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Determination of critical buckling load for elastic columns of constant and variable cross-sections using variational iteration method

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ABSTRACT

In this paper, variational iteration method (VIM) is applied to the problem of determination of critical buckling loads for Euler columns with constant and variable cross-sections. VIM is a powerful method for the solution of nonlinear ordinary and partial differential equations and integral equations. Hence it is a suitable approach for the analysis of engineering problems where an exact solution is difficult to obtain. This study presents the application of VIM to various buckling cases and results are produced for columns with different support conditions and with different variation of cross-sections. The results obtained are accurate which show that variational iteration method is a very efficient technique in the analysis of elastic stability problems.

1. Introduction

It is crucial to determine the buckling loads in structural analysis and design. Columns are basic structural forms and there are extensive studies related to the elastic stability of columns, and their static and dynamic behaviors. In this field, Euler pioneered the study of buckling of columns under their own weight [1]. Later on, Greenhill [2] made subtle contributions to this field. This problem is often called Greenhill's problem in the related literature. In the literature, it is difficult to find the exact analytical solution for the buckling problem of a non-uniform column with arbitrary distributions of flexural stiffness and axial distributed forces. Closed-form solutions for simple cases are given by Dinnik [3], Karman and Biot [4] and Timoshenko and Gere [5] and others. Wang et al. [6] give exact solutions for buckling of structural members including various cases of columns, beams, arches, rings, plates and shells.

Investigation of buckling of columns has become the center of attention and got more and more systematic during the second half of the 20th century. Exact buckling solutions for several special types of tapered columns with simple boundary conditions were obtained by Gere and Carter [7] in terms of Bessel functions. The problem of the buckling of elastic columns with step varying thickness is given by Arbabei and Li [8]. Siginer [9] studied the stability of a column whose flexural rigidity has a continuous linear variation along the column. Furthermore, the exact analytical solutions of a one-step bar and multi-step bar with varying cross-section under the action of concentrated and variably distributed axial loads were obtained by Li et al. [10–12].

Sampaio et al. [13] solved the problem of buckling behavior of inclined beam–column using energy method. They showed the exact solution using some members of the family of generalized hyper-geometric functions. Also some of the researchers who studied the mechanical behavior of beams/columns are Keller [14], Tadjbakhsh and Keller [15] and Taylor [16].

A solution technique called variational iteration method (VIM) which was originally proposed by He [17–19] has been given great importance for solving linear and nonlinear differential equations in recent years. The method can solve various

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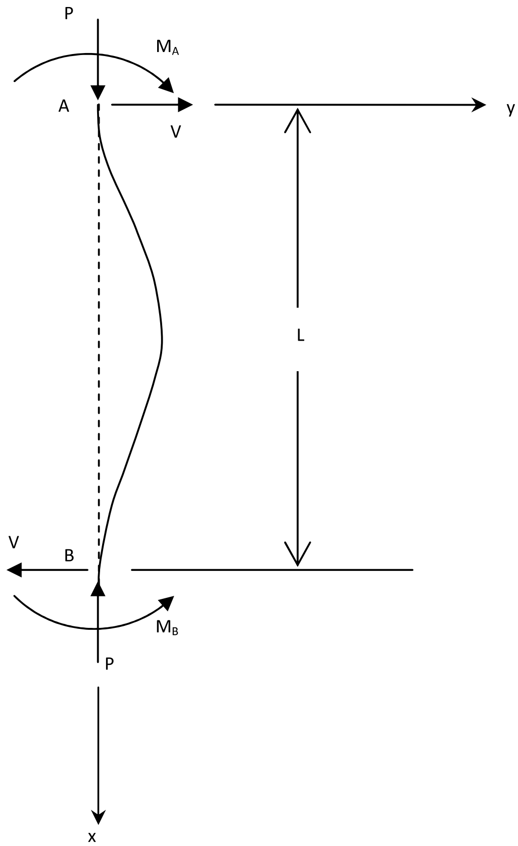

Fig. 1. General case of column buckling.

classes of linear and nonlinear equations [20–27]. VIM is a kind of variational based analytical technique efficient for finding solutions of nonlinear differential equations including boundary value and initial value problems, nonlinear system of differential equations, nonlinear partial differential equations. These successful applications of the method to the various linear and nonlinear types of problems in Physics, Mathematics and Engineering fields encourages the use of VIM in the present problem.

2. Buckling of elastic columns

Buckling investigation of columns will be explained in this section by considering the most general case of an axially loaded elastic column. To this aim, a moment and a shear may be assumed at each end of the column. Such a column and a free-body of a part of this column are shown in Figs. 1 and 2.

According to the Euler–Bernoulli beam theory, moment–displacement relation for the column in Fig. 1 is given by

$$M = EI(x)\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}. \tag{1}$$

The moment equilibrium equation may be written as follows in view of free-body in Fig. 2.

$$EI(x)w'' - Vx + Pw - M_A = 0. \tag{2}$$

Differentiating Eq. (2) twice with respect to x yields:

$$\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\left[EI(x)\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\right] + P\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} = 0. \tag{3}$$

Eq. (3) is the governing equation for buckling of columns regardless of boundary conditions and covers all the cases considered in this study.

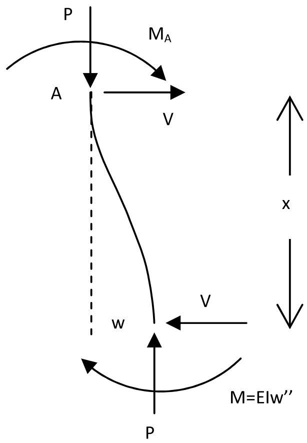

Fig. 2. Free-body of a part of the column.

3. VIM formulation of the problem

According to VIM, the following differential equation may be considered:

$$Lw + Nw = g(x) \tag{4}$$

where L is a linear operator, and N is a nonlinear operator, and $g(x)$ is an inhomogeneous term.

Based on VIM, a correct functional can be constructed as follows:

$$w_{n+1} = w_n + \int_0^x \lambda(\xi)\{Lw_n(\xi) + N\tilde{w}_n(\xi) - g(\xi)\}\mathrm{d}\xi \tag{5}$$

where λ is a general Lagrangian multiplier, which can be identified optimally via the variational theory, the subscript n denotes the nth-order approximation, $\tilde{w}$ is considered as a restricted variation i.e. $\delta\tilde{w} = 0$. By solving the differential equation for λ obtained from Eq. (5) in view of $\delta\tilde{w} = 0$ with respect to its boundary conditions, Lagrangian multiplier $\lambda(\xi)$, is obtained as follows:

$$\lambda(\xi) = \frac{1}{6}(\xi^3 - 3x\xi^2 + 3x^2\xi - x^3). \tag{6}$$

If the above VIM formulation is applied to Eq. (3), the following iteration formula can be obtained accordingly:

$$w_{n+1}(x) = w_n(x) + \int_0^x \lambda(\xi)\left\{w(\xi)^{iv} + 2\frac{[EI(\xi)]'}{EI(\xi)}\tilde{w}(\xi)''' + \left\{\frac{[EI(\xi)]''}{EI(\xi)} + \frac{P}{EI(\xi)}\right\}\tilde{w}(\xi)''\right\}\mathrm{d}\xi. \tag{7}$$

The iteration formula given in Eq. (7), is a simple approximation which can be applied to columns of any cross-section and it is expected to be an important contribution of VIM to the current problem.

4. Critical buckling loads for columns with constant cross-sections

Since the cross-section is constant which means that EI is constant, governing equation takes the following form:

$$EI\frac{\mathrm{d}^4 w}{\mathrm{d}x^4} + P\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} = 0 \tag{8}$$

where $0 < x < L$ and L is length of the column.

Non-dimensional form of this equation is:

$$\frac{\mathrm{d}^4\bar{w}}{\mathrm{d}\bar{x}^4} + \alpha\frac{\mathrm{d}^2\bar{w}}{\mathrm{d}\bar{x}^2} = 0 \tag{9}$$

where $\bar{x} = x/L$, $\bar{w} = w/L$ and α is non-dimensional critical buckling load which is

$$\alpha = \frac{pL^2}{EI}. \tag{10}$$

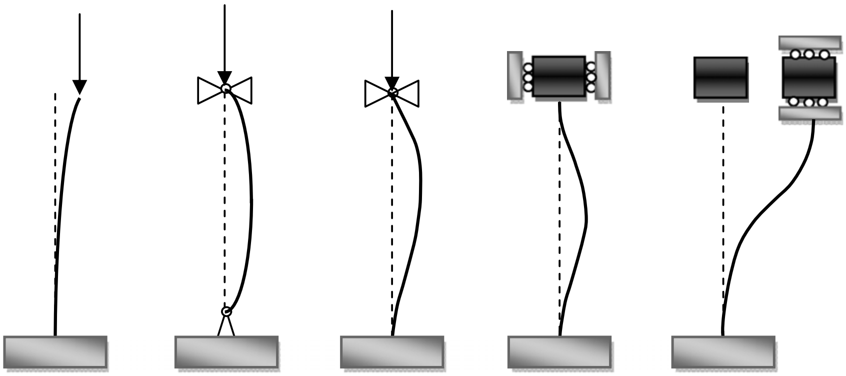

Fig. 3. Various end conditions for classical Euler Column.

Critical buckling loads will be determined for five different cases which is shown in Fig. 3. Although it is easy to obtain an exact solution for Eq. (8) which is a linear ordinary differential equation with constant coefficients, it is a good example for simulation of the use of VIM in the analysis of the problem. In this case iteration formulation given in Eq. (7) becomes

$$\bar{w}_{n+1}(\bar{x}) = \bar{w}_n(\bar{x}) + \int_0^x \frac{1}{6}(\xi^3 - 3\bar{x}\xi^2 + 3\bar{x}^2\xi - \bar{x}^3)\{\bar{w}(\bar{x})^{iv} + \alpha\bar{w}(\bar{x})''\}\mathrm{d}\xi. \tag{11}$$

An initial approximation may be chosen as a cubic polynomial with unknown coefficients. Such an approximation is:

$$\bar{w}_0 = A\bar{x}^3 + B\bar{x}^2 + C\bar{x} + D. \tag{12}$$

This approximation includes four unknown coefficients which are supposed to be found by imposing the boundary conditions of the problem considered. Classical boundary conditions for the non-dimensional equation (9), are given below [6].

$$\text{Pin support:} \quad \bar{w} = 0 \quad \text{and} \quad \frac{\mathrm{d}^2\bar{w}}{\mathrm{d}\bar{x}^2} = 0 \tag{13}$$

$$\text{Clamped support:} \quad \bar{w} = 0 \quad \text{and} \quad \frac{\mathrm{d}\bar{x}}{\mathrm{d}\bar{x}} = 0 \tag{14}$$

$$\text{Free end:} \quad \frac{\mathrm{d}^2\bar{w}}{\mathrm{d}\bar{x}^2} = 0 \quad \text{and} \quad \frac{\mathrm{d}^3\bar{w}}{\mathrm{d}\bar{x}^3} + \alpha\frac{\mathrm{d}\bar{x}}{\mathrm{d}\bar{x}} = 0 \tag{15}$$

$$\text{Sliding restraint:} \quad \frac{\mathrm{d}\bar{w}}{\mathrm{d}\bar{x}} = 0 \quad \text{and} \quad \frac{\mathrm{d}^3\bar{w}}{\mathrm{d}\bar{x}^3} + \alpha\frac{\mathrm{d}\bar{w}}{\mathrm{d}\bar{x}} = 0. \tag{16}$$

The iteration formula given in Eq. (11) produces the following results as two successive iterations.

$$\bar{w}_1 = A\left(\bar{x}^3 - \alpha\frac{\bar{x}^5}{20}\right) + B\left(\bar{x}^2 - \alpha\frac{\bar{x}^4}{12}\right) + C\bar{x} + D \tag{17}$$

$$\bar{w}_2 = A\left(\bar{x}^3 - \alpha\frac{\bar{x}^5}{20} + \alpha^2\frac{\bar{x}^7}{840}\right) + B\left(\bar{x}^2 - \alpha\frac{\bar{x}^4}{12} + \alpha^2\frac{\bar{x}^6}{360}\right) + C\bar{x} + D. \tag{18}$$

In this study computations are conducted up to ninth iteration and four boundary conditions for each case are rewritten by using the final approximation of iteration. Each boundary condition produces an equation containing four unknowns coming from the initial approximation. Hence four equations may be written with respect to the boundary conditions of the problem. These equations can be put into matrix form as follows:

$$[M(\alpha)]\{A\} = \{0\} \tag{19}$$

where $\{A\} = \langle A\ B\ C\ D\rangle^{\mathrm{T}}$. For a nontrivial solution, determinant of coefficient matrix must be zero. The problem is an eigenvalue problem and determinant of coefficient matrix yields a characteristic equation in terms of a. The smallest positive real root of this equation is the non-dimensional critical buckling load for the case considered.

Critical buckling loads for five different cases shown in Fig. 3 are computed by using VIM and compared with exact solutions [6] in Table 1. For the simplification, letters are used to define the support conditions of the column. The first letter stands for the support at the bottom and the letter for the top. Hence, C–F is *Clamped–Fixed*, P–P is *Pinned–Pinned*, C–P is *Clamped–Pinned* and C–S is *Clamped–Sliding Restraint*. The results shown in Table 1 are in perfect agreement with the exact solutions. Hence VIM efficiently solves the determination of critical buckling load problem which is an eigenvalue problem.

Table 1
Comparison of critical buckling loads for columns with constant cross-section.

Critical buckling load	C–F column	P–P column	C–P column	C–C column	C–S column
Exact [6]	2.4674	9.8696	20.1907	39.4784	9.8696
VIM	2.4674	9.8696	20.1908	39.4916	9.8696

Table 2
Comparison of critical buckling loads for columns with exponential variation of flexural rigidity.

aL	C–F		P–P		C–S		C–P		C–C	
	Exact	VIM	Exact	VIM	Exact	VIM	Exact	VIM	Exact	VIM
0.0	2.467	**2.4674**	9.870	**9.8696**	9.870	**9.8696**	20.19	**20.1908**	39.48	**39.4916**
0.1	2.394	**2.3945**	9.380	**9.3857**	9.390	**9.3881**	19.20	**19.2018**	37.55	**37.5499**
0.5	2.110	**2.1121**	7.634	**7.6345**	7.683	**7.6827**	15.64	**15.6399**	30.60	**30.5984**
1.0	1.782	**1.7823**	5.827	**5.8257**	5.973	**5.9728**	11.99	**11.9885**	23.49	**23.4901**
1.5	1.480	**1.4821**	4.389	**4.3889**	4.633	**4.6354**	9.098	**9.0996**	17.86	**17.8647**
2.0	1.209	**1.2176**	3.264	**3.2652**	3.580	**3.5919**	6.839	**6.8470**	13.46	**13.4652**

5. Critical buckling loads for columns with variable cross-sections

5.1. Variation of flexural rigidity with exponential function

With the variation of cross-section, moment of inertia of column also changes along the height of the column. Hence, the flexural rigidity $EI(x)$, is not constant. In this section an exponential variation of flexural rigidity is assumed. The analytical solutions for this case are given in [6].

The variable flexural rigidity is given in the following form:

$$EI(x) = \alpha e^{-ax}. \tag{20}$$

In the equation α has the unit of EI and the unit of a is $[L]^{-1}$. Hence, non-dimensional form of flexural rigidity is:

$$\overline{EI}(\bar{x}) = \frac{\alpha}{pL^2} e^{-(aL)\bar{x}}. \tag{21}$$

The iteration formula is given in Eq. (7). After inserting Eq. (21) into iteration formula, nine iterations are conducted for this case. To simplify the integration process, series expansion of exponential function with seven terms is used in the computations. As in constant cross-section case, a characteristic equation is obtained by means of four boundary conditions after conducting nine iterations in VIM process and normalized buckling load, PL^2/α is determined from the characteristic equation. The results are produced for the same support conditions which are previously considered in the constant cross-section case by rearranging them with respect to variation of flexural rigidity.

Table 2 compares the results obtained from VIM and analytical solutions. In the table, $aL = 0$ case corresponds to buckling loads of uniform column. From the table it can be seen that buckling load for exponential columns can be obtained by using VIM with a good accuracy and this proves the efficiency of VIM.

5.2. Variation of flexural rigidity with power law

In this section, variation of flexural rigidity of elastic columns is given by the following power function.

$$EI(x) = \alpha(1 - bx)^a \tag{22}$$

where α and a are positive constants and $bL \leq 1$. In this equation α has the unit of EI and unit of b is $[L]^{-1}$. Three analyses are conducted by assuming a linear, a quadratic and finally a cubic power consecutively with a non-dimensional variable flexural rigidity EI which may be given as follows:

$$\overline{EI}(\bar{x}) = \frac{\alpha}{pL^2}(1 - bL\bar{x})^a. \tag{23}$$

Analytical solutions for comparisons are taken from Wang et al. [6]. VIM analyses are performed up to ninth iteration and series expansion with nine terms is used for variable coefficients in the iteration equation to simplify the integration process. A characteristic equation is obtained as in previous cases for each assumed value of a and bL together with five different support conditions, i.e, C–F, P–P, C–P, C–C, C–S.

The results obtained are tabulated and compared with the analytical results in Tables 3–5. Table 3 shows the results for normalized buckling load, PL^2/α, for $a = 1$ with $bL = 0.1, 0.3, 0.5$. Table 4 compares the results of VIM with analytical solutions for $a = 2$ with the same bL values. Finally Table 5 compares VIM results with analytical solutions for $a = 3$. From these tables, it can be easily seen that VIM produces good results for normalized buckling loads in view of analytical solutions. The results show that VIM is an efficient and powerful method in the analysis of critical buckling load problems of elastic columns with variable cross-sections.

Table 3
Comparison of critical buckling loads for columns with linear variation of flexural rigidity.

bL	C–F		P–P		C–S		C–P		C–C	
	Exact	VIM	Exact	VIM	Exact	VIM	Exact	VIM	Exact	VIM
0.1	2.393	**2.3928**	9.372	**9.3716**	9.369	**9.3690**	19.17	**19.1686**	37.48	**37.4804**
0.3	2.235	**2.2351**	8.343	**8.3434**	8.317	**8.3167**	17.03	**17.0354**	33.27	**33.2733**
0.5	2.062	**2.0612**	7.256	**7.2564**	7.169	**7.1732**	14.74	**14.7423**	28.70	**28.6972**

Table 4
Comparison of critical buckling loads for columns with quadratic variation of flexural rigidity.

bL	C–F		P–P		C–S		C–P		C–C	
	Exact	VIM	Exact	VIM	Exact	VIM	Exact	VIM	Exact	VIM
0.1	2.319	**2.3191**	8.893	**8.8933**	8.893	**8.8933**	18.19	**18.1893**	35.56	**35.5618**
0.3	2.012	**2.0115**	7.005	**7.0047**	7.005	**7.0042**	14.29	**14.2912**	27.91	**27.9067**
0.5	1.683	**1.6813**	5.198	**5.1992**	5.198	**5.2031**	10.53	**10.5308**	20.48	**20.4814**

Table 5
Comparison of critical buckling loads for columns with cubic variation of flexural rigidity.

bL	C–F		P–P		C–S		C–P		C–C	
	Exact	VIM	Exact	VIM	Exact	VIM	Exact	VIM	Exact	VIM
0.1	2.246	**2.2464**	8.436	**8.4345**	8.442	**8.4416**	17.25	**17.2517**	33.73	**33.7290**
0.3	1.798	**1.7977**	5.840	**5.8405**	5.897	**5.8983**	11.92	**11.9238**	23.29	**23.2912**
0.5	1.336	**1.3368**	3.628	**3.6278**	3.758	**3.7551**	7.362	**7.3641**	14.35	**14.3498**

6. Discussion of results

In this work, application of VIM in elastic stability problems is presented with the problem of determining critical buckling loads for Euler columns with constant and variable cross-sections. Although, the governing equation for constant cross-section columns is an ordinary linear differential equation with constant coefficients, it is a good example to illustrate the application of the method to the problem. The results for this case are in very good agreement with the analytical solutions which encourages the application of VIM to the buckling problems of columns with variable cross-sections. In the variable cross-section cases, the problems are chosen for which an analytical solution exists. However, it is not possible to obtain analytical solutions for any variation in flexural rigidity which is the consequence of variation in cross-section. Since the column is assumed as made up of a single elastic material, its modulus of elasticity E is the same and as a result, the variation of flexural rigidity EI depends only on variation in cross-section. In the selected cases for variable cross-sections, some functions are selected to represent the variation along column height for which exact solutions are available and results have shown that VIM still produces very good results for the columns with changing flexural rigidity. Although, one cannot obtain an analytical solution for any variation in flexural rigidity, it may be easy to implement any variable flexural rigidity into iteration formulation of VIM to obtain a solution. This is the main advantage of VIM in the current problem and demonstrated case studies have shown that VIM is a very efficient, powerful and reliable method for obtaining the critical buckling loads for columns with variable cross-sections. This study figures out that VIM is a promising method in the analysis of various engineering problems.

7. Conclusion

In this study, the use of VIM for the analysis of determination of critical buckling loads for Euler columns with constant and variable cross-sections is presented. As a demonstration of application of the method, five different columns with constant cross-sections are analyzed first. The results for this case are accurate and show that VIM is capable of analyzing the same problems for columns of variable cross-sections. For the variable cross-section problems, the governing equation is a differential equation with variable coefficients and it is not easy to obtain analytical solutions for these types of problems. However, it is easy to put those variable parameters into the iteration equation of VIM and the result can be obtained after a few iterations. Comparisons with analytical solutions pointed out that VIM is a very efficient, powerful and reliable method in the analysis of buckling problems of columns with variable cross-sections.

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