On boundary Nevanlinna–Pick interpolation for Carathéodory matrix functions

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Abstract

A matrix version of the boundary Nevanlinna–Pick interpolation problem in the class of Carathéodory matrix functions is considered. This matrix interpolation problem is reduced to a certain matrix trigonometric moment problem with specified constraints that the nonnegative matrix-valued measure has no mass distributions at a finite number of boundary points. Based on the use of recent results due to Bolotnikov and Dym and its reduction, we obtain solvability criteria for both the boundary Nevanlinna–Pick interpolation problem and the moment problem. A parameterized description of all the solutions of each of these two problems under consideration in the nondegenerate case is given as well.

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1. Introduction

Let $p$ be a given positive integer. By a $C^{p \times p}$-valued Carathéodory matrix function $F(z)$, one means a $p \times p$ matrix-valued function which is holomorphic in the open unit disc $D = \{z \in \mathbb{C}: |z| < 1\}$ and has a nonnegative real part there.
\[ F(z) + F(z)^* \geq 0, \quad z \in \mathbb{D}. \]

We will use the notation \( \mathscr{C}_p \) to designate the class of such \( \mathbb{C}^{p \times p} \)-valued Carathéodory matrix functions. Being continued to the exterior of \( \mathbb{D} \) by the symmetry \( F(z) = -F(\bar{z}^{-1})^*, \quad |z| > 1 \), it admits the Riesz–Herglotz representation:

\[
F(z) = D + \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} d\sigma(e^{i\theta}), \quad z \in \mathbb{D}, \tag{1.1}
\]

where \( D = -D^* = \text{Im} F(0) \) and \( \sigma \) is a nonnegative finite \( p \times p \)-matrix-valued measure on the unit circle \( T = \{z \in \mathbb{C} : |z| = 1\} \), which is essentially uniquely determined by the function \( F(z) \). (See, e.g., [2,25] for the scalar case and [21] for the matrix case.) In this paper, all the measures are always assumed to be nonnegative finite \( p \times p \)-matrix-valued measures on \( T \).

In [10] we considered the solution of the multiple Nevanlinna–Pick interpolation problem in the class \( \mathscr{C}_p \) along the so-called Toeplitz block vector approach, which allows us to reduce this interpolation problem to what amounts to a certain trigonometric matrix moment problem. In the present paper, we are primarily interested in a boundary version of the multiple Nevanlinna–Pick interpolation problem in the class \( \mathscr{C}_p \) with all the interpolation nodes situated on \( T \), which can be formulated as follows.

**Problem BNP:** Given \( n \) distinct points \( z_1, \ldots, z_n \) on \( T \), given \( n \) skew-Hermitian matrices \( Y_1, \ldots, Y_n \) of order \( p \) and \( n \) Hermitian matrices \( A_1, \ldots, A_n \) of order \( p \), find all the functions \( F(z) \in \mathscr{C}_p \) which have prescribed radial boundary limits

\[
F(z_i) := \lim_{r \to 1^-} F(rz_i) = Y_i, \quad i = 1, \ldots, n \tag{1.2}
\]

and prescribed radial derivatives

\[
F'(z_i) := \lim_{r \to 1^-} \frac{F(rz_i) - F(z_i)}{(r - 1)z_i} = -\overline{z_i} A_i, \quad i = 1, \ldots, n. \tag{1.3}
\]

Various boundary Nevanlinna–Pick interpolation problems have a long history. Nevanlinna [26] did at first the boundary Nevanlinna–Pick interpolation problem in the class \( \mathscr{S}_1 \) (consisting of scalar Schur functions, which are analytic and contractive in \( \mathbb{D} \)) with the help of a boundary version of the Schur algorithm. In 1930s, Krein [24] studied a boundary Nevanlinna interpolation problem in the class \( \mathscr{K}_1 \) (consisting of scalar Nevanlinna functions, which are analytic in the open upper-half complex plane \( \mathbb{C}^+ \) and have nonnegative imaginary parts there) with real and simple nodes only, by using a method of Riesz. In 1937, Kotelyanskii [20] considered specifically a more general boundary Nevanlinna–Pick problem in the same class involving both interior and boundary data. After that, such and similar boundary interpolation problems have been intensively studied and generalized in many directions, and a number of approaches have been presented. We refer to [22,23,4–9,17–19,32,28,29,11] for more information.

We now turn to Problem BNP. Of particular interests are the solvability criteria for Problem BNP and descriptions of its solutions. It is worth noting that a necessary and sufficient condition for Problem BNP to have a solution (see Theorem 4.1 below) can be deduced fortunately from Theorem 8.4 of [8] on the boundary Nevanlinna–Pick interpolation for the class \( \mathscr{S}_p \) of \( \mathbb{C}^{p \times p} \)-valued Schur matrix functions with the aid of a simple relation between the classes \( \mathscr{C}_p \) and \( \mathscr{S}_p \). Here \( S(z) \in \mathscr{S}_p \) if and only if \( S(z) \) is a \( \mathbb{C}^{p \times p} \)-valued function analytic in \( \mathbb{D} \) and contractive there.
The main goal of this paper is to show that the afore-cited Toeplitz block vector approach is also appropriate to the solution of Problem BNP by introducing the Toeplitz block vector \( \mathbf{c} = (C_0, C_1, \ldots, C_{n-1}) \) associated with Problem BNP and setting up a basic connection between the solutions of Problem BNP and that of the following truncated trigonometric matrix moment problem with specified constraints on the bounded measure in quite a simple manner.

**Problem TTM:** Given \( n \) distinct points \( z_1, \ldots, z_n \) on the unit circle as in Problem BNP, and given a sequence of \( p \times p \) matrices \( C_0, C_1, \ldots, C_{n-1} \) with \( C_0 = C_0^* \), which consists of the Toeplitz block vector \( \mathbf{c} \) of Problem BNP, find all the measures \( \tau \) subject to

\[
C_k = \frac{1}{2\pi} \int_0^{2\pi} e^{-ik\theta} \, d\tau(e^{i\theta}), \quad k = 0, \ldots, n-1
\]

and with no mass distributions at points \( z_1, \ldots, z_n \), i.e.,

\[
\tau(\{z_i\}) = 0, \quad i = 1, \ldots, n.
\]

Observe that Problem TTM, in essence, is nothing other than a certain Carathéodory matrix coefficient problem, therefore it can be viewed as a multiple Nevanlinna–Pick interpolation problem in the class \( \mathcal{C}_p \) for which all interpolation nodes coincide with a single point, that is, the original point.

Thanks to the connection cited between Problems BNP and TTM (see Theorem 3.5), one of descriptions of all the solutions of Problem TTM in the nondegenerate case, which involves orthogonal matrix polynomials with respect to the unit circle \( T \) (see, e.g., \([3,14]\)) in the theory of moments enables one to get the same result but for Problem BNP (see Theorems 5.5 and 5.7). However, the connection is not confined to this and the advantage brought about by it is mutual since much of results in Problem BNP can serve as a starting point for the solution of Problem TTM and even more general theory of moments. The discussion of this side of the question is extremely interesting. As an example, in Section 4, we shall show how a solvability criterion for Problem TTM may be derived by means of that for Problem BNP (see Theorem 4.4).

In the scalar case \( p = 1 \), Problem BNP and Problem TTM have been considered in \([32]\). With detailed analysis of the problems originally studied by Pick \([30]\) and Nevanlinna \([27]\), Sarason posed a natural boundary version of the classical Nevanlinna–Pick interpolation problem for functions in the class \( \mathcal{C}_1 \) (i.e., Problem BNP with \( p = 1 \)), and proved that it can be reduced to a classical truncated trigonometric moment problem with finitely many constraints on the positive measures (i.e., Problem TTM with \( p = 1 \)). So the main results of this paper partially generalize those results established by Sarason in \([32]\) to the matricial case. Furthermore, there are also some results which seem to be new, especially the parameterized description of all the solutions of Problem BNP mentioned in terms of a linear fractional transformation of the \( \mathbb{C}^{p \times p} \)-valued Schur matrix functions in a certain special form occurring in Theorem 5.7 below.

It should be noted that Problem BNP is an interesting variant of the well known boundary Nevanlinna–Pick interpolation problem for the class \( \mathcal{C}_p \), referred to as Problem BNP', which is the same as Problem BNP, except with the equalities (1.3) therein replaced by the inequalities:

\[
-z_i F'(z_i) = \lim_{r \to 1^-} \frac{F(rz_i) - F(z_i)}{1 - r} \leq A_i, \quad i = 1, \ldots, n.
\]
It is well known that Problem BNP′ has a solution if and only if the Pick block matrix
\[
P = (P_{ij})_{i,j=1}^n, \quad \text{where } P_{ij} = \begin{cases} 
A_i, & \text{if } i = j, \\
\frac{Y_i + Y_j^*}{1 - z_i z_j}, & \text{if } i \neq j
\end{cases}
\]  
(1.5)
is Hermitian positive semidefinite (see, e.g., [4–6]).

On the other hand, Problem TTM with no constraints (1.4) on the measures \(\tau\), referred to as Problem TTM′, has been considered and completely solved by different authors using various approaches (see, e.g., [2,25] for the scalar case and [1,10,15,16,21] for the matricial case). Furthermore, as is well known, Problem TTM′ is solvable if and only if the Toeplitz block matrix built upon the Toeplitz block vector \(\mathbf{c} = (C_0, C_1, \ldots, C_{n-1})\) of Problem BNP
\[
T = (C_{-i})_{i,j=0}^{n-1} \quad (C_{-i} = C_i^*)
\]  
(1.6)
is Hermitian positive semidefinite. This implies that \(T \succeq 0\) is a necessary condition for Problem TTM to have a solution, but it is not sufficient, even in the scalar case. We will show actually in Section 3 that Problem BNP is solvable if and only if its associated TTM problem is solvable, and that there is a one-to-one correspondence between the solutions to Problem BNP and the solutions to its associated Problem TTM when they exist.

We point out that the Toeplitz block vector approach can also be applied to establish a similar connection between Problem BNP′ and Problem TTM′. In this paper, we will lay out the corresponding results for both Problem BNP′ and Problem TTM′ without proofs (see Section 3.2 below).

The outline of the paper is as follows. The paper consists of five sections. This introduction is the first. In Section 2, we introduce the Toeplitz block vector of Problem BNP and show that the Pick block matrix \(P\) of Problem BNP defined by Eq. (1.5) is congruent to a certain Toeplitz block matrix \(T\) defined by Eq. (1.6). With that Toeplitz block vector, there is associated the Problem TTM of Problem BNP. An intrinsic one-to-one correspondence between the solutions to Problem BNP and the solutions to its associated Problem TTM is established in Section 3. Based on the use of recent results due to Bolotnikov and Dym, we establish solvability criteria for both Problem BNP and Problem TTM in Section 4. The last section, Section 5, is devoted to parameterized descriptions of all the solutions of both Problem TTM and Problem BNP in the nondegenerate case.

2. Relation between Pick and Toeplitz block matrices

This section is partitioned into two parts. In the first we introduce the Toeplitz block vector of Problem BNP in terms of two Laurent (matrix) polynomials. In the second one, we will show that the Pick block matrix \(P\) of Problem BNP defined by Eq. (1.5) is congruent to a certain Toeplitz block matrix \(T\) of Problem TTM determined uniquely by the Toeplitz block vector of Problem BNP.

2.1. The Toeplitz block vector of Problem BNP

Let \(P(z)\) be the (unique) matrix polynomial of degree \(2n - 1\) at most, subject to \(P(z_i) = z_i^n Y_i\) and \(P'(z_i) = z_i^{n-1}(n Y_i - A_i)\) \((i = 1, \ldots, n)\). It is readily verified that the Laurent matrix polynomial \(L(z)\) of the form
\[
L(z) = z^{-n} P(z) = \sum_{i=-n}^{n-1} L_i z^i, \quad L_i \in \mathbb{C}^{p \times p}
\]
satisfies the interpolation conditions (1.2) and (1.3). Put now
\[
\Omega(z) = \frac{1}{2} \left( L(z) - L\left( \frac{1}{z} \right) \right) = \sum_{i=-n}^{n} \Omega_i z^i,
\]
(2.1)
\[
A(z) = \prod_{i=1}^{n} (z - z_i) \left( \frac{1}{z} - \bar{z}_i \right) = \sum_{i=-n}^{n} A_i z^i.
\]
We check easily that
\[
A(z) = A(\bar{z}^{-1})^* \quad \text{(or equivalently, } A_{-i} = A_i^*, \forall i) \text{ for } z \neq 0, \text{ and } \Omega(z) \text{ is a symmetric (with respect to } T) \text{ Laurent matrix polynomial: } \Omega(z) = -\Omega(\bar{z}^{-1})^* \quad \text{(or equivalently, } \Omega_{-i} = -\Omega_i^*, \forall i) \text{ for } z \neq 0 \text{ and subject to the interpolation conditions (1.2) and (1.3). Let the power series expansion of the rational matrix function } \Omega(z)/A(z) \text{ at } z = 0 \text{ be of the form:}
\]
\[
\frac{\Omega(z)}{A(z)} = -C_0 - 2C_1 z - \cdots - 2C_{n-1} z^{n-1} - 2C_n z^n - \cdots
\]
(2.2)
in which $C_0 = C_0^*$ obviously. In the sequel, we refer to the block vector $c = (C_0, C_1, \ldots, C_{n-1})$ with $C_0 = C_0^*$ and $C_k \in \mathbb{C}^{p \times p}$ as the Toeplitz block vector of Problem BNP (or Problem BNP'), compared with the Hankel block vector of the matricial Nevanlinna–Pick interpolation problem in the Nevanlinna class (see, e.g., [12]). It will play an essential role in our analysis throughout.

With the Toeplitz block vector $c = (C_0, C_1, \ldots, C_{n-1})$ and interpolation nodes $z_i (1 \leq i \leq n)$ of Problem BNP (or Problem BNP'), we introduce a certain Problem TTM (or Problem TTM', resp.) as indicated in Section 1, which is called the associated Problem TTM (or Problem TTM', resp.) of Problem BNP (or Problem BNP', resp.). We will see that there exist certain intrinsic connections of Problem BNP (or Problem BNP') with its associated Problem TTM (or Problem TTM').

2.2. Congruent relation between the Pick and Toeplitz block matrices

To show a close relation between the Pick block matrix of Problem BNP (or Problem BNP') and the Toeplitz block matrix of its associated Problem TTM (or Problem TTM'), we need to have some notation introduced in [10]. Let
\[
a(z) = \prod_{i=1}^{n} (z - z_i) = z^n + \sum_{i=0}^{n-1} a_i z^i, \quad a_i(z) = \frac{a(z)}{z - z_i}, \quad i = 1, \ldots, n,
\]
\[
W_i = \left( \frac{a_i^{(k)}(0)}{k!} \right)_{k=0}^{n-1} \in \mathbb{C}^{1 \times n}, \quad i = 1, \ldots, n
\]
(2.3)
\[
W = \begin{pmatrix} W_1 \\ \vdots \\ W_n \end{pmatrix} \in \mathbb{C}^{n \times n}.
\]
It is known (see, e.g., [13,33]) that $W$ is a nonsingular matrix.

Further let $\mathcal{L}$ be a linear operator from $\mathcal{H}$ into $\mathbb{C}^{p \times p}$, where $\mathcal{H}$ stands for the space of all rational scalar-valued functions of the complex variable $z$. More precisely, if a $R(z) \in \mathcal{H}$ has the power series expansions at $z = 0$ and at $z = \infty$, respectively:
\[ R(z) = \sum_{s=-\infty}^{\infty} R_s z^s \quad (z \to 0), \quad R(z) = \sum_{s=-\infty}^{\infty} \tilde{R}_s z^s \quad (z \to \infty), \]  
(2.4)

where the sets \{s \leq 0 : R_s \neq 0\} and \{s \geq 0 : \tilde{R}_s \neq 0\} are both finite, we define its imagine in \(C^p \times \bar{p}\) by putting

\[ L \{ R(z) \} = \frac{1}{2} C_0 (R_0 + \tilde{R}_0) + \sum_{s=1}^{n} (\tilde{R}_s C_s^* + R_{-s} C_s), \]  
(2.5)

where the \(C_s\) are the same as in (2.2), so that, in particular, \(L \{ z^k \} = C_k^*\) and \(L \{ z^{-k} \} = C_k\) for \(k = 0, 1, \ldots, n\).

**Lemma 2.1** [10]. Let \(A(z), \Omega(z)\) and \(\mathcal{L}\) be defined by Eqs. (2.1), (2.4) and (2.5), respectively. Then

\[ \mathcal{L} \left\{ \frac{z + \lambda}{z - \lambda} A(z) \right\} = \Omega(\lambda). \]

The next theorem shows that the Pick block matrix \(P\) of Problem BNP (or Problem BNP') defined by Eq. (1.5) is congruent to the Toeplitz block matrix \(T\) of the form (1.6) built on the Toeplitz block vector \(c = (C_0, C_1, \ldots, C_{n-1})\) of Problem TTM (or Problem TTM').

**Theorem 2.2.** Let \(W\) be defined by Eq. (2.3), and \(P\) be the Pick block matrix of Problem BNP (or Problem BNP'). Then

\[ P = 2(W \otimes I_p) T (W \otimes I_p)^*, \]  
(2.6)

where \(T = (C_{j-i})_{i,j=0}^{n-1}\) with \(C_{-k} = C_k^*\) is the Toeplitz block matrix of Problem TTM (or Problem TTM') and the symbol “\(\otimes\)” denotes the tensor product of two matrices.

**Proof.** Observe that the coefficients of the power series expansions for \(\frac{z + \lambda}{z - \lambda} A(z)\) at either \(z = 0\) or \(z = \infty\) are all polynomials in \(\lambda\). This fact and the linearity of \(\mathcal{L}\) imply that we can interchange \(\mathcal{L}\) with the operation of differentiating. Thus, by Lemma 2.1 the diagonal block entries \(P_{ii} (1 \leq i \leq n)\) of the Pick block matrix \(P\) can be rewritten as follows:

\[ P_{ii} = \Delta_i = -\frac{1}{z_i} \mathcal{L}'(z_i) = -\frac{1}{z_i} \mathcal{L}' \left\{ \frac{2z}{(z - z_i)^2} A(z) \right\} = 2 \mathcal{L} \left\{ a_i(z) \frac{1}{z_i} \right\} \]

\[ = 2 \left\{ \begin{pmatrix} a_i^{(0)}(0) & \cdots & a_i^{(n-1)}(0) \\ (n-1)! \end{pmatrix} \otimes I_p \right\} \left( \mathcal{L}'(z^{u-v}) \right)_{u,v=0}^{n-1} \]

\[ \times \begin{pmatrix} \underbrace{a_i^{(0)}(0)}_{(n-1)!} \\ \vdots \\ \underbrace{a_i^{(n-1)}(0)}_{(n-1)!} \end{pmatrix} \otimes I_p. \]  
(2.7)

Now we consider the nondiagonal block entries \(P_{ij}\) of the Pick block matrix \(P\). For each pair of \(i, j (1 \leq i, j \leq n)\) and \(i \neq j\), by Lemma 2.1 again, we have
\[
P_{ij} = \frac{Y_i + Y_j^*}{1 - z_i \bar{z}_j} = \frac{\Omega(z_i) - \Omega(\bar{z}_j^{-1})}{1 - z_i \bar{z}_j} = \mathcal{C} \left\{ \frac{z + z_i}{z - z_i} A(z) - \frac{z + z_j^{-1}}{z - z_j^{-1}} A(z) \right\}
\]

\[
= 2 \mathcal{C} \left\{ \frac{A(z)}{z - z_i} \left( \frac{1}{z - \bar{z}_j} \right) \right\} = 2 \mathcal{C} \left\{ a_i(z) a_j^* \left( \frac{1}{z} \right) \right\}
\]

\[
= 2 \left\{ \left( a_i^{(0)}(0), \ldots, a_i^{(n-1)}(0) \right) \otimes I_p \right\} \left( \mathcal{C} \left\{ z^{u-v} \right\} \right)_{u,v=0}^{n-1}
\]

\[
\times \left\{ \left( a_j^{(0)}(0), \ldots, a_j^{(n-1)}(0) \right) \otimes I_p \right\}.
\tag{2.8}
\]

We deduce from Eqs. (2.7) and (2.8) that

\[
P_{ij} = 2(W_i \otimes I_p)(C_{j-i})_{i,j=0}^{n-1}(W_j \otimes I_p)^*, \quad 1 \leq i, j \leq n.
\]

This leads to (2.6) immediately. Then we complete the proof of Theorem 2.2. □

From Theorem 2.2 and the nonsingularity of \( W \), we conclude that the Pick block matrix \( P \) of Problem BNP (or Problem BNP') is Hermitian positive definite (semidefinite) if and only if the Toeplitz block matrix \( T \) of Problem TTM (or Problem TTM') is positive definite (semidefinite). Thus, in particular, Problem BNP' has a solution if and only if its associated Problem TTM' has a solution. In the next section, we will present an intrinsic one-to-one correspondence between their solutions.

3. Connections between the solutions of Problem BNP and its associated Problem TTM

The object of this section is to establish an intrinsic one-to-one correspondence between the solutions to Problem BNP and the solutions to its associated Problem TTM when they exist.

To prove the main theorem of this section, we need the following two lemmas. The first is a matrix version of Lemma 1 in [32], which is originated with Riesz [31] and characterizes when \( F(z) \in \mathcal{C}_p \) has an angular derivative at \( c \in T \) with \( F(c) \) skew Hermitian.

**Lemma 3.1.** Suppose that \( F(z) \in \mathcal{C}_p \) has the Riesz–Hergoltz representation (1.1). The function \( F(z) \) has an angular derivative at \( z = c \) on \( T \) with \( F(c) \) skew Hermitian if and only if

\[
\int_0^{2\pi} |e^{i\theta} - c|^{-2} \text{tr} \sigma(e^{i\theta}) < +\infty.
\]

In this case, \(-c F'(c) = \frac{1}{\pi} \int_0^{2\pi} |e^{i\theta} - c|^{-2} \text{tr} \sigma(e^{i\theta}) \geq 0.\)

The proof of Lemma 3.1 is much the same as the proof of Lemma 1 of [32], we omit it here.
The second lemma is established by Bolotnikov and Dym [8, Lemma 7.1], which plays also an important role later on.

**Lemma 3.2.** Suppose that $F(z) \in \mathcal{C}_p$ has the Riesz–Herglotz representation (1.1) and that $z = c$ is an arbitrary point on $T$. Then the radial boundary limits

$$\lim_{r \to 1^-} (1 - r)^{k+1} F^{(k)}(rc) = \frac{k!}{\pi c^k} \sigma([c])$$

exist for all integers $k \geq 0$.

The last lemma implies that if the measure $\sigma$ has no mass distribution at point $z = c$ then

$$\lim_{r \to 1^-} (1 - r)^{k+1} F^{(k)}(rc) = 0$$

for all integers $k \geq 0$.

### 3.1. Connections between solutions of Problems BNP and TTM

For a given measure $\sigma$, the Herglotz integral of $\sigma$ is defined by

$$H_\sigma(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} d\sigma(e^{i\theta}), \quad z \in D.$$ 

By Eq. (1.1), a $C^{p \times p}$-valued function $F(z)$ belongs to the class $\mathcal{C}_p$ if and only if

$$F(z) = D + H_\sigma(z), \quad z \in D$$

in which $D$ is a certain $p \times p$ skew Hermitian matrix and $\sigma$ is a suitable measure. In virtue of Lemma 3.2, we can prove the following:

**Lemma 3.3.** Let $\tau$ be a solution of Problem TTM$'$ and let $\Phi(z) = A(z) H_\tau(z)$. Then $\Phi(z)$ has the radial boundary limits at all $z_j$

$$\Phi(z_j) := \lim_{r \to 1^-} \Phi(rz_j) = 0,$$

$$\Phi'(z_j) := \lim_{r \to 1^-} \Phi'(rz_j) = \frac{1}{\pi z_j} \tau([z_j]) \prod_{i \neq j} |z_j - z_i|^2, \quad j = 1, \ldots, n. \quad (3.1)$$

Moreover, if $\tau$ is a solution of Problem TTM, then

$$\Phi(z_j) := \lim_{r \to 1^-} \Phi(rz_j) = 0, \quad \Phi'(z_j) := \lim_{r \to 1^-} \Phi'(rz_j) = 0, \quad j = 1, \ldots, n. \quad (3.2)$$

**Proof.** Observe that for each $j$ ($1 \leq j \leq n$) we can rewrite $A(z)$ as in the form $A(z) = A_j(z)(z - z_j)^2$, in which

$$A_j(z) = -\frac{1}{z z_j} \prod_{i \neq j} (z - z_i) \left( \frac{1}{z} - \bar{z}_i \right)$$

is analytic on $T$. Then by Lemma 3.2 (with $F(z) = H_\tau(z) \in \mathcal{C}_p$, $c = z_j$ and $k = 0, 1$) we have

$$\Phi(rz_j) = A_j(rz_j)z_j^2(1 - r)^2 H_\tau(rz_j) \to 0 \quad (r \to 1^-),$$
\[
\Phi'(rz_j) = A'_j(rz_j)z_j^2(1 - r)^2H_\tau(rz_j) - 2A_j(rz_j)z_j(1 - r)H_\tau(rz_j) \\
+ A_j(rz_j)z_j^2(1 - r)^2H'_\tau(rz_j) \\
\rightarrow -\frac{1}{\pi}A_j(z_j)z_j\tau(|z_j|) = \frac{1}{\pi z_j}\tau(|z_j|)\prod_{i \neq j}|z_j - z_i|^2 \quad (r \to 1^-)
\]

for \( j = 1, \ldots, n \). Relations (3.2) follow from (3.1). Hence we complete the proof. □

The next theorem presents an intrinsic one-to-one correspondence between the solutions to Problem BNP and the solutions to its associated Problem TTM when they exist.

**Theorem 3.4.** Problem BNP is solvable if and only if its associated Problem TTM is solvable. Furthermore, the formula

\[
F(z) = \Omega(z) + A(z)H_\tau(z) \quad (z \in D)
\]

realizes a one-to-one correspondence between the solutions \( F(z) \) to Problem BNP and the solutions \( \tau \) to its associated Problem TTM.

**Proof.** Let first \( \tau \) be an arbitrary solution to Problem TTM, and \( H_\tau(z) \) be the Herglotz integral of \( \tau \). Then the power series expansion of \( H_\tau(z) \) at \( z = 0 \) is of the form:

\[
H_\tau(z) = C_0 + 2C_1z + \cdots + 2C_{n-1}z^{n-1} + \cdots \quad (z \to 0).
\]

Then the function \( F(z) \) defined as in (3.3) satisfies interpolation conditions (1.2) and (1.3) by Lemma 3.3. It remains to show that \( F(z) \in C_p \). By Eqs. (3.2) and (2.2),

\[
F(z) = A(z)\left(\frac{\Omega(z)}{A(z)} + H_\tau(z)\right) = A(z)O(z^n),
\]

so that the function \( F(z) \) is analytic in \( D \). Put now

\[
\tilde{F}(z) = \Omega(z) - \frac{1}{2\pi} \int_0^{2\pi} A(z) - A(e^{i\theta}) \frac{z - e^{i\theta}}{z - e^{i\theta}}(e^{i\theta} + z) d\tau(e^{i\theta}) \quad (3.5)
\]

and

\[
\sigma(e^{i\theta}) = \int_0^{\theta} A(e^{i\phi}) d\tau(e^{i\phi}) = \int_0^{\theta} \prod_{k=1}^{n} |e^{i\phi} - z_k|^2 d\tau(e^{i\phi}).
\]

Then \( \sigma \) is a measure and

\[
F(z) = \tilde{F}(z) + \frac{1}{2\pi} \int_0^{2\pi} e^{i\theta} + z \frac{d\sigma(e^{i\theta})}{e^{i\theta} - z},
\]

and thus \( \tilde{F}(z) \) is analytic in \( D \) by Eqs. (3.6) and (2.1), and has the form

\[
\tilde{F}(z) = \sum_{i=-n}^{n} \tilde{F}_iz^i, \quad \tilde{F}_i \in C^{p \times p}.
\]

A straightforward evaluation of Eq. (3.5) yields that \( -\tilde{F}(\bar{z}^{-1})^* = \tilde{F}(z) \), so that \( \tilde{F}_i = -\tilde{F}_i^* = 0 \) for all \( i \neq 0 \) in Eq. (3.7), and thus \( \tilde{F}(z) = \tilde{F}_0 \) is a skew Hermitian matrix. Hence, from Lemma 3.2 and Eq. (1.1), we deduce immediately that \( F(z) \in C_p \).
Conversely, let $F(z) \in \mathcal{C}_p$ be a solution to Problem BNP with an integral representation of the form (1.1). Define

$$H(z) = \frac{F(z) - \Omega(z)}{A(z)} = \frac{z^n F(z) - z^n \Omega(z)}{z^n A(z)},$$

which is analytic in $D$. Note that $z^n F(z) = O(z^n)$ ($z \to 0$) and the polynomial $z^n A(z)$ does not vanish at $z = 0$, then by Eq. (2.2) the function $H(z)$ defined by Eq. (3.8) has a power series expansion of the form (3.4).

Next, we prove that $H(z) = H_\tau(z) \in \mathcal{C}_p$, where $\tau$ is a certain solution of Problem TTM. Define a $\mathcal{C}_p \times \mathcal{C}_p$-valued function $\tau$ on $T$ by

$$d\tau(e^{i\theta}) = \begin{cases} A(e^{i\theta})^{-1} d\sigma(e^{i\theta}), & \text{if } e^{i\theta} \notin \{z_1, \ldots, z_n\}, \\ 0, & \text{if } e^{i\theta} \in \{z_1, \ldots, z_n\}. \end{cases}$$

(3.9)

By Lemma 3.1 we have

$$\int_0^{2\pi} |e^{i\theta} - z_i|^{-2} \text{tr} d\sigma(e^{i\theta}) < +\infty \quad (1 \leq i \leq n)$$
or equivalently,

$$\int_0^{2\pi} \prod_{i=1}^n |e^{i\theta} - z_i|^{-2} \text{tr} d\sigma(e^{i\theta}) = \int_0^{2\pi} \text{tr} d\tau(e^{i\theta}) < +\infty.$$

Note that $A(e^{i\theta}) > 0$ for all $e^{i\theta} \neq z_i$, the last formula implies that $\tau$ defined by (3.9) is actually a measure on $T$.

Let now

$$\tilde{\Omega}(z) = D + \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} (A(e^{i\theta}) - A(z)) d\tau(e^{i\theta}),$$

then

$$F(z) = \tilde{\Omega}(z) + A(z) H_\tau(z).$$

(3.10)

In views of Eqs. (3.8) and (3.10), we obtain

$$H(z) = \frac{\tilde{\Omega}(z) - \Omega(z)}{A(z)} + H_\tau(z).$$

It follows readily from the last equality that $\tilde{\Omega}(z)$ is an interpolant of the conditions (1.2) and (1.3), and thus $\tilde{\Omega}(z) - \Omega(z) = i\alpha A(z)$ for a suitable $\alpha = \alpha^* \in \mathcal{C}_p \times \mathcal{C}_p$. Observe that both $H(0)$ and $H_\tau(0)$ are Hermitian matrices, then $\alpha = 0$, and therefore

$$H(z) = H_\tau(z).$$

This implies that $\tau$ is a solution of Problem TTM’. Moreover, it follows from Eq. (3.9) that the $\tau$ has no mass distributions at points $z_i$ ($1 \leq i \leq n$), and thus $\tau$ is also a solution of Problem TTM. Then the proof is complete. □

From the proof of Theorem 3.4, we can obtain a precise relation between the solutions $F(z)$ to Problem BNP and the solutions $\tau$ to the associated Problem TTM.

**Theorem 3.5.** If $F(z)$ is an arbitrary solution to Problem BNP which has the Riesz–Hergoltz representation (1.1), then the measure $\tau$ defined by Eq. (3.9) is a solution to the associated Problem TTM.
Conversely, if \( \tau \) is an arbitrary solution to the associated Problem TTM, then the function \( F(z) \in \mathbb{C}_p \) given by Eq. (1.1) is a solution to Problem BNP, in which

\[
D = \text{Im} \left\{ A_{-n} \left( \frac{1}{\pi} \int_0^{2\pi} e^{-in\theta} \, d\tau(e^{i\theta}) - 2C_n \right) \right\},
\]

\[
d\sigma(e^{i\theta}) = A(e^{i\theta}) \, d\tau(e^{i\theta}),
\]

where \( A_{-n} \) and \( C_n \) are the same as in Eqs. (2.1) and (2.2), respectively.

**Proof.** To end the proof, the only thing need to be done is to verify the first formula in Eq. (3.11). From the first part of the proof of Theorem 3.4, we have \( D = \tilde{F}_0 \), the constant term of Laurent matrix polynomial \( \tilde{F}(z) \). On the other hand, since \( \tau \) is a solution of Problem TTM, the function \( \tilde{F}(z) \) defined by Eq. (3.5) can be rewritten as

\[
\tilde{F}(z) = A(z) \left( \frac{\Omega(z)}{A(z)} + \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \, d\tau(e^{i\theta}) \right) - \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} A(e^{i\theta}) \, d\tau(e^{i\theta})
\]

\[
= A(z) \left\{ z^n \left( -2C_n + \frac{1}{\pi} \int_0^{2\pi} e^{-in\theta} \, d\tau(e^{i\theta}) + o(z^n) \right) \right\}
\]

\[
- \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} A(e^{i\theta}) \, d\tau(e^{i\theta})
\]

\[
= A_{-n} \left( \frac{1}{\pi} \int_0^{2\pi} e^{-in\theta} \, d\tau(e^{i\theta}) - 2C_n \right) + \frac{1}{2\pi} \int_0^{2\pi} A(e^{i\theta}) \, d\tau(e^{i\theta}) + o(1).
\]

It follows from the last equation that

\[
\tilde{F}_0 = A_{-n} \left( \frac{1}{\pi} \int_0^{2\pi} e^{-in\theta} \, d\tau(e^{i\theta}) - 2C_n \right) + \frac{1}{2\pi} \int_0^{2\pi} A(e^{i\theta}) \, d\tau(e^{i\theta}).
\]

Furthermore, since \( \tilde{F}_0^* = -\tilde{F}_0 \), that is, \( \tilde{F}_0 \) is a skew Hermitian matrix, we have

\[
D = \tilde{F}_0 = i \text{Im} \tilde{F}_0 = i \text{Im} \left\{ A_{-n} \left( \frac{1}{\pi} \int_0^{2\pi} e^{-in\theta} \, d\tau(e^{i\theta}) - 2C_n \right) \right\},
\]

as desired. Hence the theorem is proved. \( \square \)

### 3.2. Connections between solutions of Problem BNP' and its associated Problem TTM'

By a similar argument as that of Theorem 3.4, we can prove that the solvability of Problem BNP' and that of the associated Problem TTM' are equivalent. Moreover, there exists a one-to-one correspondence between the solutions \( F(z) \) to Problem BNP' and the solutions \( \tau \) to the associated Problem TTM'.

**Corollary 3.6.** Problem BNP' is solvable, if and only if its associated Problem TTM' is solvable. Furthermore, the formula (3.3) realizes a one-to-one correspondence between the solutions \( F(z) \) to Problem BNP' and the solutions \( \tau \) to its associated Problem TTM'.
The following result is paralleled with Theorem 3.5 but for Problem BNP′ and its associated Problem TTM′.

**Corollary 3.7.** If \( F(z) \) is an arbitrary solution to Problem BNP′ which has the integral representation (1.1), then the measure \( \tau \) defined by

\[
d\tau(e^{i\theta}) = \begin{cases} A(e^{i\theta})^{-1}d\sigma(e^{i\theta}), & \text{if } e^{i\theta} \notin \{z_1, \ldots, z_n\}, \\ \pi(A_j + z_j F'(z_j)) \prod_{i \neq j} |z_j - z_i|^{-2}, & \text{if } e^{i\theta} = z_j, j = 1, \ldots, n \end{cases}
\]

is a solution to the associated Problem TTM′.

Conversely, if \( \tau \) is an arbitrary solution to the associated Problem TTM′, then the function \( F(z) \in \mathbb{C}^p \) given by Eq. (1.1) is a solution to Problem BNP′, in which \( D, \sigma \) is defined by Eq. (3.11) and \( A_n, C_n \) are the same as in Eqs. (2.1) and (2.2), respectively.

The last two corollaries imply actually that Problem BNP′ can be solved on the basis of the theory of moments.

4. Solvability criteria for Problem BNP and Problem TTM

In the scalar case, the singularity of the positive semidefinite Pick matrix \( P \) leads to the uniqueness of the solutions of Problem BNP. As well, it is easy to find out the necessary and sufficient conditions for Problem BNP to have a solution. But, in the matrix case (i.e., \( p > 1 \)) and when \( P \) is positive semidefinite, this thing becomes rather complicated, because there may be still infinitely many solutions of Problem BNP even if the Pick block matrix \( P \) is singular.

In this section, we present a necessary and sufficient condition for existence of solutions to Problem BNP, based on the use of some recent results due to Bolotnikov and Dym on boundary Nevanlinna–Pick interpolation for Schur matrix functions. Thanks to the intrinsic connections between Problem BNP and Problem TTM (see Theorems 2.2, 3.4 and 3.5), we obtain also a necessary and sufficient condition for Problem TTM and thus for Problem BNP to have a solution, a useful result in the theory of moments.

Let \( F(z) \in \mathbb{C}^p \) have the Riesz–Herglotz representation (1.1). Then for each point \( z = c \) on \( T \), we have

\[
\lim_{r \to 1^-} \frac{F(rc) + F(rc)^*}{1 - r^2} = \lim_{r \to 1^-} \frac{1}{\pi} \int_0^{2\pi} |e^{i\theta} - rc|^2 d\sigma(e^{i\theta}).
\]

By Lemma 3.1 and Eq. (4.1), we can verify that the condition (1.3) of Problem BNP can be replaced by the following radial boundary limits:

\[
\lim_{r \to 1^-} \frac{F(rz_i) + F(rz_i)^*}{1 - r^2} = A_i, \quad i = 1, \ldots, n.
\]

Let now

\[
V = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ z_1 & z_2 & \cdots & z_n \\ \vdots & \vdots & \ddots & \vdots \\ z_1^{n-1} & z_2^{n-1} & \cdots & z_n^{n-1} \end{pmatrix}, \quad C = \begin{pmatrix} I_p & Y_1 \\ I_p & Y_2 \\ \vdots & \vdots \\ I_p & Y_n \end{pmatrix}, \quad J = \text{diag}(z_1, z_2, \ldots, z_n).
\]

We check easily that the Pick block matrix \( P \) as indicated in Eq. (1.5) satisfies the following Stein matrix equation:
\[ P - (J \otimes I_p)P(J \otimes I_p)^* = C \begin{pmatrix} 0 & I_p \\ I_p & 0 \end{pmatrix} C^*. \]

By virtue of Theorem 8.4 of [8, Section 8], we can obtain a necessary and sufficient condition for Problem BNP to have a solution.

**Theorem 4.1.** Let \( J, C \) and \( P \) be defined as in Eqs. (4.2) and (1.5), respectively. Then Problem BNP has a solution if and only if \( P \geq 0 \) and, in addition,

\[
\ker P ((I_n - z_j J^*) \otimes I_p) \cap \ker C^* \subseteq \ker P, \quad j = 1, \ldots, n.
\]  \( (4.3) \)

**Proof.** First we suppose that Problem BNP is solvable and that \( F(z) \) is an arbitrary solution to Problem BNP. Define the Cayley transformation of \( F(z) \) by

\[
S(z) = (I_p + F(z))^{-1}(I_p - F(z)), \quad z \in D.
\]  \( (4.4) \)

Obviously, \( S(z) \) is analytic in \( D \). Moreover, for each \( z \in D \),

\[
I_p - S(z)S(z)^* = 2(I_p + F(z))^{-1}(F(z) + F(z)^*)(I_p + F(z)^*)^{-1} \geq 0.
\]  \( (4.5) \)

Then \( S(z) \) belongs to the Schur class \( \mathcal{S}_p \). Put now

\[
Z_i = (I_p + Y_i)^{-1}(I_p - Y_i),
\]

\[
A_i = 2(I_p + Y_i)^{-1}A_i(I_p + Y_i^*)^{-1}, \quad i = 1, \ldots, n.
\]  \( (4.6) \)

Since each \( Y_i \) is a skew Hermitian matrix, \( Z_i \) defined via Eq. (4.6) is a unitary matrix such that \(-1 \notin \sigma(Z_i) \) (the spectrum of \( Z_i \)) for all \( i \).

From Eqs. (4.4) and (4.5), we deduce that \( S(z) \) is a solution of the following boundary Nevanlinna–Pick interpolation problem for Schur matrix functions (Problem BNP\(_S\)):

\[
\lim_{r \to 1^-} S(rz_i) = Z_i, \quad \lim_{r \to 1^-} \frac{I_p - S(rz_i)S(rz_i)^*}{1 - r^2} = A_i, \quad i = 1, \ldots, n.
\]  \( (4.7) \)

Next, we suppose that Problem BNP\(_S\) is solvable and that \( S(z) \) is an arbitrary solution to Problem BNP\(_S\). Let

\[
F(z) = (I_p - S(z))(I_p + S(z))^{-1}.
\]  \( (4.8) \)

Since \(-1 \notin \sigma(Z_i) \) for all \( i \), \( F(z) \) is analytic in \( D \) and belongs to the class \( \mathcal{C}_p \). From Eqs. (4.6) and (4.7) and the nonsingularity of \( I_p + Z_i \), we conclude that \( F(z) \) determined by Eq. (4.8) is a solution of Problem BNP. Thus we have shown that Problem BNP is solvable if and only if Problem BNP\(_S\) is solvable.

It follows from Theorem 8.4 of [8, Section 8] that Problem BNP\(_S\) has a solution if and only if \( \tilde{P} \geq 0 \) and, in addition,

\[
\ker \tilde{P} ((I_n - z_j J^*) \otimes I_p) \cap \ker \tilde{C}^* \subseteq \ker \tilde{P}, \quad j = 1, \ldots, n
\]  \( (4.9) \)

in which

\[
\tilde{P} = (\tilde{P}_{ij})_{i,j=1}^n, \quad \tilde{P}_{ij} = \begin{cases} A_i, & \text{if } i = j, \\ \frac{I_p - Z_i Z_j^*}{1 - \bar{z}_i z_j}, & \text{if } i \neq j \end{cases}
\]  \( (4.10) \)

and

\[
\tilde{C} = \begin{pmatrix} I_p & Z_1 \\ I_p & Z_2 \\ \vdots & \vdots \\ I_p & Z_n \end{pmatrix}.
\]
In views of Eq. (4.5) and the definitions of the Pick block matrices $P$ and $\tilde{P}$ given in Eq. (1.5) and (4.10), respectively, we have

$$\tilde{P} = 2DPD^* \quad (4.11)$$

in which

$$D = \text{diag}((I_p + Y_1)^{-1}, \ldots, (I_p + Y_n)^{-1}),$$

is a nonsingular block diagonal matrix. Then $\tilde{P} \succeq 0$ if and only if $P \succeq 0$. Furthermore, by Eq. (4.6) and the fact that $Y_i^* = -Y_i$ ($1 \leq i \leq n$) we obtain

$$\tilde{C}^* = \left( \begin{array}{cccc}
I_p - Y_i & I_p - Y_i & \cdots & I_p - Y_i \\
I_p + Y_i & I_p + Y_i & \cdots & I_p + Y_i
\end{array} \right) D^* \quad (4.12)$$

It follows from Eqs. (4.11) and (4.12) and the nonsingularity of $D$ that

$$\text{Ker} \tilde{P} = \text{Ker} PD^*, \quad \text{Ker} \tilde{C}^* = \text{Ker} C^* D^*.$$  

Note that the block diagonal matrix $D^*$ commutes with the block diagonal matrix $(I_n - zj J^*) \otimes I_p$, so that we have

$$\text{Ker} \tilde{P}((I_n - zj J^*) \otimes I_p) = \text{Ker} P((I_n - zj J^*) \otimes I_p) D^*.$$  

The last two equations and the nonsingularity of $D$ imply that Eq. (4.9) holds true if and only if Eq. (4.3) holds. Hence we complete the proof. □

From Theorem 4.1 and the intrinsic connection between Problem BNP and Problem TTM given in Theorems 2.2, 3.4 and 3.5, we can deduce a solvability criterion for Problem TTM and further for Problem BNP. To do this, we need the following two lemmas.

**Lemma 4.2** [13, 33]. Let $W$ and $J$ be defined as in Eqs. (2.3) and (4.2), respectively. Then

$$WV = \text{diag}(a_1(z_1), \ldots, a_n(z_n)), \quad JW = WC_a$$

in which

$$C_a = \begin{pmatrix}
0 & 1 \\
& \ddots & \ddots \\
& & 0 & 1 \\
-a_0 & \cdots & -a_{n-2} & -a_{n-1}
\end{pmatrix} \quad (4.13)$$

is the first companion matrix of polynomial $a(z)$.

**Lemma 4.3.** Let $W$ and $C$ be defined as in Eqs. (2.3) and (4.2), respectively. Then

$$(W \otimes I_p)^{-1}C := B = \begin{pmatrix}
0 & K_1 \\
& \ddots & \ddots \\
& \vdots & K_2 \\
0 & \vdots \\
I_p & K_n
\end{pmatrix} \quad (4.14)$$

in which
\[
\begin{pmatrix}
K_1 \\
K_2 \\
\vdots \\
K_n
\end{pmatrix} = 2 \begin{pmatrix}
C_{n-1} \\
\vdots \\
C_1 \\
C_0
\end{pmatrix} + 2(C_{j-i-1})^{i-1}_{i, j=0} \begin{pmatrix}
\tilde{a}_0 I_p \\
\tilde{a}_1 I_p \\
\vdots \\
\tilde{a}_{n-1} I_p
\end{pmatrix}
\]
(4.15)

and all \(C_i = C^*_{-i} \ (-n \leq i \leq n)\) are determined by Eq. (2.2).

**Proof.** Since \(W\) is nonsingular and the last column of \(W\) is \((1, 1, \ldots, 1)^T\), we have

\[
(W \otimes I_p)^{-1} \begin{pmatrix}
I_p \\
I_p \\
\vdots \\
I_p
\end{pmatrix} = (W^{-1} \otimes I_p) \begin{pmatrix}
I_p \\
I_p \\
\vdots \\
I_p
\end{pmatrix} = \begin{pmatrix}
0 \\
0 \\
\vdots \\
0
\end{pmatrix}.
\]

Furthermore, by Lemma 4.2 we have

\[
\begin{pmatrix}
K_1 \\
K_2 \\
\vdots \\
K_n
\end{pmatrix} = (W \otimes I_p)^{-1} \begin{pmatrix}
Y_1 \\
Y_2 \\
\vdots \\
Y_n
\end{pmatrix} = (V \otimes I_p) \begin{pmatrix}
Y_1 \\
Y_2 \\
\vdots \\
Y_n
\end{pmatrix} = \begin{pmatrix}
\sum_{i=1}^n \frac{Y_i}{a_i(z_i)} \\
\vdots \\
\sum_{i=1}^n \frac{Y_i}{a_n(z_n)}
\end{pmatrix}.
\]
(4.16)

Let now \(\Omega_a(z)\) be the Sylvester–Lagrange polynomial subject to \(\Omega_a(z_i) = Y_i \ 1 \leq i \leq n\). Then by Eqs. (4.14) and (4.16) we check easily that the Laurent series expansion of rational matrix function \(\Omega_a(z)/a(z)\) at \(z = \infty\) has the form:

\[
\frac{\Omega_a(z)}{a(z)} = \sum_{i=1}^n \frac{Y_i}{a_i(z_i)} = \frac{K_1}{z} + \frac{K_2}{z^2} + \cdots + \frac{K_n}{z^n} + \cdots \ (z \to \infty).
\]
(4.17)

Now it remains only to prove \(K_i \ (1 \leq i \leq n)\) determined by Eq. (4.17) satisfy Eq. (4.15).

Note that there exists a \(p \times p\) matrix polynomial of the form \(H(z) = H_0 + H_1 z + \cdots + H_n z^n\) obeying

\[
z^n \Omega_a(z) - z^n \Omega(z) = a(z) H(z)
\]
or equivalently,

\[
\frac{\Omega_a(z)}{a(z)} = \frac{\Omega(z)}{A(z)} \left( \frac{1}{z^n} + \frac{\tilde{a}_{n-1}}{z^{n-1}} + \cdots + \frac{\tilde{a}_1}{z} + \tilde{a}_0 \right) + \frac{H_0}{z^n} + \cdots + \frac{H_{n-1}}{z} + H_n.
\]
(4.18)

Since \(\Omega(z) = -\Omega(\bar{z}^{-1})^*\), \(A(z) = A(\bar{z}^{-1})^*\) and Eq. (2.2) is true at \(z = 0\), the Laurent series expansion of rational matrix function \(\Omega(z)/A(z)\) at \(z = \infty\) has the form:

\[
\frac{\Omega(z)}{A(z)} = C_0 + \frac{2C_1^*}{z} + \cdots + \frac{2C_{n-1}^*}{z^{n-1}} + \frac{2C_n^*}{z^n} + O(z^{-n-1}).
\]
(4.19)

Inserting Eqs. (4.17) and (4.19) into Eq. (4.18) and comparing the coefficients of \(z^{-i} \ (1 \leq i \leq n)\) in both sides of Eq. (4.18), we obtain

\[
\begin{pmatrix}
K_1 \\
K_2 \\
\vdots \\
K_n
\end{pmatrix} = \begin{pmatrix}
H_{n-1} \\
\vdots \\
H_1 \\
H_0 + C_0
\end{pmatrix} + \begin{pmatrix}
2C_1^* & C_0 \\
2C_2^* & 2C_1^* \\
\vdots & \vdots & \ddots & \ddots \\
2C_n^* & \cdots & 2C_2^* & C_1^*
\end{pmatrix} \begin{pmatrix}
\tilde{a}_0 I_p \\
\tilde{a}_1 I_p \\
\vdots \\
\tilde{a}_{n-1} I_p
\end{pmatrix}.
\]
(4.20)
On the other hand, $\Omega_a(z)/a(z)$ being analytic at $z = 0$, then inserting Eq. (2.2) into (4.18) we obtain

$$(-C_0 - 2C_1 z - \cdots - 2C_n z^n + O(z^{n+1})) \left( \frac{1}{z^{n-1}} + \cdots + \frac{\tilde{a}_1}{z} + \tilde{a}_0 \right) + \frac{H_0}{z^n} + \cdots + \frac{H_{n-1}}{z} + H_n = O(1).$$

The last equation yields immediately that

$$\begin{bmatrix} H_{n-1} \\ \vdots \\ H_1 \\ H_0 \end{bmatrix} = \begin{bmatrix} 2C_{n-1} \\ \vdots \\ 2C_1 \\ C_0 \end{bmatrix} + \begin{bmatrix} 0 & C_0 & 2C_1 & \cdots & 2C_{n-2} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & C_0 & 2C_1 \\ 0 & \cdots & 0 & 0 & C_0 \end{bmatrix} \begin{bmatrix} \tilde{a}_0 I_p \\ \tilde{a}_1 I_p \\ \vdots \\ \tilde{a}_{n-1} I_p \end{bmatrix}.$$

Then Eq. (4.15) follows directly from Eqs. (4.20) and (4.21). We complete the proof. □

Now by virtue of what have been proved, we present a solvability criterion for Problem TTM and further for Problem BNP.

**Theorem 4.4.** Let $T$, $B$ and $C_a$ be defined as in Eqs. (1.6), (4.14) and (4.13), respectively. Problem TTM or/and Problem BNP has a solution if and only if $T \geq 0$ and, in addition,

$$\text{Ker } T((I_n - z_j C_a^* \otimes I_p) \cap \text{Ker } B^* \subseteq \text{Ker } T, \quad j = 1, \ldots, n. \quad (4.22)$$

**Proof.** From the first statement of Theorem 3.4 (or Theorem 3.5), we know that both Problem BNP and its associated Problem TTM have the same solvability. By Theorem 2.2, we have $T \geq 0$ if and only if $P \geq 0$. Now we must only prove that the conditions (4.3) and (4.22) are equivalent.

First by Theorem 2.2 and Lemma 4.2, we have

$$\text{Ker } P = \text{Ker } T(W \otimes I_p)^* \quad (4.23)$$

and

$$\text{Ker } P((I_n - z_j J^*) \otimes I_p) = \text{Ker } T(W^* \otimes I_p)((I_n - z_j J^*) \otimes I_p) = \text{Ker } T((W^* - z_j (JW)^*) \otimes I_p) = \text{Ker } T((W^* - z_j (WC_a)^*) \otimes I_p) = \text{Ker } T((I_n - z_j C_a^* \otimes I_p)(W \otimes I_p)^*). \quad (4.24)$$

Next, by Lemma 4.3, we have

$$\text{Ker } C^* = \text{Ker } C^*(W^{-1} \otimes I_p)^*(W \otimes I_p)^* = \text{Ker } ((W \otimes I_p)^{-1} C)^*(W \otimes I_p)^* = \text{Ker } B^*(W \otimes I_p)^*. \quad (4.25)$$

Eqs. (4.25)–(4.27) imply that Eq. (4.3) holds if and only if

$$\text{Ker } T((I_n - z_j C_a^* \otimes I_p)(W \otimes I_p)^* \cap \text{Ker } B^*(W \otimes I_p)^* \subseteq \text{Ker } T(W \otimes I_p)^*.$$

Since $W$ is nonsingular, the last equation is equivalent to Eq. (4.22). Hence we finish the proof. □

We remark that the Toeplitz block matrix $T$ as indicated in Eq. (1.6) satisfies the following Stein matrix equation:
\[ T - (C_a \otimes I_p)T(C_a \otimes I_p)^* = \frac{1}{2} B \begin{pmatrix} 0 & I_p \\ I_p & 0 \end{pmatrix} B^* \]
in which \( C_a \) and \( B \) are determined by Eqs. (4.13) and (4.14), respectively.

5. Solutions of Problem BNP: in the nondegenerate case

Problem BNP is termed nondegenerate if the Pick block matrix \( P \) defined by Eq. (1.5) is positive definite. Similarly, Problem TTM is termed nondegenerate if the Toeplitz block matrix \( T \) defined by Eq. (1.6) is positive definite.

In this section, we always assume that both Problem BNP and its associated Problem TTM are nondegenerate. Under this assumption, we will give a description of all solutions of Problem BNP, based on the use of the theory of moments and the connection between the solutions of Problem BNP and the solutions of Problem TTM established in Section 3.

To begin with, we need some preliminary results for our discussion.

5.1. Some preliminary results

The following lemma is a direct consequence of Lemma 3.2, which characterizes the mass of a measure \( \tau \) at a given point \( z = c \) on \( T \) in terms of its Hergoltz integral.

**Lemma 5.1.** Let \( H_\tau(z) \) be the Hergoltz integral of a measure \( \tau \). Then for each point \( z = c \) on \( T \), we have the radial boundary limit

\[ \lim_{r \to 1^-} (1 - r)H_\tau(rc) = \frac{1}{\pi} \tau(\{c\}). \]

By Lemma 5.1, we have

**Lemma 5.2.** A measure \( \tau \) has no mass at point \( z = c \) on \( T \) if and only if the radial boundary limit

\[ \lim_{r \to 1^-} (1 - r)H_\tau(rc) = 0. \]

Let now \( T \) be as in Eq. (1.6) and \( T^{-1} = (\Gamma_{ij})_{i,j=1}^n \) denote the block decomposition of \( T^{-1} \) into \( p \times p \) blocks, and let in turn \( \alpha(z) \), \( \beta(z) \), \( \Gamma(z) \) and \( \delta(z) \) be the \( p \times p \) matrix polynomials defined by

\[
\begin{align*}
\alpha(z) &= (\Gamma_{nn} + z\Gamma_{n-1,n} + \cdots + z^{n-1}\Gamma_{1n})\Gamma_{nn}^{-\frac{1}{2}}, \\
\beta(z) &= (\Gamma_{n1} + z\Gamma_{n-1,1} + \cdots + z^{n-1}\Gamma_{11})\Gamma_{11}^{-\frac{1}{2}}, \\
\gamma(z) &= [(C_0 + 2C_1^*z^{-1} + \cdots + 2C_n^*z^{1-n})\alpha(z)]_+, \\
\delta(z) &= [(C_0 + 2C_1^*z^{-1} + \cdots + 2C_n^*z^{1-n})\beta(z)]_+, 
\end{align*}
\]

where the symbol \([\cdot]_+\) stands for the matrix polynomial part of \((\cdot)\).

In the sequel, denote by \( \hat{M}(z) \) the reciprocal of a \( p \times p \) matrix polynomial \( M(z) = M_0 + zM_1 + \cdots + z^sM_s \) with \( M_s \neq 0 \), i.e.,

\[ \hat{M}(z) = z^sM^* \left( \frac{1}{z} \right) = M_s^* + zM_{s-1}^* + \cdots + z^sM_0^*. \]
We can check that these matrix polynomials obey the following important properties. For more
details, see e.g. [14,16,3].

**Lemma 5.3.** The matrix polynomials $\alpha(z)$, $\beta(z)$, $\gamma(z)$, $\delta(z)$ defined by Eqs. (5.1) and their reciprocals $\hat{\alpha}(z)$, $\hat{\beta}(z)$, $\hat{\gamma}(z)$, $\hat{\delta}(z)$ are all nonsingular on $T$, and satisfy the following remarkable identities:

\[
\begin{align*}
\alpha(z)\hat{\alpha}(z) & = \beta(z)\hat{\beta}(z), \\
\gamma(z)\hat{\gamma}(z) & = \delta(z)\hat{\delta}(z), \\
\hat{\alpha}(z)\gamma(z) + \hat{\gamma}(z)\alpha(z) & = 2z^{n-1}I_p, \\
\hat{\delta}(z)\beta(z) + \hat{\beta}(z)\delta(z) & = 2z^{n-1}I_p.
\end{align*}
\]

Remark that since $\det(\hat{\alpha}(z))$ and $\det(\beta(z))$ have finitely many zeros in the complex plane and are nonzero on $T$, $\hat{\alpha}(rz_i)$ and $\beta(rz_i)$ are nonsingular when $r$ tends sufficiently to 1 from the left.

5.2. Solutions of Problem BNP: in the nondegenerate case

We first recall a basic result on the solutions of Problem TTM. See, e.g., [1,10,16] for details.

**Lemma 5.4.** Let $\alpha(z)$, $\beta(z)$, $\gamma(z)$ and $\delta(z)$ be as in Eqs. (5.1). Then the formula

\[
H_\tau(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} d\tau(e^{i\theta})
= (\gamma(z) + z\delta(z)S(z))(\alpha(z) - z\beta(z)S(z))^{-1}
\]  

(5.2)

establishes a one-to-one correspondence between all the solutions $\tau$ of Problem TTM and all the functions $S(z)$ of in the Schur class $S_p$.

Note that each solution $\tau$ of Problem TTM is also a solution of Problem TTM', so that it can be represented as a linear fractional transformation of the form (5.2) with a certain $S(z) \in S_p$. However, each solution $\tau$ of Problem TTM' is not necessary a solution of Problem TTM.

According to Lemma 5.2, a solution $\tau$ of Problem TTM' is also a solution of Problem TTM if and only if

\[
\lim_{r \to 1^-} (1 - r)H_\tau(rz_i) = 0, \quad i = 1, \ldots, n. \tag{5.3}
\]

Thus, in order to describe all the solutions of Problem TTM, we are only to determine the functions $S(z) \in S_p$ in Eq. (5.2) subject to Eq. (5.3).

From Lemma 5.3 and Eq. (5.3), we deduce

\[
\lim_{r \to 1^-} (1 - r)H_\tau(rz_i) = \lim_{r \to 1^-} (1 - r)[H_\tau(rz_i) + \hat{\alpha}(rz_i)^{-1}\hat{\gamma}(rz_i)]
= \lim_{r \to 1^-} (1 - r)\hat{\alpha}(rz_i)^{-1}[(\hat{\alpha}(rz_i)\gamma(rz_i) + \hat{\gamma}(rz_i)\alpha(rz_i))
+ rz_i(\hat{\alpha}(rz_i)\delta(rz_i) - \hat{\gamma}(rz_i)\beta(rz_i))S(rz_i)]
\times (\alpha(rz_i) - rz_i\beta(rz_i)S(rz_i))^{-1}
\]


The last equation implies that \( z = z_i \) on \( T \) is not a mass point of the measure \( \tau \) if and only if
\[
\lim_{r \to 1^-} (1 - r) (\beta(rz_i) - rz_i S(rz_i))^{-1} = 0
\]
or equivalently,
\[
\lim_{r \to 1^-} (1 - r) \left[ S(rz_i) - \frac{1}{rz_i} \beta(rz_i)^{-1} \alpha(rz_i) \right]^{-1} = 0. 
\quad (5.4)
\]

Summarizing what has been established, we can obtain a parameterized description of the solutions of Problem TTM.

**Theorem 5.5.** Let the matrix polynomials \( \alpha(z), \beta(z), \gamma(z) \) and \( \delta(z) \) be defined by Eq. (5.1). The solutions \( \tau \) of Problem TTM can be parameterized by the linear fractional transformation of the form (5.2), in which the parameter \( S(z) \) runs over the class \( S_p \) and obeys Eq. (5.4) for \( i = 1, \ldots, n \).

Remark that in the scalar case: \( p = 1 \), we can verify that Eq. (5.4) is equivalent to
\[
\lim_{r \to 1^-} \frac{S(rz_i) - \frac{1}{z_i} \beta(z_i)^{-1} \alpha(z_i)}{1 - r} = \infty,
\]
where each \( \frac{1}{z_i} \beta(z_i)^{-1} \alpha(z_i) \) lies on \( T \).

From Theorems 5.5 and 3.5, we obtain immediately a description of the solutions of Problem BNP in the nondegenerate case.

**Theorem 5.6.** All the solutions \( F(z) \) of Problem BNP are parameterized by the formula
\[
F(z) = D + H_{\sigma}(z)
\]
in which the measure \( \sigma \) is of the form
\[
d\sigma(e^{i\theta}) = A(e^{i\theta}) \, d\tau(e^{i\theta}),
\]
where \( \tau \) is determined by Eq. (5.2) with the parameter \( S(z) \in S_p \) obeying Eq. (5.4) for \( i = 1, \ldots, n \), and \( D \) is the skew hermitian matrix given by Eq. (3.11).

By Theorems 5.5 and 3.4, we can formulate the general solution \( F(z) \) of Problem BNP.

**Theorem 5.7.** The general solution \( F(z) \) of Problem BNP can be parameterized by a linear fractional transformation:
\[
F(z) = (\Phi_{11}(z) + \Phi_{12}(z) S(z)) (\Phi_{21}(z) + \Phi_{22}(z) S(z))^{-1}
\]
in which the coefficient matrix
\[
\begin{pmatrix}
\Phi_{11}(z) & \Phi_{12}(z) \\
\Phi_{21}(z) & \Phi_{22}(z)
\end{pmatrix} = \begin{pmatrix}
A(z) I_p & \Omega(z) \\
0 & I_p
\end{pmatrix} \begin{pmatrix}
\gamma(z) & z \delta(z) \\
\alpha(z) & -z \beta(z)
\end{pmatrix}
\]
and the parameter \( S(z) \) runs over the class \( S_p \) and obeys Eq. (5.4) for \( i = 1, \ldots, n \).
Finally we should point out that in the nondegenerate case the general solution $F(z)$ of Problem BNP has the same form as in Theorem 5.7 (or Theorem 5.6) except that the parameter $S(z)$ runs over the class ${\mathcal S}_p$ only.

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References


