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Discrete schemes for Gaussian curvature and their convergence ☆

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ABSTRACT

The popular angular defect schemes for Gaussian curvature only converge at the regular vertex with valence 6. In this paper, we present a new discrete scheme for Gaussian curvature, which converges at the regular vertex with valence greater than 4. We show that it is impossible to build a discrete scheme for Gaussian curvature which converges at the regular vertex with valence 4 by a counterexample. We also study the convergence property of other discrete schemes for Gaussian curvature and compare their asymptotic errors by numerical experiments.

1. Introduction

Estimation of intrinsic geometric invariants is important in a number of applications such as in computer vision, computer graphics, geometric modelling and computer aided design. It is well known that Gaussian curvature is one of the most essential geometric invariants for surfaces. However, in the classical differential geometry, this invariant is well defined only for C^2 smooth surfaces. In the field of modern-computer-related geometry, one often uses C^0 continuous discrete triangular meshes to represent smooth surfaces approximately. Hence, estimation of accurately Gaussian curvature for triangular meshes is demanded strongly.

In the past years, a wealth of different methods for estimating Gaussian curvature have been proposed in the vast literature of applied geometry. These methods can be divided into two classes. The first class is for computing Gaussian curvature based on the local fitting or interpolation technique [1–5], while the second class is for giving discretization formulas which represent the information about the Gaussian curvature [6–9]. In this paper, our focus is on the methods in the second class and our main aim is to present a new discretization scheme for Gaussian curvature which has better convergence property than the previous discretization schemes.

Let M be a triangulation of the smooth surface S in $\mathbb{R}^3$. For a vertex $\mathbf{p}$ of M, suppose that $\{\mathbf{p}_i\}_{i=1}^n$ is the set of the one-ring neighbor vertexes of $\mathbf{p}$. The set $\{\mathbf{p}_i\mathbf{p}\mathbf{p}_{i+1}\}(i = 1, \ldots, n)$ of n Euclidean triangles forms a piecewise linear approximation of S around $\mathbf{p}$. Throughout the paper, we use the following conventions $\mathbf{p}_{n+1} = \mathbf{p}_1$ and $\mathbf{p}_0 = \mathbf{p}_n$. Let γ_i denote the angle $\angle\mathbf{p}_i\mathbf{p}\mathbf{p}_{i+1}$ and let the angular defect at $\mathbf{p}$ be $2\pi - \sum_i \gamma_i$.

A popular discretization scheme for computing Gaussian curvature is in the form of $(2\pi - \sum_i \gamma_i)/E$, where E is a geometric quantity. In general, one takes E as $A(\mathbf{p})/3$ and obtains the following approximation

$$G^{(1)} := \frac{3(2\pi - \sum\limits_i \gamma_i)}{A(\mathbf{p})}, \tag{1}$$

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where $A(\mathbf{p})$ is the sum of the areas of triangles $\mathbf{p}_i\mathbf{p}\mathbf{p}_{i+1}$. In [6], another scheme

$$G^{(2)} := \frac{2\pi - \sum_i \gamma_i}{S_p} \tag{2}$$

is given, where

$$S_p := \sum_i \frac{1}{4\sin\gamma_i}\left[\eta_i\eta_{i+1} - \frac{\cos\gamma_i}{2}(\eta_i^2 + \eta_{i+1}^2)\right]$$

is called the module of the mesh at $\mathbf{p}$. In [10,11], the discretization approximation $G^{(1)}$ is modified as

$$G^{(3)} := \frac{2\pi - \sum_i \gamma_i}{\frac{1}{2}\sum_i \mathrm{area}(\mathbf{p}_i\mathbf{p}\mathbf{p}_{i+1}) - \frac{1}{8}\sum_i \cot(\gamma_i)d_i^2}, \tag{3}$$

where d_i is the length of edge $\mathbf{p}_i\mathbf{p}_{i+1}$. There are different viewpoints for explaining the reason why the angular defect closely relates to the Gaussian curvature, including the viewpoints of the Gaussian–Bonnet theorem, Gaussian map and Legendre's formula (see the next section in details).

Asymptotic analysis for the discretization schemes have been given in [4,6,9]. In [4], the authors show that the discretization scheme $G^{(1)}$ is not always convergent to the true Gaussian curvature for the non-uniform data. In [6], the authors prove that the angular defect is asymptotically equivalent to a homogeneous polynomial of degree two in the principal curvatures and show that the scheme $G^{(2)}$ converges to the exact Gaussian curvature in a linear rate provided $\mathbf{p}$ is a regular vertex with valence six. Moreover, in [6], the authors show that 4 is the only value of the valence such that the angular defect depends upon the principal directions. In [9], Xu proves that the discretization scheme $G^{(1)}$ has a quadratic convergence rate if the mesh satisfies the so-called parallelogram criterion, which requires valence 6. Therefore, one hopes to construct a discretization scheme which converges over any discrete mesh. But in [12], Xu et al. show that it is impossible to construct a discrete scheme which is convergent over any discrete mesh. Hence, we have to be content with the discretization schemes which converge under some conditions. According to past experiences [6,12,13], we regard a discretization scheme as desirable if it has the following properties:

(1) It converges at regular vertexes, at least for sufficiently large valence (the definition of the regular vertex will be given in Section 2);
(2) It converges at umbilical points, i.e., the points satisfying $k_m = k_M$ where k_m and k_M are two principal curvatures.

As stated before, the previous discretization schemes, including $G^{(1)}$, $G^{(2)}$ and $G^{(3)}$, only converge at the regular vertex with valence 6. In [6], a method for computing the Gaussian curvature at the regular vertex with valence unequal to 4 is described. But the method requires two meshes with valences n_1 and n_2 ($n_1 \neq 4, n_2 \neq 4, n_1 \neq n_2$). In this paper, we will construct a discretization scheme which converges at the regular vertex with valence not less than 5, and also at umbilical points with any valence. Moreover, the discretization scheme requires only a single mesh. Hence, the new scheme is more desirable. Furthermore, we show that it is impossible to construct a discretization scheme which is convergent at the regular vertex with valence 4. Therefore, the convergence problem remains open for the regular vertexes with valence 3. Here, it should be noted that the pointwise convergence discussed in this paper is different from the convergence in norm as discussed in [14,15].

The rest of the paper is organized as follows. Section 2 describes some notations and definitions and Section 3 shows three viewpoints for expressing the relation between the angular defect and Gaussian curvature. In Section 4, we study the convergence property of a modified discretization Gaussian curvature scheme. We present in Section 5 a new discretization scheme and prove that the scheme has a good convergence property, which is the central result of the paper. In Section 6, for the regular vertex with valence 4, we show that it is impossible to build a discretization scheme which is convergent to the real Gaussian curvature. Some numerical results are given in Section 7.

2. Preliminaries

In this section, we introduce some notations and definitions used throughout the paper (see also Fig. 1). Let S be a given smooth surface and $\mathbf{p}$ be a point over S. Suppose that the set $\{\mathbf{p}_i\mathbf{p}\mathbf{p}_{i+1}\}$, $i = 1, \ldots, n$, of n Euclidean triangles forms a piecewise linear approximation of S around $\mathbf{p}$. The vector from $\mathbf{p}$ to $\mathbf{p}_i$ is denoted as $\overrightarrow{\mathbf{p}\mathbf{p}_i}$. The normal vector and tangent plane of S at the point $\mathbf{p}$ are denoted by $\mathbf{n}$ and Π, respectively. We denote the projection of $\mathbf{p}_i$ onto Π as $\mathbf{q}_i$, and define the plane containing $\mathbf{n}$, $\mathbf{p}$ and $\mathbf{p}_i$ as Π_i. Then we let κ_i denote the curvature of the plane curve $S \cap \Pi_i$ at $\mathbf{p}$. The distances from $\mathbf{p}$ to $\mathbf{p}_i$ and $\mathbf{q}_i$ are denoted as η_i and l_i, respectively. Let γ_i and β_i denote $\angle\mathbf{p}_i\mathbf{p}\mathbf{p}_{i+1}$ and $\angle\mathbf{q}_i\mathbf{p}\mathbf{q}_{i+1}$. The two principal curvatures at $\mathbf{p}$ are denoted as k_m and k_M. Let $\eta = \max_i \eta_i$. The following results are presented in [6,9,13]:

$$\frac{l_i}{\eta_i} = 1 + O(\eta), \qquad \beta_i = \gamma_i + O(\eta^2), \tag{4}$$

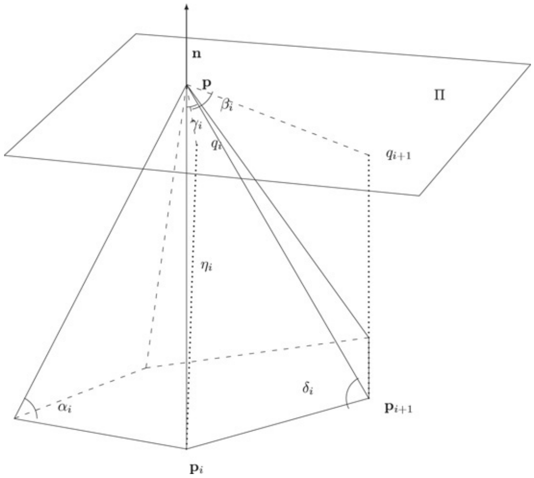

Fig. 1. Notations.

$$\left\|\sum_i w_i \overrightarrow{\mathbf{p}\mathbf{p}_i}\right\| = \sum_i \frac{w_i \kappa_i \eta_i^2}{2} + O(\eta^3), \tag{5}$$

where $w_i \in \mathbb{R}$.

Now we give the definition of the regular vertex using the notations introduced above. Similar definitions also appear in [6,13].

Definition 1. Let $\mathbf{p}$ be a point of a smooth surface S and let $\mathbf{p}_i$, $i = 1, \ldots, n$ be its one-ring neighbor. The point $\mathbf{p}$ is called a regular vertex if it satisfies the following conditions:
(1) the $\beta_i = \frac{2\pi}{n}$,
(2) the η_i's all take the same value η.

Remark 1. We can replace (1) in Definition 1 by requiring that the γ_i's all take the same value. Since $\beta_i = \gamma_i + O(\eta^2)$, all the results in this paper hold also for the alternative definition.

3. The angular defect and Gaussian curvature

In this section, we summarize three different viewpoints for expressing the relation between the angular defect and Gaussian curvature. These viewpoints have been described in the literature [4,9]. We collect them here.

3.1. Gaussian–Bonnet theorem viewpoint

Let D be a region of the surface S, whose boundary consists of piecewise smooth curves Γ_j's. Then the local Gaussian–Bonnet theorem is as follows

$$\int\int_D G(\mathbf{p})\mathrm{d}A + \sum_j \int_{\Gamma_j} k_g(\Gamma_j)\mathrm{d}s + \sum_j \alpha_j = 2\pi,$$

where $G(\mathbf{p})$ is the Gaussian curvature at p, $k_g(\Gamma_j)$ is the geodesic curvature of the boundary curve Γ_j and α_j is the exterior angle at the jth corner point $\mathbf{p}_j$ of the boundary. If all the Γ_j's are the geodesic curves, the above formula reduces to

$$\int\int_D G(\mathbf{p})\mathrm{d}A = 2\pi - \sum_j \alpha_j. \tag{6}$$

Let M be a triangulation of the surface S. For the vertex $\mathbf{p}$, each triangle $\mathbf{p}_i\mathbf{p}\mathbf{p}_{i+1}$ can be partitioned into three equal parts, one corresponding to each of its vertexes. We let D be the union of the part corresponding to $\mathbf{p}$ of triangles $\mathbf{p}_i\mathbf{p}\mathbf{p}_{i+1}$. Note that $\sum_i \gamma_i = \sum_j \alpha_j$. Assuming $G(\mathbf{p})$ is a constant on D, and using (6), we can see that $G(\mathbf{p})$ can be approximated by $G^{(1)}(\mathbf{p})$, where $G^{(1)}(\mathbf{p})$ is the discrete Gaussian curvature obtained using $G^{(1)}$ at $\mathbf{p}$.

3.2. Spherical image viewpoint

We now introduce another definition of Gaussian curvature. Let D be a small patch of area A including point $\mathbf{p}$ on the surface S. There will be a corresponding patch of area I on the Gaussian map. Gaussian curvature at $\mathbf{p}$ is the limit of ratio $\lim_{A\to 0} \frac{I}{A}$.

Let us consider a discrete version of the definition. The Gaussian map image, i.e. the spherical image, of the triangle $\mathbf{p}_i\mathbf{p}\mathbf{p}_{i+1}$ is the point $\frac{(\mathbf{p}-\mathbf{p}_i)\times(\mathbf{p}-\mathbf{p}_{i+1})}{\|(\mathbf{p}-\mathbf{p}_i)\times(\mathbf{p}-\mathbf{p}_{i+1})\|}$. Join these points by a great circle forming a spherical polygon on the unit sphere. The area of this spherical polygon is $2\pi - \sum_i \gamma_i$. Similarly to the above, each triangle is partitioned into three parts, one corresponding to each vertex. Then the Gaussian curvature can be approximated by $G^{(1)}(\mathbf{p})$.

3.3. Geodesic triangle viewpoint

Let $T = ABC$ be a geodesic triangle on the surface S with angles α, β, γ and geodesic edge lengths a, b, c. Let $A'B'C'$ be a corresponding Euclidean triangle with angles α', β', γ' and edge lengths a, b, c. Legendre presents the following formula

$$\alpha - \alpha' = \text{area}(T)\frac{G(A)}{3} + o(a^2 + b^2 + c^2),$$

where area(T) is the area of the geodesic triangle ABC and $G(A)$ is the Gaussian curvature at A.

Using Legendre's formula for each triangles with $\mathbf{p}$ as a vertex, we arrive at the estimating formula $G^{(1)}(\mathbf{p})$ again.

4. Convergence of angular defect schemes

In [9], Xu gives an analysis about the scheme $G^{(1)}$ and proves that the scheme converges at the vertexes satisfying the so-called parallelogram criterion. In [6], the authors give an elegant analysis about the angular defect and they show that the angular deficit is asymptotically equivalent to a homogeneous polynomial of degree two in the principal curvatures with closed form coefficients if the vertex $\mathbf{p}$ is regular. Moreover, they present another angular scheme $G^{(2)} := \frac{2\pi - \sum_i \gamma_i}{S_p}$. In fact, using the law of cosine, we have

$$\begin{aligned}\frac{1}{2}\sum_i \text{area}(\mathbf{p}_i\mathbf{p}\mathbf{p}_{i+1}) - \frac{1}{8}\sum_i \cot(\gamma_i)d_i^2 &= \sum_i\left[\frac{1}{4}\eta_i\eta_{i+1}\sin\gamma_i - \frac{1}{8}\frac{\cos\gamma_i}{\sin\gamma_i}(\eta_i^2 + \eta_{i+1}^2 - 2\eta_i\eta_{i+1}\cos\gamma_i)\right]\\ &= \sum_i \frac{1}{4\sin\gamma_i}\left[\eta_i\eta_{i+1} - \frac{\cos\gamma_i}{2}(\eta_i^2 + \eta_{i+1}^2)\right] = S_p.\end{aligned}$$

This shows that $G^{(2)}$ and $G^{(3)}$ are equivalent, which means that these two schemes output the same value for the same triangular mesh.

In [9], the author proves that the discretization scheme $G^{(1)}$ has quadratic convergence rate under the parallelogram criterion. In the following theorem, we shall show that the discretization scheme $G^{(3)}$ has also quadratic convergence rate under the same criterion.

Theorem 1. *Let* $\mathbf{p}$ *be a vertex of* M *with valence six, and let* $\mathbf{p}_j, j = 1, \ldots, 6$ *be its neighbor vertexes. Suppose that* $\mathbf{p}$ *and* $\mathbf{p}_j, j = 1, \ldots, 6$ *are on a sufficiently smooth parametric surface* $\mathbf{F}(\xi_1, \xi_2) \in \mathbb{R}^3$*, and there exist* $\mathbf{u}$ *and* $\mathbf{u}_j \in \mathbb{R}^2$ *such that*

$$\mathbf{p} = \mathbf{F}(\mathbf{u}), \qquad \mathbf{p}_j = \mathbf{F}(\mathbf{u}_j) \quad \text{and} \quad \mathbf{u}_j - \mathbf{u} = (\mathbf{u}_{j-1} - \mathbf{u}) + (\mathbf{u}_{j+1} - \mathbf{u}), \quad j = 1, \ldots, 6.$$

Then

$$\frac{2\pi - \sum_i \gamma_i}{\frac{1}{2}A(\mathbf{p}, r) - \frac{1}{8}\sum_i \cot(\gamma_i(r))d_i^2(r)} = G(\mathbf{p}) + O(r^2),$$

where, $G(\mathbf{p})$ *is the real Gaussian curvature of* $\mathbf{F}(\mathbf{u})$ *at* $\mathbf{p}$*,*

$$A(\mathbf{p}, r) := \sum_i \text{area}[\mathbf{p}_i(r)\mathbf{p}\mathbf{p}_{i+1}(r)], \qquad \mathbf{p}_i(r) := \mathbf{F}(\mathbf{u}_i(r)),$$

and $\mathbf{u}_i(r) = \mathbf{u} + r(\mathbf{u}_i - \mathbf{u}), i = 1, \ldots, 6.$

Proof. Let

$$A(\mathbf{p}, r) = a_0 r^2 + a_1 r^3 + O(r^4) \tag{7}$$

and

$$\frac{A(\mathbf{p}, r)}{2} - \frac{1}{8}\sum_i \cot(\gamma_i(r))d_i^2(r) = b_0 r^2 + b_1 r^3 + O(r^4)$$

be the Taylor expansions with respect to r. According to Theorem 4.1 in [9],

$$\frac{3(2\pi - \sum_i \gamma_i)}{A(\mathbf{p}, r)} = G(\mathbf{p}) + O(r^2).$$

Hence, to prove the theorem, we need to show that $b_0 = a_0/3$ and $b_1 = a_1/3$. According to [9], we have $a_1 = 0$, which implies that we only need to prove $b_0 = a_0/3$ and $b_1 = 0$.

Note that the $\mathbf{u}$ and $\mathbf{u}_j, j = 1, \ldots 6$, satisfy the parallelogram criterion. Without loss of generality, we may assume $\mathbf{u} = [0, 0]^T$ and $\mathbf{u}_1 = [1, 0]^T$. Then there exist a constant $a > 0$ and an angle θ such that

$$\mathbf{u}_2 = [a\cos\theta, a\sin\theta]^T.$$

Hence, $\mathbf{u}_3 = [a\cos\theta - 1, a\sin\theta]^T$ and $\mathbf{u}_{j+3} = -\mathbf{u}_j,\ j = 1, 2, 3$. Let

$$\mathbf{u}_j = s_j\mathbf{d}_j = s_j[g_j, l_j]^T,\ j = 1, \ldots, 6,$$

where $s_j = \|\mathbf{u}_j\|$ and $\|\mathbf{d}_j\| = 1$. Then, we have

$$\begin{array}{llllll} s_1 = 1, & s_2 = a, & s_3 = \sqrt{a^2 - 2ac + 1}, & s_4 = s_1, & s_5 = s_2, & s_6 = s_3, \\ g_1 = 1, & g_2 = c, & g_3 = (ac - 1)/s_3, & g_4 = -g_1, & g_5 = -g_2, & g_6 = -g_3, \\ l_1 = 0, & l_2 = t, & l_3 = at/s_3, & l_4 = -l_1, & l_5 = -l_2, & l_6 = -l_3, \end{array}$$

where $(c, t) := (\cos\theta, \sin\theta)$. Note that

$$A(\mathbf{p}, r) = \frac{1}{2}\sum_{j=1}^{6}\sqrt{\|\mathbf{p}_j(r) - \mathbf{p}\|^2\|\mathbf{p}_{j+1}(r) - \mathbf{p}\|^2 - \langle\mathbf{p}_j(r) - \mathbf{p}, \mathbf{p}_{j+1}(r) - \mathbf{p}\rangle^2}, \tag{8}$$

$$\cot(\gamma_j(r)) = \frac{\langle\mathbf{p}_j(r) - \mathbf{p}, \mathbf{p}_{j+1}(r) - \mathbf{p}\rangle}{\sqrt{\|\mathbf{p}_j(r) - \mathbf{p}\|^2\|\mathbf{p}_{j+1}(r) - \mathbf{p}\|^2 - \langle\mathbf{p}_j(r) - \mathbf{p}, \mathbf{p}_{j+1}(r) - \mathbf{p}\rangle^2}}, \tag{9}$$

and

$$d_j^2(r) = \|\mathbf{p}_j(r) - \mathbf{p}\|^2 + \|\mathbf{p}_{j+1}(r) - \mathbf{p}\|^2 - 2\langle\mathbf{p}_j(r) - \mathbf{p}, \mathbf{p}_{j+1}(r) - \mathbf{p}\rangle. \tag{10}$$

Let $\mathbf{F}^k_{\mathbf{d}_j}$ denote the kth order directional derivative of $\mathbf{F}$ in the direction $\mathbf{d}_j$. Then using the Taylor expansion with respect to r, we have that

$$\begin{aligned} \|\mathbf{p}_j(r) - \mathbf{p}_j\|^2 &= s_j^2r^2\langle\mathbf{F}_{d_j}, \mathbf{F}_{d_j}\rangle + s_j^3r^3\langle\mathbf{F}_{d_j}, \mathbf{F}^2_{d_j}\rangle + \frac{1}{4}s_j^4r^4\langle\mathbf{F}^2_{d_j}, \mathbf{F}^2_{d_j}\rangle \\ &\quad + \frac{1}{3}s_j^4r^4\langle\mathbf{F}_{d_j}, \mathbf{F}^3_{d_j}\rangle + \frac{1}{6}s_j^5r^5\langle\mathbf{F}^2_{d_j}, \mathbf{F}^3_{d_j}\rangle + \frac{1}{12}s_j^5r^5\langle\mathbf{F}_{d_j}, \mathbf{F}^4_{d_j}\rangle + O(r^6), \end{aligned} \tag{11}$$

and

$$\begin{aligned} \langle\mathbf{p}_j(r) - \mathbf{p}, \mathbf{p}_{j+1}(r) - \mathbf{p}\rangle &= s_js_{j+1}r^2\langle\mathbf{F}_{d_j}, \mathbf{F}_{d_{j+1}}\rangle + \frac{1}{2}s_js_{j+1}^2r^3\langle\mathbf{F}_{d_j}, \mathbf{F}^2_{d_{j+1}}\rangle + \frac{1}{2}s_j^2s_{j+1}r^3\langle\mathbf{F}_{d_{j+1}}, \mathbf{F}^2_{d_j}\rangle \\ &\quad + \frac{1}{4}s_j^2s_{j+}^2r^4\langle\mathbf{F}^2_{d_{j+1}}, \mathbf{F}^2_{d_j}\rangle + \frac{1}{6}s_js_{j+}^3r^4\langle\mathbf{F}_{d_j}, \mathbf{F}^3_{d_{j+1}}\rangle + \frac{1}{6}s_j^3s_{j+1}r^4\langle\mathbf{F}_{d_{j+1}}, \mathbf{F}_{d_{j+1}}\rangle \\ &\quad + \frac{1}{12}s_j^2s_{j+1}^3r^5\langle\mathbf{F}^2_{d_j}, \mathbf{F}^3_{d_{j+1}}\rangle + \frac{1}{12}s_{j+1}^2s_j^3r^5\langle\mathbf{F}^2_{d_j}, \mathbf{F}^3_{d_{j+1}}\rangle \\ &\quad + \frac{1}{24}s_{j+1}^4s_jr^5\langle\mathbf{F}_{d_j}, \mathbf{F}^4_{d_{j+1}}\rangle + \frac{1}{24}s_{j+1}s_j^4r^5\langle\mathbf{F}^4_{d_j}, \mathbf{F}_{d_{j+1}}\rangle + O(r^6). \end{aligned} \tag{12}$$

To compute all the inner products in the two equations above, we let

$$\mathbf{t}_i = \frac{\partial\mathbf{F}(\xi_1, \xi_2)}{\partial\xi_i}, \qquad \mathbf{t}_{ij} = \frac{\partial^2\mathbf{F}(\xi_1, \xi_2)}{\partial\xi_i\partial\xi_j}, \qquad \mathbf{t}_{ijk} = \frac{\partial^3\mathbf{F}}{\partial\xi_i\partial\xi_j\partial\xi_k}, \ \mathbf{t}_{ijkl} = \frac{\partial^4\mathbf{F}}{\partial\xi_i\partial\xi_j\partial\xi_k\partial\xi_l}$$

for $i, j, k, l = 1, 2$, and let

$$g_{ij} = \mathbf{t}_i^T\mathbf{t}_j, \qquad g_{ijk} = \mathbf{t}_i^T\mathbf{t}_{jk}, \qquad e_{ijkl} = \mathbf{t}_i^T\mathbf{t}_{jkl}, \qquad e_{ijklm} = \mathbf{t}_i^T\mathbf{t}_{jklm}, \qquad f_{ijklm} = \mathbf{t}_{ij}^T\mathbf{t}_{klm}.$$

Since $\mathbf{F}^k_{\mathbf{d}_j}$ can be written as a linear combination of $\mathbf{t}_i$, $\mathbf{t}_{ij}$, $\mathbf{t}_{ijk}$ and $\mathbf{t}_{ijkl}$, all the inner products in (11) and (12) can be expressed as a linear combination of $g_{ij}, g_{ijk}, g_{ijkl}, e_{ijkl}, e_{ijklm}$ and f_{ijklm}.

Substituting (11) and (12) into (8)–(10), and then substituting (8)–(10) into the expression $\frac{1}{2}A(\mathbf{p}, r) - \frac{1}{8}\sum_i\cot(\gamma_i(r))d_i^2(r)$, and using Maple to conduct all the symbolic calculations, we have

$$b_0 = a_0/3 = \sqrt{a^2t^2(g_{11}g_{22} - g_{12}^2)}, \qquad b_1 = 0.$$

The theorem follows. □

Remark 2. The calculation of b_0 and b_1 involves a huge number of terms. It is almost impossible to finish the derivation by hand. Maple completes all the computation in 26 s on a PC equipped with a 3.0 GHZ Intel(R) CPU. The Maple code that conducts all derivations of the theorem is available at http://lsec.cc.ac.cn/~xuzq/maple.html. The interested readers are encouraged to perform the computation.

Remark 3. It should be pointed out that there is another discretization scheme

$$G^{(4)} := \frac{2\pi - \sum_i \gamma_i}{A_M(\mathbf{p})},$$

where $A_M(\mathbf{p})$ is the area of the Voronoi region. Since $\sum_i \text{area}(\mathbf{p}_i\mathbf{p}\mathbf{p}_{i+1})$ could be approximated by $3A_M(p)$ under some conditions, for example the conditions of Theorem 1, $G^{(4)}$ is easily derived from $G^{(1)}$.

5. A new discretization scheme of the Gaussian curvature and its convergence

In this section, we introduce a new discretization scheme for Gaussian curvature which converges at umbilical points and regular vertexes with valence greater than 4. This is the main result of the paper.

Before introducing the new discretization, we discuss some properties about the discrete mean curvature. Setting $\alpha_i = \angle \mathbf{p}_i\mathbf{p}_{i-1}\mathbf{p}$ and $\delta_i = \angle \mathbf{p}_i\mathbf{p}_{i+1}\mathbf{p}$, we let

$$H^{(1)} := 2\left\| \frac{\sum_i (\cot\alpha_i + \cot\delta_i)\overrightarrow{\mathbf{p}\mathbf{p}}_i}{\sum_i (\cot\alpha_i + \cot\delta_i)\eta_i^2} \right\|, \tag{13}$$

which is a popular discrete scheme for the mean curvature at vertex $\mathbf{p}$ (c.f. [16]). Moreover, the real mean curvature and the real Gaussian curvature at $\mathbf{p}$ are denoted as H and G respectively. Then, we have:

Lemma 1. *Suppose that* $\mathbf{p}$ *is a regular vertex or a umbilical point. The discrete scheme* $H^{(1)}$ *converges linearly to the mean curvature* H *as* $\eta = \eta_i \to 0$.

Proof. Firstly, let us consider the regular vertex. Since $\mathbf{p}$ is a regular vertex, one has $\frac{\cot\alpha_i+\cot\delta_i}{\cot\alpha_j+\cot\delta_j} = 1 + O(\eta^2)$, for the different i and j. By (5), we have

$$\left\| \sum_i (\cot\alpha_i + \cot\delta_i)\overrightarrow{\mathbf{p}\mathbf{p}}_i \right\| = \sum_i \frac{(\cot\alpha_i + \cot\delta_i)\eta_i^2 k_i}{2} + O(\eta^3).$$

Hence, one has

$$H^{(1)} = \sum_i \frac{(\cot\alpha_i + \cot\delta_i)\eta_i^2}{\sum_j (\cot\alpha_j + \cot\delta_j)\eta_j^2}\kappa_i + O(\eta) = \frac{1}{n}\sum_i \kappa_i + O(\eta) = H + O(\eta). \tag{14}$$

Secondly, we consider the umbilical points. According to the definition of umbilical points, one has $k_i = k_j = H$ for any i and j. Hence,

$$\begin{aligned} H^{(1)} &:= 2\left\| \frac{\sum_i (\cot\alpha_i + \cot\delta_i)\overrightarrow{\mathbf{p}\mathbf{p}}_i}{\sum_i (\cot\alpha_i + \cot\delta_i)\eta_i^2} \right\| \\ &= \frac{\sum_i (\cot\alpha_i + \cot\delta_i)\eta_i^2 k_i + O(\eta^3)}{\sum_i (\cot\alpha_i + \cot\delta_i)\eta_i^2} = H + O(\eta). \end{aligned} \tag{15}$$

Combining (14) and (15), the theorem follows. $\square$

Now, we turn to a new discrete scheme for Gaussian curvature. Set $\varphi_i := \sum_{j=1}^{i} \gamma_j$ and

$$G^{(5)} := \frac{2\pi - \sum_i \gamma_i - 2(S_p - A)(H^{(1)})^2}{2A - S_p},$$

where

$$A := \sum_i \frac{1}{4\sin\gamma_i}\left(\frac{\eta_i\eta_{i+1}}{2}(1 - \cos 2\varphi_i \cos 2\varphi_{i+1}) - \frac{\cos\gamma_i}{4}(\eta_i^2 \sin^2\varphi_i + \eta_{i+1}^2 \sin^2\varphi_{i+1})\right),$$

$$S_p := \sum_i \frac{1}{4\sin(\gamma_i)}\left[\eta_i\eta_{i+1} - \frac{\cos(\gamma_i)}{2}(\eta_i^2 + \eta_{i+1}^2)\right].$$

Then, we have:

Theorem 2. *Suppose that* $\mathbf{p}$ *is a regular vertex with valence greater than* 4 *or a umbilical point. The discretization scheme* $G^{(5)}$ *converges towards the Gaussian curvature* G *as* $\eta_i \to 0$ *at* $\mathbf{p}$.

Proof. We firstly consider the case where $\mathbf{p}$ is a regular vertex. Set $\theta(n) := \frac{2\pi}{n}$. Since $\mathbf{p}$ is a regular vertex, we have $\gamma_i = \theta(n) + O(\eta^2)$ for any i according to (4). After a brief calculation, we have $A = A' + O(\eta^4)$ and $S_p = S_p' + O(\eta^4)$, where

$$A' = \frac{1}{16\sin(\theta(n))}[2n - n\cos(2\theta(n)) - n\cos(\theta(n))]\eta^2,$$
$$S_p' = \frac{n}{4\sin(\theta(n))}[1-\cos(\theta(n))]\eta^2.$$

Hence, we have

$$\left(2\pi - \sum_i \gamma_i - 2(S_p - A)(H^{(1)})^2\right)/(2A - S_p) = \left(2\pi - \sum_i \gamma_i - 2(S_p' - A')(H^{(1)})^2\right)/(2A' - S_p') + O(\eta^2).$$

Note that $\frac{\eta_{\max}}{\eta_{\min}} = 1 + O(\eta)$. According to Theorem 3 in [6], we have

$$2\pi - \sum_i \gamma_i = A'G + B'(k_M^2 + k_m^2) + o(\eta^2),$$

where, $B' = \frac{1}{16\sin(\theta(n))}[n + \frac{n}{2}\cos(2\theta(n)) - \frac{3n}{2}\cos(\theta(n))]\eta^2$.

Note that

$$\begin{aligned} A'G + B'(k_M^2 + k_m^2) &= A'G + B'[(k_M + k_m)^2 - 2k_M k_m] \\ &= A'G + 4B'H^2 - 2B'G \\ &= (A' - 2B')G + 4B'H^2. \end{aligned}$$

Hence, we have

$$2\pi - \sum_i \gamma_i = (A' - 2B')G + 4B'H^2 + o(\eta^2). \tag{16}$$

Note that $A' = O(\eta^2)$, $B' = O(\eta^2)$ and $A' - 2B' \neq 0$ provided $n \neq 3$. Combining (16), Lemma 1 and $S_p' = A' + 2B'$, one has

$$\begin{aligned} G &= \frac{2\pi - \sum_i \gamma_i - 4B'H^2}{A' - 2B'} + o(1) \\ &= \frac{2\pi - \sum_i \gamma_i - 2(S_p' - A')(H^{(1)})^2}{2A' - S_p'} + o(1) = G^{(5)} + o(1) \end{aligned}$$

provided $n \geq 5$. Therefore, $G^{(5)}$ converges to the Gaussian curvature.

Now, let us turn to the umbilical point. For umbilical points, each direction is the principal direction. According to Lemma 4 in [6], we have

$$2\pi - \sum_i \gamma_i = (AG + (S_p - A)k_m^2) + o(\eta^2)$$

at umbilical points. Since $k_m^2 = H^2 = G$, we have

$$\begin{aligned} 2\pi - \sum_i \gamma_i &= (AG + (S_p - A)k_m^2) + o(\eta^2) \\ &= (AG + 2(S_p - A)H^2 - (S_p - A)G) + o(\eta^2). \end{aligned}$$

From the equation above, we arrive at

$$G = \frac{2\pi - \sum_i \gamma_i - 2(S_p - A)(H^{(1)})^2}{2A - S_p} + o(1) = G^{(5)} + o(1).$$

The theorem follows. $\square$

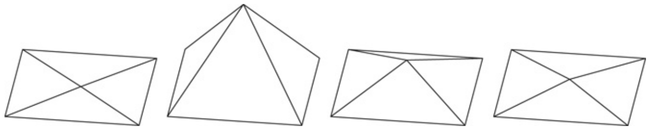

Fig. 2. A sequence of the regular vertexes with valence $n = 4$ for the function $f(x, y) = x^2 + xy + y^2$. At these regular vertexes, it is impossible to construct a discrete Gaussian curvature scheme which converges to the correct value.

Remark 4. Theorem 2 shows that the new scheme $G^{(5)}$ converges at regular vertexes with valence greater than 4. And hence the new scheme has better convergence properties than the available scheme.

Remark 5. In [13], the authors also prove that the discrete scheme $H^{(1)}$ converges to the real mean curvature at regular vertexes. However, the definition of the regular vertex in [13] is different from our definition.

Remark 6. According to the conclusions above, the Gaussian curvature and the mean curvature can be approximated at regular vertexes with valence greater than 4. Hence, using the formulas $k_m = H - \sqrt{H^2 - G}$ and $k_M = H + \sqrt{H^2 - G}$, one can approximate the principal curvatures at regular vertexes with valence greater than 4.

6. A counterexample for the regular vertex with valence 4

In [12], we have constructed a triangular mesh and shown that it is impossible to construct a discrete Gaussian curvature scheme which converges for the mesh. But the vertex in the mesh is not regular. In this section, we shall show that it is also impossible to build a discrete Gaussian curvature scheme which converges at regular vertexes with valence 4.

Suppose that the xy plane is triangulated around $(0, 0)$ by choosing 4 points $\mathbf{q}_1 = (r_1, 0)$, $\mathbf{q}_2 = (0, r_1)$, $\mathbf{q}_3 = (-r_1, 0)$ and $\mathbf{q}_4 = (0, -r_1)$. For a bivariate function $f(x, y)$, the graph of $f(x, y)$, i.e. $\mathbf{F}(x, y) = [x, y, f(x, y)]^{\mathrm{T}}$, can be regarded as a parametric surface. Let $\mathbf{p}_0 = \mathbf{F}(0, 0)$ and $\mathbf{p}_i = \mathbf{F}(\mathbf{q}_i), i = 1, 2, 3, 4$. The set of triangles $\mathbf{p}_i\mathbf{p}_0\mathbf{p}_{i+1}$ forms a triangular mesh approximation of $\mathbf{F}$ at $\mathbf{p}_0$ and we denote the triangular mesh as M_f. When $f(x, y)$ is in the form of $x^2 + cxy + y^2$ where $c \in \mathbb{R}$, it is easy to prove that $\mathbf{p}_0 := (0, 0, 0)^{\mathrm{T}}$ is a regular vertex with valence 4. Moreover, we can see that $\mathbf{p}_1 = (r_1, 0, r_1^2)^{\mathrm{T}}$, $\mathbf{p}_2 = (0, r_1, r_1^2)^{\mathrm{T}}$, $\mathbf{p}_3 = (-r_1, 0, r_1^2)^{\mathrm{T}}$ and $\mathbf{p}_4 = (0, -r_1, r_1^2)^{\mathrm{T}}$. Now we show that it is impossible to construct a discretization scheme for Gaussian curvature which converges over the vertex $\mathbf{p}_0$ (see Fig. 2).

Suppose for the purpose of contradiction that there is a discrete scheme for Gaussian curvature, which is denoted as $G(M_f, \mathbf{p}_0; \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4)$ and is convergent at the regular vertex $\mathbf{p}$ with valence 4. It is easy to calculate that the Gaussian curvature of $\mathbf{F}(x, y, z)$ at $\mathbf{p}_0$ is $4 - c^2$. According to the assumption, we have $\lim_{r_1\to 0} G(M_f, \mathbf{p}_0; \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4) = 4 - c^2$. Note that the triangular mesh M_f is independent of c, i.e. for any function $f(x, y)$ which is in the form of $x^2 + cxy + y^2$, the triangular mesh M_f is the same. Hence, $\lim_{r_1\to 0} G(M_f, \mathbf{p}_0; \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4)$ is independent of c. A contradiction occurs. The assumption of $G(M_f, \mathbf{p}_0; \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4)$ being convergent at the regular vertex $\mathbf{p}$ with valence 4 does not hold.

Remark 7. The counterexample in this section justifies the conclusion in [6], which says that 4 is the only value of valence such that $2\pi - \sum_i \gamma_i$ depends upon the principal directions.

Remark 8. An open problem is to find a discretization scheme for Gaussian curvature which converges at the regular vertex with valence 3.

7. Numerical experiments

The aim of this section is to exhibit the numerical behaviors of the discrete schemes mentioned above. For a vector $\mathbf{a} = (a_{20}, a_{11}, a_{02}) \in \mathbb{R}^3$, we define a bivariate function $f_{\mathbf{a}}(x, y) := a_{20}x^2 + a_{11}xy + a_{02}y^2$, and regard the graph of the function $f_{\mathbf{a}}(x, y)$ as a parametric surface

$$\mathbf{F}_{\mathbf{a}}(x, y) = [x, y, f_{\mathbf{a}}(x, y)]^{\mathrm{T}} \in \mathbb{R}^3.$$

The Gaussian curvature of $\mathbf{F}_{\mathbf{a}}(x, y)$ at the origin is $4a_{20}a_{02} - a_{11}^2$. The domain around $(0, 0)$ is triangulated locally by choosing n points:

$$\mathbf{q}_k = l_k(\cos\theta_k, \sin\theta_k), \qquad \theta_k = 2(k-1)\pi/n, \quad k = 1, \ldots, n.$$

Let $\mathbf{p}_k = \mathbf{F}_{\mathbf{a}}(\mathbf{q}_k)$ and $\mathbf{p}_0 = (0, 0, 0)^{\mathrm{T}}$. Hence, the set of triangles $\{\mathbf{p}_k\mathbf{p}_0\mathbf{p}_{k+1}\}$ forms a piecewise linear approximation of $\mathbf{F}_{\mathbf{a}}$ around $\mathbf{p}_0$. We set $e_k := f_a(\cos\theta_k, \sin\theta_k)$ and select

$$l_k = \sqrt{\frac{\sqrt{1 + 4e_k^2(l_{k-1}^2 + l_{k-1}^4 e_{k-1}^2)} - 1}{2e_k^2}}, \quad k \geq 2 \tag{17}$$

so that $\mathbf{p}_0$ is a regular vertex.

Table 1
The asymptotic maximal error $\varepsilon^{(i)}(n)$.

n	$\varepsilon^{(1)}(n)$	$\varepsilon^{(2)}(n)$	$\varepsilon^{(4)}(n)$	$\varepsilon^{(5)}(n)$
4	$4.6016e+01$	$3.3571e+01$	$3.3570e+01$	$3.3593e+01$
5	$8.2000e+00$	$9.3792e+00$	$9.3792e+00$	$4.1631e+01\eta$
6	$1.2226e+01\eta$	$1.2903e+01\eta$	$1.2903e+01\eta$	$1.1488e+01\eta$
7	$3.8464e+00$	$4.5783e+00$	$4.5783e+00$	$9.0676e-01\eta$
8	$5.8387e+00$	$7.7628e+00$	$7.7628e+00$	$6.5630e+01\eta$

Table 2
The asymptotic error $\varepsilon^{(i)}$ over a sphere.

N	η	$\varepsilon^{(1)}$	$\varepsilon^{(2)}$	$\varepsilon^{(4)}$	$\varepsilon^{(5)}$
30	0.710	3.798e−01	1.905e−01	1.905e−01	2.126e−01
100	0.383	3.517e−01	5.480e−02	5.480e−02	1.192e−01
400	0.196	2.673e−01	1.280e−02	1.280e−02	1.730e−02
1300	0.109	2.812e−01	3.801e−03	3.801e−03	6.500e−03
5000	0.056	2.669e−01	9.648e−04	9.648e−04	2.703e−03

We let $G^{(i)}(\mathbf{p}_0 : \mathbf{F_a})$ denote the approximated Gaussian curvatures of $\mathbf{F_a}$ at $\mathbf{p}_0$, which are obtained by using the discretization scheme $G^{(i)}$. Suppose that $\mathcal{A}$ is a set consisting of M randomly chosen vectors $\mathbf{a}$. Then, we let

$$\varepsilon^{(i)}(n) = \sum_{\mathbf{a}\in\mathcal{A}} |G^{(i)}(\mathbf{p}_0 : \mathbf{F_a}) - (4a_{20}a_{02} - a_{11}^2)|/M.$$

Since $\mathbf{p}$ is a regular vertex, each edge has the same length η. Table 1 shows the asymptotic maximal error $\varepsilon^{(i)}(n)$ when $M = 10^4$. The convergence property and the convergence rate are checked by taking $l_1 = 1/8, 1/16, 1/32, \ldots$ (when $k \geq 2$, l_k can be obtained by (17)).

From Table 1, we can see that all methods work well on valence 6 but only new method works well for valence ≥ 5.

We compute the Gaussian curvature over a randomly triangulated unit sphere by the discretization schemes to test their convergent property at umbilical points. The vertexes of the random triangulation are uniform distribution on the sphere. Denote the vertexes in the random triangulation as $\mathbf{p}_i$, $i = 1, \ldots, N$ where N is the number of the vertexes in the random triangulation. We let $G^{(j)}(\mathbf{p}_i)$ denote the approximate Gaussian curvature at the vertex $\mathbf{p}_i$ which is obtained by the discrete scheme $G^{(j)}$. Similarly to the above, we use $\varepsilon^{(j)} = \sum_{i=1}^{N} |(G^{(j)}(\mathbf{p}_i) - 1)|/N$ to measure the error of discretization scheme $G^{(j)}$ and use η to denote the average length of the edges. Table 2 lists $\varepsilon^{(j)}$ for different N.

From these numerical results, we can draw the following conclusions: For the regular vertexes with the valence greater than 4, or the umbilical points, the new discretization scheme $G^{(5)}$ converges to the real Gaussian curvature, which agrees with the theoretical result.

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