e_{\sigma'(n,j,(k)^2_1)}) \\
&\le d(e_{\sigma'(n,j,(k)^2_1)}, e_{\sigma(\sigma'(n,j,(k)^2_1),\rho'(n,j,(k)^2_1),(k)^2_2)}) + d(x, e_{\sigma(\sigma'(n,j,(k)^2_1),\rho'(n,j,(k)^2_1),(k)^2_2)}) \\
&< d(e_{\sigma'(n,j,(k)^2_1)}, e_{\sigma(\sigma'(n,j,(k)^2_1),\rho'(n,j,(k)^2_1),(k)^2_2)}) + 1/2^{\rho(\sigma'(n,j,(k)^2_1),\rho'(n,j,(k)^2_1),(k)^2_2)} \\
&< 1/2^{\rho'(n,j,(k)^2_1)} - 1/2^{\rho(\sigma'(n,j,(k)^2_1),\rho'(n,j,(k)^2_1),(k)^2_2)}.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
d(y, e_{\sigma'(n,j,(k)^2_1)}) &\le d(x, e_{\sigma'(n,j,(k)^2_1)}) + d(x,y) \\
&< 1/2^{\rho'(n,j,(k)^2_1)} - 1/2^{\rho(\sigma'(n,j,(k)^2_1),\rho'(n,j,(k)^2_1),(k)^2_2)} + 1/2^{\alpha(n,j)} \\
&\le 1/2^{\rho'(n,j,(k)^2_1)}.
\end{aligned}$$

Therefore, $x, y \in B(e_{\sigma'(n,j,(k)^2_1)}, 1/2^{\rho'(n,j,(k)^2_1)})$. Due to (4), this implies $f_n(x), f_n(y) \in (c_{n,j,(k)^2_1} - 1/2^{j+1},\ c_{n,j,(k)^2_1} + 1/2^{j+1})$. Hence $|f(x) - f(y)| < 1/2^j$. This shows that α is an effective modulus of continuity of (f_n). □

The condition (ii) in Lemma 3.4 is equivalent to δ_6-computability defined by Weihrauch in [7]. Lemma 3.4 shows that for a real-valued function on an effectively compact metric space, computability defined by Mori, Tsujii, and Yasugi in [5] and used in this paper is equivalent to δ_6-computability.

Now we are ready to show the main theorem.

Theorem 3.5 *Let $(M, d, \mathcal{S})$ be an effectively compact metric space. Let (f_n) be a computable sequence of real-valued functions on M and f a computable real-valued function on M. If f_n converges pointwise monotonically to f as $n \to \infty$, then f_n converges effectively uniformly to f.*

Proof. Let (e_l) be a computable sequence dense in M.

Let $U_{j,n} = (f_n - f)^{-1}((-1/2^j, 1/2^j))$. Since f_n converges pointwise to f, it holds that $(\forall j)\ M = \bigcup_{n=0}^{\infty} U_{j,n}$. Due to Lemma 3.4, there exists a recursively enumerable set $S \subset \mathbb{N}^4$ such that $(\forall j)(\forall n)\ U_{j,n} = \bigcup_{(i,l)\in S(j,n)} B(e_l, 1/2^i)$. From these two equalities, we obtain

$$(\forall j) \quad M = \bigcup_{(n,i,l)\in S(j)} B(e_l, 1/2^i).$$

This implies $(\forall j)\ S(j) \neq \emptyset$. Therefore, there exist recursive functions $\theta, \rho, \sigma : \mathbb{N}^2 \to \mathbb{N}$ such that $S = \{(j, \theta(j,k), \rho(j,k), \sigma(j,k)) \mid j,k \in \mathbb{N}\}$. Using these functions, we can rewrite the formula above as follows:

$$(\forall j) \quad M = \bigcup_{k=0}^{\infty} B(e_{\sigma(j,k)}, 1/2^{\rho(j,k)}).$$

Since M itself is a compact subset of M, application of Lemma 3.3 yields that there exists a recursive total function $\beta : \mathbb{N} \to \mathbb{N}$ such that

$$(\forall j) \quad M = \bigcup_{k=0}^{\beta(j)} B(e_{\sigma(j,k)}, 1/2^{\rho(j,k)}).$$

Since $B(e_{\sigma(j,k)}, 1/2^{\rho(j,k)}) \subset U_{j,\theta(j,k)}$, this implies $(\forall j)\ M = \bigcup_{k=0}^{\beta(j)} U_{j,\theta(j,k)}$. Since f_n converges monotonically to f, it holds that $U_{j,n} \subset U_{j,n'}$ if $n \leq n'$. Let $\alpha : \mathbb{N} \to \mathbb{N}$ be a recursive function defined by $\alpha(j) = \max_{k\leq\beta(j)} \theta(j,k)$. We have $(\forall j)(\forall n \geq \alpha(j))\ M = U_{j,n}$, which is equivalent to:

$$(\forall j)(\forall n \geq \alpha(j))(\forall x \in M) \quad |f_n(x) - f(x)| < 1/2^j.$$

Thus f_n converges effectively uniformly to f as $n \to \infty$. □

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