Some Löwner partial orders of Schur complements and Kronecker products of matrices

Jianzhou Liu

Department of Mathematics, Xiangtan University, Xiangtan, Hunan 411005, People's Republic of China

Received 10 May 1998; accepted 7 November 1998

Abstract

In this paper, we define generalized Schur complements of matrices. Further, combining Schur complements of matrices with Kronecker products of matrices, we study their Löwner partial order and obtain some important inequalities, that improve recent results. © 1999 Elsevier Science Inc. All rights reserved.

AMS classification: 15A15; 15A45

Keywords: Generalized Schur complement; Kronecker product; Löwner partial order

1. Introduction

The research of Kronecker products and Schur complements have obtained many important results respectively (cf. [1-6]). Recently, M. Fiedler and T.L. Markham [7] obtained some Löwner partial order of sum of Schur complements for some special type matrices. J.Z. Liu and L. Zhu [8] obtained some estimates of eigenvalues for Schur complements of $BAB^r$, where $B$ is a $n \times n$ complex matrix and $A$ is a positive definite Hermiltian matrix. In this paper, we shall study Löwner partial orders of Schur complements and Kronecker products of matrices and their applications.

| E-mail: icam@pdns.xtu.edu.cn |

0024-3795/99/$ - see front matter © 1999 Elsevier Science Inc. All rights reserved.

Pii: S 0 0 2 4 - 3 7 9 5 ( 9 8 ) 1 0 2 4 1 - 0
Let $C^{n \times n}$ denote the set of $m \times n$ complex matrices. Let $H_n$ denote the set of $n \times n$ Hermitian matrices, and let $H_n^+$ ($H_n^-$) denote the subset of consisting of positive semidefinite (positive definite) matrices. For $A, B \in H_n \geq 1$, we will write $A \preceq B$ if $B - A \in H_n^+$. Relation $\preceq$ is called the Löwner partial order. For $A \in C^{m \times n}$, we assume that the eigenvalues of $A$ are arranged so that $\lambda_1(A) \geq \cdots \geq \lambda_n(A)$. For $A \in C^{m \times n}$, we assume that the singular values of $A$ are arranged so that $\sigma_1(A) \geq \cdots \geq \sigma_{\min(m,n)}(A)$.

Let $N = \{1, 2, \ldots, n\}$. If $x \subset N$, $|x|$ equals the cardinality of $x$. Let $A \in C^{m \times n}$, for nonempty index sets $x \subset M = \{1, 2, \ldots, m\}$; $\beta \subset N$, we denote by $A(x, \beta)$ that submatrix of $A$ lying in the rows indicated by $x$ and the columns indicated by $\beta$, the submatrix of $A(x, \beta)$ (if $|x| \leq \min\{m, n\}$) is abbreviated to $A(x)$. Let $A = (a_{ij}) \in C^{m \times n}, B = (b_{ij}) \in C^{p \times q}$, we denote $(a_{ij}B)$ by $A \otimes B$, that is said to be the Kronecker product of $A$ and $B$.

**Definition 1.** Let $A \in C^{m \times n}, l = \min\{m, n\}, x \subset L = \{1, 2, \ldots, l\}$ and $x' = M - x$, $\beta' = N - x$, then

$$A/x = A(x', \beta') - A(x', x)A(x)A(x', \beta')^{-1} A(x, \beta')$$

is called the generalized Schur complement with respect to $A(x)$, where $A(x)$ is a nonsingular submatrix of $A$. (Note: In this paper, $A(x)$ in $A/x$ indicates that $A(x)$ is a nonsingular submatrix of $A$. No further mention is to be made of this.) We, of course, adopt the convention that $A/x = A$.

**Lemma 1** [1]. Let $A \in C^{l \times m}, B \in C^{m \times n}, C \in C^{p \times q}, D \in C^{q \times s}$, then

$$AB \otimes CD = (A \otimes C)(B \otimes D).$$  \hspace{1cm} (1)

**Lemma 2** [1]. Let $A \in C^{l \times m}, B \in C^{n \times p}$, then

$$(A \otimes B)^* = A^* \otimes B^*.$$  \hspace{1cm} (2)

If $l = m, n = p$, and both the $A$ and $B$ are nonsingular matrices, then

$$(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}.$$  \hspace{1cm} (3)

**Lemma 3** [1]. Let $A, B \in H_n^+, C, D \in H_n^+$, and $A \succeq B, C \succeq D$, then

$$A \otimes C \succeq B \otimes D.$$  \hspace{1cm} (4)

**Lemma 4** [2]. Let $A \in H_n^+$ and $x \subset N$, where $|x| = k$, then

$$\lambda_i(A) \geq \lambda_i(A/x) \geq \lambda_{i,k}(A), \quad i = 1, 2, \ldots, n - k.$$  \hspace{1cm} (5)
2. Partial orders

In this section, we obtain some Löwner partial orders of Schur complements and Kronecker products of matrices.

**Theorem 1.** Let \( A \in C^{m \times m} \) and \( B \in C^{n \times n} \) be nonsingular, \( x \subset M, \beta \subset N, \alpha' = M - x, \beta' = N - \beta, \) and let \( \gamma = \{1, 2, \ldots, mn\} - \gamma', \) where \( \gamma' = \{n(i - 1) + j : i \in \alpha', j \in \beta'\}. \) Then

\[
(A/x) \otimes (B/\beta) = (A \otimes B)/\gamma.
\]

**Proof.** For nonsingular \( A \) with nonsingular \( A(x), \) it is well known that \( A/x \) is also nonsingular and [5]

\[
(A/x)^{-1} = (A^{-1})(x').
\]

Hence, it is straightforward to deduce that

\[
(A/x)^{-1} \otimes (B/\beta)^{-1} = [(A^{-1})(x') \otimes ((B^{-1})(\beta'))] = (A^{-1} \otimes B^{-1})(\gamma').
\]

Taking inverse of both sides, by Eqs. (3) and (7), we obtain

\[
(A/x) \otimes (B/\beta) = [(A/x)^{-1} \otimes (B/\beta)^{-1}]^{-1} \quad \text{(by Eq. (3))}
\]

\[
= [(A^{-1} \otimes B^{-1})(\gamma')]^{-1} = [(A \otimes B)^{-1}(\gamma')]^{-1} \quad \text{(by Eq. (7))}
\]

\[
= (A \otimes B)/\gamma. \quad \square
\]

We give the following example to illustrate Theorem 1.

**Example 1.** Let

\[
A = \begin{pmatrix}
1 & 1 & 1 \\
0 & 1 & -1 \\
1 & 0 & -1
\end{pmatrix}, \\
B = \begin{pmatrix}
1 & 0 & -1 & 0 \\
0 & 2 & 0 & -1 \\
-1 & 0 & -1 & 0 \\
0 & 1 & 0 & 1
\end{pmatrix},
\]

\( x = \{1\}, \beta = \{3, 4\}, \) then \( x' = \{2, 3\}, \beta' = \{1, 2\}. \) By Theorem 1, we have

\[
\gamma' = \{5, 6, 9, 10\}, \quad \gamma = \{1, 2, 3, 4, 7, 8, 11, 12\}.
\]
By calculation, we have

\[
A/\gamma = \begin{pmatrix} 1 & -1 \\ -1 & -2 \end{pmatrix}, \quad E/\beta = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix},
\]

\[
(A \otimes B)/\gamma = (A/\gamma) \otimes (B/\beta) = \begin{pmatrix} 2 & 0 & -2 & 0 \\ 0 & 3 & 0 & -3 \\ -2 & \cdot & -4 & 0 \\ 0 & -3 & 0 & -6 \end{pmatrix}.
\]

Corollary 1. Let \( A_i \in C^{n \times n}, \forall i \in \mathbb{N} = \{1, 2, \ldots, m\} \) (\( i = 1, 2, \ldots, m \)). Then there exists \( \gamma \in \{1, 2, \ldots, \prod_{i=1}^{m} n_i\} \) with \( |\gamma| = \prod_{i=1}^{m} n_i - \prod_{i=1}^{m} (n_i - |\gamma_i|) \) such that

\[
(A_1/\gamma_1) \otimes \cdots \otimes (A_m/\gamma_m) = (A_1 \otimes \cdots \otimes A_m)/\gamma.
\]

Theorem 2. Let \( A \in C^{m \times n}, B \in C^{p \times q}, s = \min(m, n), t = \min(p, q), \) and \( \alpha \in S = \{1, 2, \ldots, s\}, \beta \in T = \{1, 2, \ldots, t\}, M = \{1, 2, \ldots, m\}, P = \{1, 2, \ldots, p\}. \alpha' = M - \alpha, \beta' = P - \beta, \) and \( \gamma = \{1, 2, \ldots, mp\} \setminus \gamma', \) where \( \gamma' = \{p(i - 1) + j: i \in \alpha', j \in \beta'\}. \) Then

\[
[(A/\gamma) \otimes (B/\beta)][(A/\gamma) \otimes (B/\beta)]^* \geq [(A \otimes B)(A \otimes B)^*]/\gamma.
\]

Proof. Let \( \alpha' = M - \alpha, \beta' = N - \alpha, \) then there exist two permutation matrices \( U \in C^{m \times m} \) and \( V \in C^{n \times n}, \) such that

\[
UAV^* = \begin{pmatrix} A(\alpha) & A(\alpha, \delta') \\ A(\alpha', \alpha) & A(\alpha', \delta') \end{pmatrix},
\]

\[
UAA^*U^* = \begin{pmatrix} (AA^*)(\alpha) & (AA^*)(\alpha, \alpha') \\ (AA^*)(\alpha, \alpha') & (AA^*)(\alpha') \end{pmatrix}.
\]

Let \( \tilde{\alpha} = \{1, 2, \ldots, |\alpha|\}, \tilde{\alpha'} = M - \tilde{\alpha}, \tilde{\delta'} = N - \tilde{\alpha}, \) then

\[
UAV^*/\tilde{\alpha} = A/\alpha. \quad UAA^*U/\tilde{\alpha} = (AA^*)/\alpha.
\]

Hence, we may assume that \( \alpha = \{1, 2, \ldots, |\alpha|\}, \alpha' = M - \alpha, \delta' = N - \alpha. \) Let

\[
X = -A(\alpha', \alpha)[A(\alpha)]^{-1},
\]

\[
Y = -[(AA^*)(\alpha', \alpha)][AA^*](\alpha)]^{-1}.
\]

then

\[
(X, I)A = (0, A/\alpha).
\]
we have

\[
(A/x)(A/x)^* = (0/A/x)(0/A/x)^* = (X \mathbf{1})AA^*(X \mathbf{1})^*
\]

\[
= (AA^*)(x') + X[(AA^*)(x, x')] + [(AA^*)(x', x)]X^*
\]

\[
+ X[(AA^*)(x)]X^*
\]

\[
= (AA^*/x + [(AA^*)(x, x)][(AA^*)(x)]^{-1}[(AA^*)(x, x')]
\]

\[
+ X[(AA^*)(x, x')] + [(AA^*)(x', x)]X^* + X[(AA^*)(x)]X^*
\]

\[
= (AA^*/x - Y[(AA^*)(x, x')] + X[(AA^*)(x, x')]
\]

\[
+ [(AA^*)(x', x)]X^* + X[(AA^*)(x)]X^*
\]

\[
= \begin{pmatrix}
(AA^* + (X - Y)[(AA^*)(x)][(AA^*)(x)]^{-1}[(AA^*)(x, x')]
\end{pmatrix}
\]

\[
+ [(AA^*)(x, x)][(AA^*)(x)]^{-1}[(AA^*)(x), x']X^* + X[(AA^*)(x)]X^*
\]

\[
= (AA^*/x - (X - Y)[(AA^*)(x)])Y' - Y[(AA^*)(x)]X^*
\]

\[
+ X[(AA^*)(x)]X^* = (AA^*/x + (X - Y)[(AA^*)(x)](X - Y)^*
\]

\[
\geq (AA^*)/x.
\]

Similarly, we can prove

\[
(B/\beta)(B/\beta)^* \geq (BB^*)/\beta.
\]

By Eqs. (1), (2), (4), (6), (10) and (11), we have

\[
[(A/x) \otimes (B/\beta)][(A/x) \otimes (B/\beta)]^*
\]

\[
= [(A/x) \otimes (B/\beta)][(A/x)^* \otimes (B/\beta)^*]
\]

(by Eq. (2))

\[
= [(A/x)(A/x)^*] \otimes [(B/\beta)(B/\beta)^*]
\]

(by Eq. (1))

\[
\geq [(AA^*)/x] \otimes [(BB^*)/\beta]
\]

(by Eqs. (10), (11) and (4))

\[
\geq [(AA^*) \otimes (BB^*)]/\gamma
\]

(by Eq. (6))

\[
= [(A \otimes B)(A \otimes B)^*]/\gamma.
\]

**Corollary 2.** Let \( A_i \in \mathbb{C}^{m_i \times n}, \) \( p_i = \min(m_i, n_i), \) and \( \alpha_i \subset P_i = \{1, 2, \ldots, p_i\}, \)

\((i = 1, 2, \ldots, l).\) Then there exists \( \gamma \subset \{1, 2, \ldots, \prod_{i=1}^m p_i\} \) with \( |\gamma| = \prod_{i=1}^m m_i - \prod_{i=1}^m (m_i - |\alpha_i|) \) such that

\[
[(A_1/\alpha_1) \otimes \cdots \otimes (A_l/\alpha_l)][(A_1/\alpha_1) \otimes \cdots \otimes (A_l/\alpha_l)]^*
\]

\[
\geq [(A_1 \otimes \cdots \otimes A_l)(A_1 \otimes \cdots \otimes A_l)^*/\gamma.
\]

(12)
3. Inequalities for singular values and eigenvalues of matrices

In this section, we obtain some inequalities for singular values and eigenvalues of Schur complements and Kronecker products of matrices.

**Theorem 3.** Assume the hypothesis of Theorem 2. Then

\[
\sigma_i[(A/\alpha) \otimes (B/\beta)] \geq \sigma_{i+|\alpha|+|\beta|}(A \otimes B), \quad i = 1, 2, \ldots, (m-|\alpha|)(p-|\beta|).
\]  

(13)

**Proof.** By Theorem 2 and Lemma 4, we have

\[
\sigma_i^2[(A/\alpha) \otimes (B/\beta)]
\]

\[
= \lambda_i\{[(A/\alpha) \otimes (B/\beta)][(A/\alpha) \otimes (B/\beta)]^*\}
\]

\[
\geq \lambda_i\{[(A \otimes B)(A \otimes B)^*/\gamma] \} \quad \text{(by Eq. (9))}
\]

\[
\geq \lambda_{i+|\alpha|+|\beta|}\{[(A \otimes B)(A \otimes B)]^*/\gamma\} \quad \text{(by Eq. (5)}
\]

\[
= \sigma_{i+|\alpha|+|\beta|}^2(A \otimes B).
\]

Setting \(B = (1), \beta = \phi\) in Eq. (13), we obtain

\[
\sigma_i(A/\alpha) \geq \sigma_{i+|\alpha|}(A), \quad (i = 1, 2, \ldots, m-|\alpha|).
\]

(14)

Setting \(A \in H_m\) in Eq. (14), then

\[
|\lambda_i(A/\alpha)| \geq |\lambda_{i+|\alpha|}(A)|, \quad (i = 1, 2, \ldots, m-|\alpha|).
\]

(15)

We give the following example to illustrate Theorem 3.

**Example 2.** Let all the assumption of Example 1 be satisfied. By calculation, we have

\[
\sigma_1[(A/\alpha) \otimes (B/\beta)] = \frac{3(1+\sqrt{13})}{2}, \quad \sigma_2[(A/\alpha) \otimes (B/\beta)] = 1 + \sqrt{13},
\]

\[
\sigma_3[(A/\alpha) \otimes (B/\beta)] = \frac{3\sqrt{13} - 1}{2}, \quad \sigma_4[(A/\alpha) \otimes (B/\beta)] = \sqrt{13} - 1,
\]

\[
\sigma_1(A \otimes B) = \sigma_2(A \otimes B) = \frac{\sqrt{3}(1 + \sqrt{13})}{2},
\]

\[
\sigma_3(A \otimes B) = \sigma_4(A \otimes B) = \sigma_5(A \otimes B) = \sigma_6(A \otimes B) = \sqrt{6},
\]

\[
\sigma_7(A \otimes B) = \frac{1 + \sqrt{13}}{2}, \quad \sigma_8(A \otimes B) = \sigma_9(A \otimes B) = \frac{\sqrt{3}(\sqrt{13} - 1)}{2},
\]

\[
\sigma_{10}(A \otimes B) = \sigma_{11}(A \otimes B) = \sqrt{2}, \quad \sigma_{12}(A \otimes B) = \frac{\sqrt{13} - 1}{2}.
\]
Thus,

\[
\begin{align*}
\sigma_1[(A/\alpha) \otimes (B/\beta)] &= \frac{3(1 + \sqrt{13})}{2} > \frac{\sqrt{3}(\sqrt{13} - 1)}{2} \\
&= \sigma_{1,3,1,4,1,1,2}(A \otimes B) = \sigma_{9}(A \otimes B), \\
\sigma_2[(A/\alpha) \otimes (B/\beta)] &= 1 + \sqrt{13} > \sqrt{2} = \sigma_{10}(A \otimes B), \\
\sigma_3[(A/\alpha) \otimes (B/\beta)] &= \frac{3(\sqrt{13} - 1)}{2} > \sqrt{2} = \sigma_{11}(A \otimes B), \\
\sigma_4[(A/\alpha) \otimes (B/\beta)] &= \sqrt{13} - 1 > \frac{\sqrt{13} - 1}{2} = \sigma_{12}(A \otimes B).
\end{align*}
\]

Acknowledgements

The author is grateful to the referee for helpful comments.

References