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Randomly weighted self-normalized Lévy processes

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Abstract

Let (U_t, V_t) be a bivariate Lévy process, where V_t is a subordinator and U_t is a Lévy process formed by randomly weighting each jump of V_t by an independent random variable X_t having cdf F. We investigate the asymptotic distribution of the self-normalized Lévy process U_t / V_t at 0 and at ∞. We show that all subsequential limits of this ratio at 0 (∞) are continuous for any nondegenerate F with finite expectation if and only if V_t belongs to the centered Feller class at 0 (∞). We also characterize when U_t / V_t has a non-degenerate limit distribution at 0 and ∞.

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1. Introduction and statements of two main results

We begin by defining the bivariate Lévy process (U_t, V_t), $t \geq 0$, that will be the object of our study. Let F be a cumulative distribution function [cdf] satisfying

$$\int_{-\infty}^{\infty} |x| \, F(\mathrm{d}x) < \infty \tag{1}$$

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and Λ be a Lévy measure on $\mathbb{R}$ with support in $(0, \infty)$ such that

$$\int_0^1 y\Lambda(\mathrm{d}y) < \infty. \tag{2}$$

We define the *Lévy function* $\overline{\Lambda}(x) = \Lambda(x, \infty)$ for $x \geq 0$. Using Corollary 15.8 on page 291 of Kallenberg [10] and assumptions (1) and (2), we can define via F and Λ the bivariate Lévy process (U_t, V_t), $t \geq 0$, having the joint characteristic function

$$\begin{aligned} E \exp\left(\mathrm{i}\theta_1 U_t + \mathrm{i}\theta_2 V_t\right) &=: \phi\left(t, \theta_1, \theta_2\right) \\ &= \exp\left(t \int_{(0,\infty)} \int_{-\infty}^{\infty} \left(e^{\mathrm{i}(\theta_1 u + \theta_2 v)} - 1\right) \Pi(\mathrm{d}u, \mathrm{d}v)\right), \end{aligned} \tag{3}$$

with

$$\Pi\left(\mathrm{d}u, \mathrm{d}v\right) = F\left(\mathrm{d}u/v\right) \Lambda\left(\mathrm{d}v\right). \tag{4}$$

From the form of $\phi(t, \theta_1, \theta_2)$ it is clear that V_t is a driftless subordinator.

Throughout this paper (U_t, V_t), $t \geq 0$, denotes a Lévy process satisfying (1) and (2) and having joint characteristic function (3).

Now let $\{X_s\}_{s\geq 0}$ be a class of i.i.d. F random variables independent of the V_t process. We shall soon see that for each $t \geq 0$ the bivariate process

$$(U_t, V_t) \stackrel{\mathrm{D}}{=} \left(\sum_{0\leq s\leq t} X_s \Delta V_s, \sum_{0\leq s\leq t} \Delta V_s\right), \tag{5}$$

where $\Delta V_s = V_s - V_{s-}$. Notice that in the representation (5) each jump of V_t is weighted by an independent X_t so that U_t can be viewed as a randomly weighted Lévy process.

Here is a graphic way to picture this bivariate process. Consider ΔV_s as the intensity of a random shock to a system at time $s > 0$ and $X_s \Delta V_s$ as the cost of repairing the damage that it causes. Then V_t, U_t and U_t/V_t represent, respectively, up to time t, the total intensity of the shocks, the total cost of repair and the average cost of repair with respect to shock intensity. For instance, ΔV_s can represent a measure of the intensity of a tornado that comes down in a Midwestern American state at time s during tornado season and X_s the cost of the repair of the damage per intensity that it causes. Note that X_s is a random variable that depends on where the tornado hits the ground, say a large city, a medium size town, a village, an open field, etc. It is assumed that a tornado is equally likely to strike anywhere in the state.

We shall be studying the asymptotic distributional behavior of the randomly weighted self-normalized Lévy process U_t/V_t near 0 and infinity. Note that $\overline{\Lambda}(0+) = \infty$ implies that $V_t > 0$ a.s. for any $t > 0$. Whereas if $\overline{\Lambda}(0+) < \infty$, then, with probability 1, $V_t = 0$ for all t close enough to zero. For such $t > 0$, $U_t/V_t = 0/0 := 0$. Therefore to avoid this triviality, when we consider the asymptotic behavior of U_t/V_t near 0 we shall always assume that $\overline{\Lambda}(0+) = \infty$.

Our study is motivated by the following results for weighted sums. Let $\{Y, Y_i : i \geq 1\}$ denote a sequence of i.i.d. random variables, where Y is non-negative and nondegenerate with cdf G. Now let $\{X, X_i : i \geq 1\}$ be a sequence of i.i.d. random variables, independent of $\{Y, Y_i : i \geq 1\}$. Assume that X has cdf F and is in the class $\mathcal{X}$ of nondegenerate random variables X satisfying

$E|X| < \infty$. Consider the self-normalized sums

$$T(n) = \frac{\sum_{i=1}^{n} X_i Y_i}{\sum_{i=1}^{n} Y_i}.$$

We define $0/0 := 0$. Theorem 4 of Breiman [5] says that $T(n)$ converges in distribution along the full sequence $\{n\}$ for *every* $X \in \mathcal{X}$ with at least one limit law being nondegenerate if and only if $Y \in D(\beta)$, with $0 \le \beta < 1$, which means that for some function L slowly varying at infinity, $P\{Y > y\} = y^{-\beta} L(y)$, $y > 0$. In the case $0 < \beta < 1$ this is equivalent to $Y \ge 0$ being in the domain of attraction of a positive stable law of index β. Breiman [5] has shown in his Theorem 3 that in this case the limit has a distribution related to the arcsine law. At the end of his paper Breiman conjectured that $T(n)$ converges in distribution to a nondegenerate law for *some* $X \in \mathcal{X}$ if and only if $Y \in D(\beta)$, with $0 \le \beta < 1$. Mason and Zinn [16] partially verified his conjecture. They established the following.

Whenever X is nondegenerate and satisfies $E|X|^p < \infty$ for some $p > 2$, then $T(n)$ converges in distribution to a nondegenerate random variable if and only if $Y \in D(\beta)$, $0 \le \beta < 1$.

Recently, Kevei and Mason [11] investigated the subsequential limits of $T(n)$. To state their main result we need some definitions. A random variable Y (not necessarily non-negative) is said to be in the *Feller class* if there exist sequences of centering and norming constants $\{a_n\}_{n\ge1}$ and $\{b_n\}_{n\ge1}$ such that if $Y_1, Y_2, \ldots$ are i.i.d. Y then for every subsequence of $\{n\}$ there exists a further subsequence $\{n'\}$ such that

$$\frac{1}{b_{n'}} \left\{ \sum_{i=1}^{n'} Y_i - a_{n'} \right\} \xrightarrow{\mathrm{D}} W, \qquad \text{as } n' \to \infty,$$

where W is a nondegenerate random variable. We shall denote this by $Y \in \mathcal{F}$. Furthermore, Y is in the *centered Feller class*, if Y is in the *Feller class* and one can choose $a_n = 0$, for all $n \ge 1$. We shall denote this as $Y \in \mathcal{F}_c$. The main theorem in [11] connects $Y \in \mathcal{F}_c$ with the continuity of all of the subsequential limit laws of $T(n)$. It says that all of the subsequential distributional limits of $T(n)$ are continuous for any X in the class $\mathcal{X}$, if and only if $Y \in \mathcal{F}_c$.

The notions of Feller class and centered Feller class carry over to Lévy processes. In particular, a Lévy process Y_t is said to be in the *Feller class* at infinity if there exist a norming function $B(t)$ and a centering function $A(t)$ such that for each sequence $t_k \to \infty$ there exists a subsequence $t'_k \to \infty$ such that

$$\left(Y_{t'_k} - A(t'_k) \right) / B(t'_k) \xrightarrow{\mathrm{D}} W, \quad \text{as } k \to \infty,$$

where W is a nondegenerate random variable. The Lévy process Y_t belongs to the *centered Feller class* at infinity if it is in the Feller class at infinity and the centering function $A(t)$ can be chosen to be identically zero. For the definitions of *Feller class* at zero and *centered Feller class* at zero replace $t_k \to \infty$ and $t'_k \to \infty$, by $t_k \searrow 0$ and $t'_k \searrow 0$, respectively. See Maller and Mason [13,14] for more details.

In this paper, we consider the continuous time analog of the results described above, i.e. we investigate the asymptotic properties of the self-normalized Lévy process

$$T_t = U_t / V_t, \tag{6}$$

as $t \searrow 0$ or $t \to \infty$. The expression *continuous time analog* is justified by Remark 2 in [11], where it is pointed out that under appropriate regularity conditions, norming sequence $\{b_n\}_{n\geq 1}$ and subsequences $\{n'\}$,

$$\left(\frac{\sum_{1\leq i\leq n't} X_i Y_i}{b_{n'}}, \frac{\sum_{1\leq i\leq n't} Y_i}{b_{n'}} \right) \xrightarrow{\mathrm{D}} (a_1 t + U_t, a_2 t + V_t), \quad \text{as } n' \to \infty. \tag{7}$$

In light of (7) the results that we obtain in the case $t \to \infty$ are perhaps not too surprising given those just described for weighted sums. However, we find our results in the case $t \searrow 0$ unexpected.

Our main goal is to establish the following two theorems about the asymptotic distributional behavior of U_t/V_t. In the process we shall uncover a lot of information about its subsequential limit laws. First, assuming that $E|X|^p < \infty$, for some $p > 2$, we obtain a partial solution to the continuous time version of the Breiman conjecture, i.e. the continuous time version of the result of Mason and Zinn [16].

Theorem 1. *Assume that X is nondegenerate and for some $p > 2$, $E|X|^p < \infty$. Also assume that Λ satisfies (2) and, in the case $t \searrow 0$, that $\overline{\Lambda}(0+) = \infty$. The following are necessary and sufficient conditions for U_t/V_t to converge in distribution as $t \searrow 0$ (as $t \to \infty$) to a random variable T, in which case it must happen that $(EX)^2 \leq ET^2 \leq EX^2$.*

(i) *$U_t/V_t \xrightarrow{\mathrm{D}} T$ and $(EX)^2 < ET^2 < EX^2$ if and only if $\overline{\Lambda}$ is regularly varying at zero (infinity) with index $-\beta \in (-1, 0)$, in which case the random variable T has cumulative distribution function*

$$P\{T \leq x\} = \frac{1}{2} + \frac{1}{\pi\beta} \arctan\left[\frac{\int |u-x|^\beta \operatorname{sgn}(x-u) F(\mathrm{d}u)}{\int |u-x|^\beta F(\mathrm{d}u)} \tan\frac{\pi\beta}{2} \right],$$
$$x \in (-\infty, \infty); \tag{8}$$

(ii) *$U_t/V_t \xrightarrow{\mathrm{D}} T$ and $ET^2 = EX^2$ if and only if $\overline{\Lambda}$ is slowly varying at zero (infinity), in which case $T \stackrel{\mathrm{D}}{=} X$;*

(iii) *$U_t/V_t \xrightarrow{\mathrm{D}} T$ and $ET^2 = (EX)^2$ if and only if $\overline{\Lambda}$ is regularly varying at zero (infinity) with index -1, in which case $T = EX$.*

Remark 1. The assumption that $E|X|^p < \infty$ for some $p > 2$ is only used in the proof of necessity in Theorem 1. For the sufficiency parts of the theorem we only need to assume that X is nondegenerate and $E|X| < \infty$. In line with the Breiman [5] conjecture we suspect that $U_t/V_t \xrightarrow{\mathrm{D}} T$, as $t \searrow 0$ (as $t \to \infty$), where T is nondegenerate only if $\overline{\Lambda}$ satisfies the conclusion of part (i) or (ii), and in the case that T is degenerate only if $\overline{\Lambda}$ satisfies the conclusion of (iii).

Remark 2. A special case of Theorem 1 shows that if $W_t, t > 0$, is standard Brownian motion, $V_t = \inf\{s \geq 0 : W_s > t\}$ and each X_t in (5) is a zero/one random variable with $P\{X_t = 1\} = 1/2$, then U_t/V_t converges in distribution to the arcsine law as $t \searrow 0$ or $t \to \infty$. This is a consequence of the fact that V_t is a stable process of index $1/2$, since in this case we can set $\beta = 1/2$ and let F be the cdf of X in (8), which yields after a little calculation that T has the arcsine density $g_T(t) = \pi^{-1}(t(1-t))^{-1/2}$ for $0 < t < 1$. Moreover, $U_t/V_t \stackrel{\mathrm{D}}{=} U_1/V_1$, for all $t > 0$, which can be seen by using the self-similar property of the $1/2$-stable process.

Remark 3. Theorem 1 has an interesting connection to some results of Barlow et al. [3] and Watanabe [22]. Suppose V_t is a strictly stable process of index $0 < \beta < 1$ and each X_t in (5) is a zero/one random variable with $P\{X_t = 1\} = p$, with $0 < p < 1$. Then Theorem 1 implies that U_t/V_t converges in distribution as $t \searrow 0$ or $t \to \infty$ to a random variable $Y_{\beta,p}$ with density defined for $0 < x < 1$, by

$$g_{Y_{\beta,p}}(x) = \frac{\sin(\pi\beta)}{\pi} \frac{p(1-p)x^{\beta-1}(1-x)^{\beta-1}}{p^2(1-x)^{2\beta} + (1-p)^2 x^{2\beta} + 2p(1-p)x^{\beta}(1-x)^{\beta}\cos(\pi\beta)}.$$

Furthermore, since V_t is self-similar, one sees that $U_t/V_t \stackrel{\mathrm{D}}{=} U_1/V_1$, for all $t > 0$. Barlow et al. [3] and Watanabe [22] show that $g_{Y_{\beta,p}}$ is the density of the random variable

$$p^{1/\beta}V_1 / \left(p^{1/\beta}V_1 + (1-p)^{1/\beta}V_1'\right),$$

where $V_1 \stackrel{\mathrm{D}}{=} V_1'$ with V_1 and V_1' independent. Moreover, Theorem 2 of Watanabe [22] says that if A_t is the occupation time of Z_s, a p-skewed Bessel process of dimension $2-2\beta$, defined as

$$A_t = \int_0^t \mathbf{1}\{Z_s \geq 0\}\,\mathrm{d}s,$$

then for all $t > 0$, A_t/t has a distribution with density $g_{Y_{\beta,p}}$. We point out that two additional representations can be given for $Y_{\beta,p}$ using Propositions 1 and 2 in the next section. For more about the distribution of $Y_{\beta,p}$ as well as that of closely related random variables refer to James [9].

Remark 4. Let V_t be a subordinator and for each $x \geq 0$ let $T(x)$ denote $\inf\{t \geq 0 : V_t > x\}$. Theorem 1 is analogous to Theorem 6, Chapter 3, of Bertoin [1], which says that $x^{-1}V_{T(x)-}$ converges in distribution as $x \searrow 0$, (as $x \to \infty$) if and only if V_t satisfies the necessary assumptions of Theorem 1 for some $-\beta \in [-1, 0]$. The $\beta = 0$ case corresponds to $\overline{\Lambda}$ being slowly varying at zero (infinity). When $-\beta \in (-1, 0)$, the limiting distribution is the generalized arcsine law.

Our most significant result about subsequential laws of U_t/V_t is the following. Note that in contrast to Theorem 1 we only assume finite expectation of X.

Theorem 2. *Assume that (U_t, V_t), $t \geq 0$, satisfies (1) and (2) and has joint characteristic function (3). All subsequential distributional limits of U_t/V_t, as $t \searrow 0$, (as $t \to \infty$) are continuous for any cdf F in the class $\mathcal{X}$, if and only if V_t is in the centered Feller class at 0 (∞).*

Remark 5. The proof of Theorem 2 shows that if F is in the class $\mathcal{X}$ and V_t is in the centered Feller class at 0 (∞), all of the subsequential limit laws of U_t/V_t, as $t \searrow 0$, (as $t \to \infty$) are not only continuous, but also have Lebesgue densities on $\mathbb{R}$.

The rest of the paper is organized as follows. Section 2 contains two representations of the 2-dimensional Lévy process (U_t, V_t). The first one plays a crucial role in the proof of Theorem 1, while the second one points out the connection between the continuous and discrete time versions of V_t. We provide a fairly exhaustive list of properties of the subsequential limit laws of (U_t, V_t) in Section 3, and we prove our main results in Section 4. Appendix contains some technical results needed in the proofs.

2. Preliminaries

2.1. Representations for (U_t, V_t)

Let (U_t, V_t) , $t \geq 0$, be a Lévy process satisfying (1) and (2) with joint characteristic function (3). We establish two representations for the bivariate Lévy process.

Let $\varpi_1, \varpi_2, \dots$ be a sequence of i.i.d. exponential random variables with mean 1, and for each integer $i \geq 1$ set $S_i = \sum_{j=1}^{i} \varpi_j$. Independent of $\varpi_1, \varpi_2, \dots$ let $X_1, X_2, \dots$ be a sequence of i.i.d. random variables with cdf F, which by (1) satisfies $E|X_1| < \infty$. Consider the Poisson process $N(t)$ on $[0, \infty)$ with rate 1,

$$N(t) = \sum_{j=1}^{\infty} \mathbf{1}_{\{S_j \leq t\}}, \quad t \geq 0. \tag{9}$$

Define for $s > 0$,

$$\varphi(s) = \sup\left\{y : \overline{\Lambda}(y) > s\right\}, \tag{10}$$

where the supremum of the empty set is taken as 0. It is easy to check that (2) and Lemma 1 below imply that for all $\delta > 0$,

$$\int_{\delta}^{\infty} \varphi(s)\, \mathrm{d}s < \infty. \tag{11}$$

We have the following distributional representation of (U_t, V_t).

Proposition 1. *For each fixed $t > 0$,*

$$(U_t, V_t) \stackrel{\mathrm{D}}{=} \left(\sum_{i=1}^{\infty} X_i \varphi\left(\frac{S_i}{t}\right), \sum_{i=1}^{\infty} \varphi\left(\frac{S_i}{t}\right) \right). \tag{12}$$

It is important to note that this representation only holds for fixed $t > 0$ and not for the process in t. As a first consequence of this representation we obtain that $E|U_t|/V_t \leq E|X| < \infty$, in particular, by Markov's inequality, U_t/V_t is stochastically bounded.

Now let $\{X_s\}_{s \geq 0}$ be a class of i.i.d. F random variables. Consider for each $t \geq 0$ the process

$$\left(\sum_{0 \leq s \leq t} X_s \Delta V_s, \sum_{0 \leq s \leq t} \Delta V_s \right),$$

where $\Delta V_s = V_s - V_{s-}$. The following representation reveals the analogy between the continuous and the discrete time self-normalization.

Proposition 2. *For each fixed $t \geq 0$,*

$$(U_t, V_t) \stackrel{\mathrm{D}}{=} \left(\sum_{0 \leq s \leq t} X_s \Delta V_s, \sum_{0 \leq s \leq t} \Delta V_s \right). \tag{13}$$

Remark 6. Notice that the process on the right hand side of (13) is a stationary independent increment process. Since it has the same characteristic function as (U_t, V_t), the distributional representation in (13) holds as a process in $t \geq 0$.

2.2. Proofs of Propositions 1 and 2

In the proofs of Propositions 1 and 2 we shall assume that $\Lambda((0,\infty)) = \infty$. The case $\Lambda((0,\infty)) < \infty$ follows by the same methods.

First we state a useful lemma giving a well-known change of variables formula (see Revuz and Yor [19], Proposition 4.9, p. 8, or Brémaud [6], p. 301), where the integrals are understood to be Riemann–Stieltjes integrals.

Lemma 1. *Let h be a measurable function defined on $(a,b]$, $0 < a < b < \infty$, and R a measure on $(0,\infty)$ such that*

$$\overline{R}(x) := R\{(x,\infty)\}, \quad x > 0,$$

is right continuous and $\overline{R}(\infty) = 0$. Assume that $\int_0^\infty |h(x)| R(\mathrm{d}x) < \infty$, and define for $s > 0$

$$\varphi(s) = \sup\left\{y : \overline{R}(y) > s\right\},$$

where the supremum of the empty set is defined to be 0. Then we have

$$\int_0^\infty h(x) R(\mathrm{d}x) = \int_0^\infty h(\varphi(s))\,\mathrm{d}s. \tag{14}$$

Proof of Proposition 1. We only consider the process on [0, 1].

Applying the Lévy–Itô integral representation of a Lévy process to our case we have that a.s. for each $t \geq 0$

$$(U_t, V_t) = \int_{\mathbb{R}^2\setminus\{0\}} (u,v) N([0,t], \mathrm{d}u, \mathrm{d}v), \tag{15}$$

where N is a Poisson point process on $(0,1) \times \mathbb{R} \times [0,\infty)$, with intensity measure Leb $\times\, \Pi$, where Π is the Lévy measure as in (4).

For the Poisson point process we have the representation

$$N = \sum_{i=1}^\infty \delta_{(U_i, X_i\varphi(S_i), \varphi(S_i))}, \tag{16}$$

where $\{U_i\}$ are i.i.d. Uniform(0, 1) random variables, independent of $\{X_i\}$ and $\{\varpi_i\}$. (At this step we consider the Lévy process on [0, 1].) To see this, let

$$M = \sum_{i=1}^\infty \delta_{(U_i, X_i, S_i)},$$

which is a marked Poisson point process on $[0,1] \times \mathbb{R} \times (0,\infty)$, with intensity measure $\nu = \text{Leb} \times F \times \text{Leb}$. Put $h(u,x,s) = (u, x\varphi(s), \varphi(s))$. Then $\nu \circ h^{-1} = \text{Leb} \times \Pi$. Thus Proposition 2.1 in Rosiński [20] implies that the sequences $\{U_i\}$, $\{X_i\}$, $\{S_i\}$ can be defined on the same space as N such that (16) holds.

Using (16) for N, from (15) we obtain that a.s. for each $t \in [0,1]$

$$(U_t, V_t) = \sum_{i=1}^\infty (X_i\varphi(S_i), \varphi(S_i))\, \mathbf{1}_{\{U_i \leq t\}}. \tag{17}$$

To finish the proof note that if $\sum_{i=1}^{\infty} \delta_{x_i}$ is a Poisson point process and independently $\{\beta_i\}$ is an i.i.d. Bernoulli(t) sequence, then

$$\sum_{i=1}^{\infty} \delta_{x_i} \mathbf{1}_{\{\beta_i=1\}} \stackrel{\mathrm{D}}{=} \sum_{i=1}^{\infty} \delta_{x_i/t},$$

i.e. for a Poisson point process independent Bernoulli thinning and scaling are distributionally the same.

Since the process representation (17) can be extended to any finite interval $[0, T]$ (see the final remark in [20]), this completes the proof. □

We point out that Proposition 1 can also be proved by the same way as Proposition 5.1 in Maller and Mason [12].

Next we turn to the proof of the second representation.

Proof of Proposition 2. Let $\{N_n\}_{n\geq 1}$ be a sequence of independent Poisson processes on $[0, \infty)$ with rate 1. Independent of $\{N_n\}_{n\geq 1}$ let $\left\{\xi_{i,n}\right\}_{i\geq 1, n\geq 1}$ be an array of independent random variables such that for each $i \geq 1$, $n \geq 1$, $\xi_{i,n}$ has distribution $P_{i,n}$ defined for each Borel subset of A of $\mathbb{R}$ by

$$P_{i,n}(A) = P\left\{\xi_{i,n} \in A\right\} = \Lambda\left(A \cap \left[a_n, a_{n-1}\right)\right)/\mu_n,$$

where a_n is a strictly decreasing sequence of positive numbers converging to zero such that $a_0 = \infty$ and for all $n \geq 1$, $0 < \mu_n = \Lambda\left(\left[a_n, a_{n-1}\right)\right) < \infty$.

The process V_t, $t \geq 0$, has the representation as the Poisson point process

$$V_t = \sum_{n=1}^{\infty} \sum_{i \leq N_n(t\mu_n)} \xi_{i,n} =: \sum_{n=1}^{\infty} V_t^{(n)}.$$

See Bertoin [1], page 16. In this representation

$$V_t^{(n)} = \sum_{0\leq s\leq t} \Delta V_s \mathbf{1}_{\{a_n \leq \Delta V_s < a_{n-1}\}}$$

and

$$\Delta V_s \mathbf{1}_{\{a_n \leq \Delta V_s < a_{n-1}\}} = \sum_{i \leq N_n(s\mu_n)} \xi_{i,n} - \sum_{i \leq N_n(s\mu_n-)} \xi_{i,n}.$$

Moreover if $\Delta V_s > 0$ there exists a unique pair (i, n) such that $\Delta V_s = \xi_{i,n}$. Clearly

$$\begin{aligned} &\left(\sum_{0\leq s\leq t} X_s \Delta V_s \mathbf{1}_{\{a_n \leq \Delta V_s < a_{n-1}\}}, \sum_{0\leq s\leq t} \Delta V_s \mathbf{1}_{\{a_n \leq \Delta V_s < a_{n-1}\}}\right) \\ &\stackrel{\mathrm{D}}{=} \left(\sum_{i \leq N_n(t\mu_n)} X_{i,n}\, \xi_{i,n}, \sum_{i \leq N_n(t\mu_n)} \xi_{i,n}\right) =: \left(U_t^{(n)}, V_t^{(n)}\right), \end{aligned} \tag{18}$$

where $\{X_{i,n}\}_{i\geq 1, n\geq 1}$ is an array of i.i.d. random variables with common distribution function F. Notice that the process $\left(U_t^{(n)}, V_t^{(n)}\right)$ in (18) is a compound Poisson process. Keeping this in mind, we see after a little calculation that

$$E \exp\left(\mathrm{i}\left(\theta_1 U_t^{(n)} + \theta_2 V_t^{(n)}\right)\right)$$

$$= \exp\left(t\int_{[a_n,a_{n-1})}\int_{-\infty}^{\infty}\left(e^{\mathrm{i}(\theta_1 u+\theta_2 v)}-1\right)F\left(\mathrm{d}u/v\right)\Lambda\left(\mathrm{d}v\right)\right).$$

Since the random variables $\left\{\left(U_t^{(n)}, V_t^{(n)}\right)\right\}_{n\geq 1}$ are independent we readily conclude that (3) holds. $\square$

3. Additional asymptotic distribution results along subsequences

Let $\mathrm{id}(a, b, \nu)$ denote an infinitely divisible distribution on $\mathbb{R}^d$ with characteristic exponent

$$\mathrm{i}u'b - \frac{1}{2}u'au + \int\left(e^{\mathrm{i}u'x} - 1 - \mathrm{i}u'x\mathbf{1}_{\{|x|\leq 1\}}\right)\nu(\mathrm{d}x),$$

where $b \in \mathbb{R}^d$, $a \in \mathbb{R}^{d\times d}$ is a positive semidefinite matrix, ν is a Lévy measure on $\mathbb{R}^d$ and u' stands for the transpose of u. In our case d is 1 or 2. For any $h > 0$ put

$$a^h = a + \int_{|x|\leq h} xx'\nu(\mathrm{d}x) \quad \text{and} \quad b^h = b - \int_{h<|x|\leq 1} x\nu(\mathrm{d}x).$$

When $d = 1$, $\mathrm{id}(b, \Lambda)$, with Lévy measure Λ on $(0, \infty)$, such that (2) holds, and $b \geq 0$, denotes a non-negative infinitely divisible distribution with Laplace transform

$$\exp\left(-\theta b - \int_0^{\infty}\left(1 - e^{-\theta u}\right)\Lambda\left(\mathrm{d}u\right)\right).$$

In this section it will be convenient to use the following representation for the joint characteristic function of the Lévy process (U_t, V_t), $t \geq 0$, satisfying (1) and (2) and having joint characteristic function (3):

$$\begin{aligned}\phi\left(t, \theta_1, \theta_2\right) &= \exp\left(\mathrm{i}t\left(\theta_1 b_1 + \theta_2 b_2\right)\right)\\ &\times \exp\left(t\int_{(0,\infty)}\int_{-\infty}^{\infty}\left(e^{\mathrm{i}(\theta_1 u+\theta_2 v)} - 1 - \left(\mathrm{i}\theta_1 u + \mathrm{i}\theta_2 v\right)\mathbf{1}_{\{u^2+v^2\leq 1\}}\right)\Pi\left(\mathrm{d}u, \mathrm{d}v\right)\right), \qquad (19)\end{aligned}$$

where $\Pi\left(\mathrm{d}u, \mathrm{d}v\right)$ is as in (4) and

$$\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} \int_{0<u^2+v^2\leq 1} u\Pi\left(\mathrm{d}u, \mathrm{d}v\right) \\ \int_{0<u^2+v^2\leq 1} v\Pi\left(\mathrm{d}u, \mathrm{d}v\right) \end{pmatrix}. \qquad (20)$$

Note that assumptions (1) and (2) ensure that (20) is well defined.

First we investigate the possible subsequential distributional limits of (U_t, V_t). The following theorem is an analog of Theorem 1 in [11].

Theorem 3. *Consider the bivariate Lévy process* (U_t, V_t), $t \geq 0$, *satisfying* (1) *and* (2) *with joint characteristic function* (19). *Assume that for some deterministic sequences* $t_k \searrow 0$ ($t_k \to \infty$) *and* B_k *the distributional convergence*

$$\frac{V_{t_k}}{B_k} \xrightarrow{\mathrm{D}} V \qquad (21)$$

holds, where V has $\mathrm{id}(b, \Lambda_0)$ *distribution with Lévy measure* Λ_0 *on* $(0, \infty)$*. Then*

$$\left(\frac{U_{t_k}}{B_k}, \frac{V_{t_k}}{B_k}\right) \xrightarrow{\mathrm{D}} (U, V), \tag{22}$$

where (U, V) *has* $\mathrm{id}(\mathbf{0}, \mathbf{c}, \Pi_0)$ *distribution with Lévy measure* $\Pi_0\,(\mathrm{d}u, \mathrm{d}v) = F\,(\mathrm{d}u/v)\,\Lambda_0\,(\mathrm{d}v)$ *on* $\mathbb{R} \times (0, \infty)$ *and*

$$\mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} bEX + \displaystyle\int_{0<u^2+v^2\le 1} u\,\Pi_0\,(\mathrm{d}u, \mathrm{d}v) \\ b + \displaystyle\int_{0<u^2+v^2\le 1} v\,\Pi_0\,(\mathrm{d}u, \mathrm{d}v) \end{pmatrix}, \tag{23}$$

i.e. it has characteristic function

$$\Psi(\theta_1, \theta_2) = Ee^{\mathrm{i}(\theta_1 U + \theta_2 V)} = \exp\Bigg\{\mathrm{i}(\theta_1 c_1 + \theta_2 c_2) + \int_0^\infty \int_{-\infty}^\infty \left(e^{\mathrm{i}(\theta_1 u + \theta_2 v)} - 1 - (\mathrm{i}\theta_1 u + \mathrm{i}\theta_2 v)\,\mathbf{1}_{\{u^2+v^2\le 1\}}\right) F\,(\mathrm{d}u/v)\,\Lambda_0\,(\mathrm{d}v)\Bigg\}. \tag{24}$$

Theorem 3 has some immediate consequences concerning the subsequential limits of (U_t, V_t). The first part of the following corollary is deduced from Theorem 3 and classical theory, i.e. Theorem 15.14 in [10]. The second part follows by Fourier inversion.

Corollary 1. *Let* $(U_t, V_t), t \ge 0$*, be as in Theorem 3. For deterministic constants* t_k, B_k *the vector* $B_k^{-1}(U_{t_k}, V_{t_k})$ *converges in distribution to* (U, V) *as* $t_k \searrow 0$ *(as* $t_k \to \infty$*) having characteristic function (24) if, and only if* $t_k \overline{\Lambda}(vB_k) \to \overline{\Lambda}_0(v)$ *for every continuity point of* Λ_0*, and* $\int_0^h x t_k \Lambda(\mathrm{d}B_k x) \to \int_0^h x \Lambda_0(\mathrm{d}x) + b$ *for some continuity point h of* Λ_0*. Moreover, if* $\overline{\Lambda}(0+) = \infty$*, or* $b > 0$ *then* $V > 0$ *a.s., and so* $U_{t_k}/V_{t_k} \xrightarrow{\mathrm{D}} U/V$*, and with* Ψ *as in (24) for all* x

$$P\,\{U/V \le x\} = \frac{1}{2} - \frac{1}{2\pi\mathrm{i}} \int_{-\infty}^\infty \frac{\Psi(u, -ux)}{u} \mathrm{d}u.$$

The remaining results in this section, though interesting in their own right, are crucial for the proof of Theorem 2.

The following proposition provides a sufficient condition for (U, V) to have a C^∞ 2-dimensional density. It also gives an alternative proof for Theorem 3 in [11]. We require the following notation: put for $v > 0$,

$$V_2\,(v) = \int_{0<u\le v} u^2 \Lambda(\mathrm{d}u). \tag{25}$$

Proposition 3. *Assume that* (U, V) *has joint characteristic function*

$$Ee^{\mathrm{i}(\theta_1 U + \theta_2 V)} = \exp\left\{\int_{(0,\infty)} \int_{\mathbb{R}} \left(e^{\mathrm{i}(\theta_1 u + \theta_2 v)} - 1\right) F\left(\frac{\mathrm{d}u}{v}\right) \Lambda(\mathrm{d}v)\right\},$$

where $\int_0^1 v\Lambda(\mathrm{d}v) < \infty$ *and* F *is in the class* $\mathcal{X}$. *Whenever*

$$\limsup_{v\searrow 0} \frac{v^2\overline{\Lambda}(v)}{V_2(v)} < \infty \tag{26}$$

holds, then (U, V) *has a* C^∞ *density.*

As a consequence we obtain the following.

Corollary 2. *Let* $(U_t, V_t), t \geq 0$, *be as in Theorem 3. Assume that* V_t *is in the centered Feller class at zero (infinity) and* F *is in the class* $\mathcal{X}$. *Then for a suitable norming function* $B(t)$ *any subsequential distributional limit of*

$$\left(\frac{U_{t_k}}{B(t_k)}, \frac{V_{t_k}}{B(t_k)}\right)$$

along a subsequence $t_k \searrow 0\, (t_k \to \infty)$, *say* (W_1, W_2), *has a* C^∞ *Lebesgue density* f *on* $\mathbb{R}^2$, *which implies that the asymptotic distribution of the corresponding ratio along the subsequence* $\{t_k\}$ *has a Lebesgue density* g_T *on* $\mathbb{R}$.

The following corollary is an immediate consequence of Theorem 3. Note that a Lévy process Y_t that is in the Feller class at zero (infinity) but not in the centered Feller class at zero (infinity) has the required property.

Corollary 3. *Let* $(U_t, V_t), t \geq 0$, *be as in Theorem 3. Suppose along a subsequence* $t_k \searrow 0$ $(t_k \to \infty)$

$$\frac{V_{t_k} - A(t_k)}{B(t_k)} \xrightarrow{\mathrm{D}} W,$$

where W *is nondegenerate and* $A(t_k)/B(t_k) \to \infty$, *as* $k \to \infty$. *Then*

$$\frac{U_{t_k}}{V_{t_k}} \xrightarrow{\mathrm{D}} EX, \quad \text{as } k \to \infty.$$

For $t > 0$ and $\varepsilon \in (0, 1)$ put

$$A_t(\varepsilon) = \left\{ \frac{\varphi(S_1/t)}{\sum_{i=1}^{\infty} \varphi(S_i/t)} > 1 - \varepsilon \right\}, \tag{27}$$

and

$$\Delta_t = \left| \frac{\sum_{i=1}^{\infty} X_i \varphi(S_i/t)}{\sum_{i=1}^{\infty} \varphi(S_i/t)} - X_1 \right|.$$

Proposition 4. *Assume that for a subsequence* $t_k \searrow 0$ *or* $t_k \to \infty$

$$\lim_{\varepsilon\to 0} \liminf_{k\to\infty} P\{A_{t_k}(\varepsilon)\} = \delta > 0, \tag{28}$$

then

$$\lim_{\varepsilon \to 0} \liminf_{k \to \infty} P\{\Delta_{t_k} \leq \varepsilon\} \geq \delta.$$

Together with the stochastic boundedness of U_t / V_t this implies the following.

Corollary 4. *Let $(U_t, V_t), t \geq 0$, be as in Theorem 3. Assume that (28) holds for V_t, and $P\{X = x_0\} > 0$ for some x_0. Then there exists a subsequence $t_k \searrow 0\,(t_k \to \infty)$ such that $U_{t_k}/V_{t_k} \xrightarrow{\text{D}} T$, with $P\{T = x_0\} > 0$.*

Put

$$R_t = \frac{\sum_{i=1}^{\infty} \varphi^2\left(\frac{S_i}{t}\right)}{\left(\sum_{i=1}^{\infty} \varphi\left(\frac{S_i}{t}\right)\right)^2}. \tag{29}$$

Proposition 5. *Assume that $R_t^{-1} \neq O_P(1)$ as $t \searrow 0$ or $t \to \infty$, then there exists a subsequence $t_k \searrow 0$ or $t_k \to \infty$ such that $U_{t_k}/V_{t_k} \xrightarrow{\text{D}} T$, with $P\{T = EX\} > 0$.*

The proofs of Propositions 4 and 5 are adaptations of those of Theorems 4 and 5 in [11]. Therefore we only sketch the proof of the first one, and omit the proof of the second one.

4. Proofs of results

Recall that throughout this paper (U_t, V_t), $t \geq 0$, denotes a Lévy process satisfying (1) and (2) and having joint characteristic function (3). We start with the proof of Theorem 3 since this result is crucial for both the proofs of Theorems 1 and 2.

4.1. *Proof of Theorem 3*

Recall the notation at the beginning of Section 3. Since V_t is a driftless subordinator, by Theorem 15.14 (ii) in [10], (21) is equivalent to the convergences

$$t_k \overline{\Lambda}(v B_k) \to \overline{\Lambda}_0(v), \quad \text{as } k \to \infty, \tag{30}$$

for any $v > 0$ continuity point of $\overline{\Lambda}_0$, and

$$\int_0^v x t_k \Lambda(\mathrm{d}B_k x) \to \int_0^v x \Lambda_0(\mathrm{d}x) + b, \quad \text{as } k \to \infty, \tag{31}$$

where $v > 0$ is a fixed continuity point of $\overline{\Lambda}_0$.

Notice that using (19) we have that

$$E e^{\mathrm{i}\left(\theta_1 \frac{U_{t_k}}{B_k} + \theta_2 \frac{V_{t_k}}{B_k}\right)} = \exp\left\{\mathrm{i}\frac{t_k}{B_k}(\theta_1 b_1 + \theta_2 b_2)\right\}$$
$$\times \exp\left\{\int\!\!\int \left[e^{\mathrm{i}(\theta_1 u + \theta_2 v)/B_k} - 1 - \frac{\mathrm{i}}{B_k}(\theta_1 u + \theta_2 v)\mathbf{1}_{\{0 < u^2 + v^2 \leq 1\}}\right] t_k \Pi(\mathrm{d}u, \mathrm{d}v)\right\}$$

$$= \exp\left\{ \mathrm{i} \frac{t_k}{B_k} (\theta_1 b_1 + \theta_2 b_2) \right\}$$
$$\times \exp\left\{ \int\!\!\int \left[e^{\mathrm{i}(\theta_1 x + \theta_2 y)} - 1 - \mathrm{i}(\theta_1 x + \theta_2 y) \mathbf{1}_{\{0 < x^2 + y^2 \le B_k^{-2}\}} \right] \Pi_k(\mathrm{d}x, \mathrm{d}y) \right\},$$

where Π is the Lévy measure on $(0, \infty) \times \mathbb{R}$ defined by (4) and for each $k \ge 1$, Π_k is the Lévy measure on $(0, \infty) \times \mathbb{R}$ defined by

$$\Pi_k(\mathrm{d}x, \mathrm{d}y) = t_k \Pi(B_k \mathrm{d}x, B_k \mathrm{d}y).$$

Further, for each $k \ge 0$ and $h > 0$ with $\Pi_0(\{x : |x| = h\}) = 0$, in accordance with the notation at the beginning of Section 3, let

$$a_k^h = \int_{x^2 + y^2 \le h^2} \begin{pmatrix} x^2 & xy \\ xy & y^2 \end{pmatrix} \Pi_k(\mathrm{d}x, \mathrm{d}y),$$
$$\begin{aligned} b_k^h &= \frac{t_k}{B_k} \mathbf{b} - \int_{1 < x^2 + y^2 \le B_k^{-2}} (x, y) \Pi_k(\mathrm{d}x, \mathrm{d}y) - \int_{h^2 < x^2 + y^2 \le 1} (x, y) \Pi_k(\mathrm{d}x, \mathrm{d}y) \\ &= \int_{x^2 + y^2 \le h^2} (x, y) \Pi_k(\mathrm{d}x, \mathrm{d}y), \end{aligned}$$

where we used (20). We set $a^h := a_0^h$ and $b^h := b_0^h$.

To show (22), by Theorem 15.14(i) in [10] we have to prove that as $k \to \infty$,

$$\Pi_k \xrightarrow{v} \Pi_0, \text{ on } \mathbb{R}^2 - \{\mathbf{0}\} \tag{32}$$

and for some (any) $h > 0$ with $\Pi_0(\{x : |x| = h\}) = 0$, as $k \to \infty$,

$$a_k^h \to a^h, \tag{33}$$
$$b_k^h \to b^h. \tag{34}$$

To establish (32) it suffices to show that for each (u, v) with $u \ge 0$, $v > 0$, and (u, v), with $u > 0$, $v = 0$, that when (u, v) is a continuity point of $\overline{\Pi}_0$,

$$t_k \overline{\Pi}(B_k u, B_k v) \to \overline{\Pi}_0(u, v), \quad \text{as } k \to \infty,$$

and when $(-u, v)$ is a continuity point of Π_0,

$$t_k \Pi(-B_k u, B_k v) \to \Pi_0(-u, v), \quad \text{as } k \to \infty;$$

where for $u \ge 0$, $v > 0$,

$$t_k \overline{\Pi}(B_k u, B_k v) = \int_v^\infty \overline{F}(u/y) t_k \Lambda(\mathrm{d}B_k y),$$
$$\overline{\Pi}_0(u, v) = \int_v^\infty \overline{F}(u/y) \Lambda_0(\mathrm{d}y),$$
$$t_k \Pi(-B_k u, B_k v) = \int_v^\infty F(-u/y) t_k \Lambda(\mathrm{d}B_k y)$$

and

$$\Pi_0(-u, v) = \int_v^\infty F(-u/y)\Lambda_0(\mathrm{d}y).$$

This follows with obvious changes of notation exactly as in the proof of Proposition 1 in [11].

The proofs that (33) and (34) hold follow exactly as in Propositions 2 and 3 in [11]. It turns out that a^h converges to the zero matrix as $h \searrow 0$ and by (31)

$$b^h = \begin{pmatrix} bEX + \int_0^h \psi(v)\Lambda_0(\mathrm{d}v) \\ b + \int_0^h \phi(v)v\Lambda_0(\mathrm{d}v) \end{pmatrix},$$

where ψ and ϕ are the following functions of $v \in (0, h]$:

$$\phi(v) = \int_{\left[-\sqrt{h^2-v^2}, \sqrt{h^2-v^2}\right]} F\left(\frac{\mathrm{d}u}{v}\right) \quad \text{and} \quad \psi(v) = \int_{\left[-\sqrt{h^2-v^2}, \sqrt{h^2-v^2}\right]} uF\left(\frac{\mathrm{d}u}{v}\right).$$

(Refer to [11] for details.) Thus

$$\lim_{h\to 0} b^h = \begin{pmatrix} bEX \\ b \end{pmatrix},$$

and the theorem follows with the stated constants. □

4.2. *Proof of Theorem 1*

The following three lemmas establish the "in which case" parts of (i), (ii) and (iii) of Theorem 1.

Lemma 2. *If $\overline{\Lambda}$ is regularly varying at zero (infinity) with index $-\beta$ with $\beta \in (0, 1)$, then for an appropriate norming function B_t the random variable $B_t^{-1}(U_t, V_t)$ converges in distribution as $t \searrow 0$ (as $t \to \infty$) to (U, V), having joint characteristic function*

$$\phi(\theta_1, \theta_2) = \exp\left(\int_{(0,\infty)} \int_{-\infty}^{\infty} \left(e^{\mathrm{i}(\theta_1 u + \theta_2 v)} - 1\right) F(\mathrm{d}u/v)\, \beta v^{-1-\beta} \mathrm{d}v\right) \tag{35}$$

and thus

$$T_t = \frac{U_t}{V_t} \xrightarrow{\mathrm{D}} \frac{U}{V}, \quad \text{as } t \searrow 0 \text{ (as } t \to \infty). \tag{36}$$

Moreover, the cdf of U/V is given by (8).

Proof. We can find a function B_t on $[0, \infty)$ such that

$$B_t = L^*(t)\, t^{1/\beta}, \quad t > 0,$$

with L^* defined on $[0, \infty)$ slowly varying at zero (infinity) satisfying for all $y > 0$,

$$\overline{\mu}_t(y) := t\overline{\Lambda}(yB_t) \to \overline{\Lambda}_0(y) = y^{-\beta}, \quad \text{as } t \searrow 0 \text{ (as } t \to \infty).$$

It is routine to show using well-known properties of regularly varying functions that for any $y > 0$, as $t \searrow 0$ (as $t \to \infty$)

$$a_t^h := \int_{0<y\le h} y\mu_t(\mathrm{d}y) \to \frac{\beta h^{1-\beta}}{1-\beta} = \int_{0<y\le h} y\Lambda_0(\mathrm{d}y) =: a^h.$$

Thus by applying Theorem 15.14(ii) in [10] we find that $B_t^{-1} V_t$ converges in distribution as $t \searrow 0$ (as $t \to \infty$) to V, having characteristic function $\phi(0, \theta_2)$. This says that V is a subordinator with an id(0, Λ_0) distribution. Theorem 3 completes the proof of (35).

Next, using Fubini's theorem and the explicit formula for the β-stable characteristic function (Meerschaert and Scheffler [17] p. 266), we have for an appropriate constant $c > 0$

$$\begin{aligned}
&\int_{(0,\infty)} \int_{-\infty}^{\infty} \left(e^{\mathrm{i}(\theta_1 u + \theta_2 v)} - 1\right) F(\mathrm{d}u/v)\, \beta v^{-1-\beta} \mathrm{d}v \\
&\quad = \int_{-\infty}^{\infty} F(\mathrm{d}u) \int_0^{\infty} \left[e^{\mathrm{i}(\theta_1 u + \theta_2) y} - 1\right] \Lambda_0(\mathrm{d}y) \\
&\quad = -c \int_{-\infty}^{\infty} |\theta_1 u + \theta_2|^{\beta} \left(1 - \mathrm{i}\, \mathrm{sgn}(\theta_1 u + \theta_2) \tan \frac{\pi\beta}{2}\right) F(\mathrm{d}u).
\end{aligned}$$

We see now that the characteristic function of $U - Vx$ is

$$\begin{aligned}
E e^{\mathrm{i}t(U - Vx)} &= \exp\left\{ -|t|^{\beta} c \int |u - x|^{\beta} F(\mathrm{d}u) \right. \\
&\qquad \left. \times \left[1 - \mathrm{i}\,\mathrm{sgn}(t) \tan \frac{\pi\beta}{2} \frac{\int |u - x|^{\beta} \mathrm{sgn}(u - x) F(\mathrm{d}u)}{\int |u - x|^{\beta} F(\mathrm{d}u)}\right]\right\}. \qquad (37)
\end{aligned}$$

Proposition 4 in [5] now shows that T has cdf (8). □

Lemma 3. *If $\overline{\Lambda}$ is slowly varying at zero (at infinity), then*

$$T_t = \frac{U_t}{V_t} \xrightarrow{\mathrm{D}} X, \quad \text{as } t \searrow 0 \ (\text{as } t \to \infty), \qquad (38)$$

where in the $t \searrow 0$ case we also assume that $\overline{\Lambda}(0+) = \infty$.

Proof. The proof follows the lines of that of Lemma 5.3 in [12].

We shall only prove the $t \to \infty$ case. The $t \searrow 0$ case is nearly identical. Now $\overline{\Lambda}$ slowly varying at infinity implies that φ is non-increasing and rapidly varying at 0 with index $-\infty$. (See the argument in Item 5 on p. 22 of de Haan [8].) This means that for all $0 < \lambda < 1$

$$\varphi(x\lambda)/\varphi(x) \to \infty, \quad \text{as } x \searrow 0.$$

By Theorem 1.2.1 on p. 15 of [8], rapidly varying at 0 with index $-\infty$ implies that

$$\frac{\int_x^{\overline{\Lambda}(0+)} \varphi(y)\, \mathrm{d}y}{x\varphi(x)} \to 0, \quad \text{as } x \searrow 0. \qquad (39)$$

By Lemma 8 in Appendix, we have

$$\begin{aligned}
E\left(\frac{\sum_{i=2}^{\infty} |X_i| \varphi\left(\frac{S_i}{t}\right)}{\varphi\left(\frac{S_1}{t}\right)} \middle| S_1 \right) &= E|X|\, E\left(\frac{\sum_{i=2}^{\infty} \varphi\left(\frac{S_i}{t}\right)}{\varphi\left(\frac{S_1}{t}\right)} \middle| S_1 \right) \\
&= E|X|\, S_1 \frac{\int_{S_1/t}^{\overline{\Lambda}(0+)} \varphi(y)\, \mathrm{d}y}{\frac{S_1}{t} \varphi\left(\frac{S_1}{t}\right)},
\end{aligned}$$

and by (39)

$$E\,|X|\,S_1 \frac{\int_{S_1/t}^{\overline{\Lambda}(0+)} \varphi\,(y)\,\mathrm{d}y}{\frac{S_1}{t}\varphi\left(\frac{S_1}{t}\right)} \xrightarrow{\mathrm{P}} 0, \qquad \text{as } t \to \infty.$$

From this we can readily conclude that

$$\sum_{i=1}^{\infty} \varphi\left(\frac{S_i}{t}\right) = \varphi\left(\frac{S_1}{t}\right)(1 + o_P\,(1))\,, \qquad \text{as } t \to \infty, \tag{40}$$

and

$$\sum_{i=1}^{\infty} X_i \varphi\left(\frac{S_i}{t}\right) = X_1 \varphi\left(\frac{S_1}{t}\right)(1 + o_P\,(1))\,, \qquad \text{as } t \to \infty. \tag{41}$$

From the representation (12) and the stochastic identities (40) and (41) we see that

$$\frac{U_t}{V_t} \stackrel{D}{=} \frac{X_1 \varphi\left(\frac{S_1}{t}\right)(1 + o_P\,(1))}{\varphi\left(\frac{S_1}{t}\right)(1 + o_P\,(1))} = X_1 + o_P\,(1)\,, \qquad \text{as } t \to \infty.$$

Obviously T_t converges in distribution as $t \to \infty$ to X. $\square$

Lemma 4. *If $\overline{\Lambda}$ is regularly varying at zero (at infinity) with index -1,*

$$T_t = \frac{U_t}{V_t} \xrightarrow{\mathrm{D}} EX, \quad \textit{as } t \searrow 0 \ (\textit{as } t \to \infty). \tag{42}$$

Proof. Since $\overline{\Lambda}$ is regularly varying at zero (at infinity) with index -1, we can find norming and centering functions $b\,(t)$ and $a\,(t)$ such that $b(t)/a(t) \to 0$ as $t \searrow 0$ (as $t \to \infty$) and

$$b\,(t)^{-1}\,(V_t - a\,(t)) \xrightarrow{\mathrm{D}} W, \quad \text{as } t \searrow 0 \ (\text{as } t \to \infty),$$

where W is a nondegenerate random variable. (Here we apply part (i) of Theorem 15.14 in [10].) From this we see that

$$V_t/a(t) \xrightarrow{\mathrm{P}} 1, \quad \text{as } t \searrow 0 \ (\text{as } t \to \infty).$$

A straightforward application of Theorem 3 now shows that

$$\left(\frac{U_t}{a(t)}, \frac{V_t}{a(t)}\right) \xrightarrow{\mathrm{P}} (EX, 1)\,, \quad \text{as } t \searrow 0 \ (\text{as } t \to \infty). \quad \square$$

Next we turn to the necessary and sufficient parts of (i), (ii) and (iii). Assume that for some random variable T

$$T_t \xrightarrow{\mathrm{D}} T, \quad \text{as } t \searrow 0 \ (\text{as } t \to \infty), \tag{43}$$

where in the case $t \searrow 0$ we assume that $\overline{\Lambda}(0+) = \infty$. Our basic tool will be Proposition 1, which says that

$$T_t = \frac{U_t}{V_t} \stackrel{\mathrm{D}}{=} \frac{\sum_{i=1}^{\infty} X_i \varphi\left(\frac{S_i}{t}\right)}{\sum_{i=1}^{\infty} \varphi\left(\frac{S_i}{t}\right)}. \tag{44}$$

Since we assume that

$$E\,|X|^p < \infty \tag{45}$$

for some $p > 2$, we get by Jensen's inequality that

$$E\,|T_t|^p \le E\,|X|^p < \infty.$$

(This is the only place in the proof that we use assumption (45).) Notice that (43) and (45) imply that

$$ET_t^2 \to ET^2, \quad \text{as } t \searrow 0 \text{ (as } t \to \infty). \tag{46}$$

Obviously $ET_t = EX$ and a little calculation gives that

$$ET_t^2 = (EX)^2 + Var(X)ER_t,$$

where R_t is defined as in (29). Clearly, $R_t \in [0, 1]$ and whenever (46) holds, then for some $0 \le \beta \le 1$

$$ER_t \to 1 - \beta, \quad \text{as } t \searrow 0 \text{ (as } t \to \infty), \tag{47}$$

which is equivalent to

$$(EX)^2 \le ET^2 \le EX^2. \tag{48}$$

It turns out that the value of $0 \le \beta \le 1$ determines the asymptotic distribution of T_t as $t \searrow 0$ (as $t \to \infty$) and the behavior of the Lévy function $\overline{\Lambda}$ near zero (at infinity). For instance, when $\beta = 1$, $Var\,(T_t) \to 0$, which implies that

$$T_t \xrightarrow{\mathrm{P}} EX, \quad \text{as } t \searrow 0 \text{(as } t \to \infty). \tag{49}$$

In general we have the following result, which in combination with Lemmas 2–4 will complete the proof of Theorem 1.

Proposition 6. *If (47) holds for some $0 \le \beta \le 1$, then $\overline{\Lambda}$ is regularly varying at zero (infinity) with index $-\beta$. (In the case $t \searrow 0$ we assume that $\overline{\Lambda}(0+) = \infty$.)*

Proof. Recall the definition of $N(t)$ in (9) and notice that by (29) for any $t > 0$ we can write

$$R_t = \frac{\int_0^\infty \varphi^2(s)\, N(\mathrm{d}ts)}{\left(\int_0^\infty \varphi(s)\, N(\mathrm{d}ts)\right)^2}.$$

Define for $T > 0$ its truncated version

$$R_t(T) = \frac{\int_0^T \varphi^2(s)\, N(\mathrm{d}ts)}{\left(\int_0^T \varphi(s)\, N(\mathrm{d}ts)\right)^2}. \tag{50}$$

Given that $N(Tt) = n$

$$R_t(T) \stackrel{\mathrm{D}}{=} \frac{\sum_{i=1}^{n} \varphi^2(V_i)}{\left(\sum_{i=1}^{n} \varphi(V_i)\right)^2},$$

where $V_1, \ldots, V_n$ are i.i.d. Uniform$(0, T)$. The same computation as in Maller and Mason [12] gives

$$E R_t(T) = t \int_0^\infty u \left(\int_0^T \varphi^2(s) e^{-u\varphi(s)} \mathrm{d}s \right) e^{-t \int_0^T (1-e^{-u\varphi(s)}) \mathrm{d}s} \, \mathrm{d}u.$$

Clearly $R_t(T) \le 1$. Also $R_t(T) \xrightarrow{\mathrm{D}} R_t$ as $T \to \infty$ and thus

$$E R_t(T) \to E R_t, \quad \text{as } T \to \infty. \tag{51}$$

For each $T > 0$ and $u > 0$, set

$$\Phi_T(u) = \int_0^T (1 - e^{-u\varphi(s)}) \mathrm{d}s, \qquad \Phi(u) = \int_0^\infty (1 - e^{-u\varphi(s)}) \mathrm{d}s \quad \text{and}$$

$$f_{T,t}(u) = -tu \Phi_T''(u) e^{-t\Phi_T(u)} = tu \left(\int_0^T \varphi^2(s) e^{-u\varphi(s)} \mathrm{d}s \right) e^{-t \int_0^T (1-e^{-u\varphi(s)}) \mathrm{d}s}. \tag{52}$$

Also for $u > 0$, set

$$f_{(t)}(u) = -tu \Phi''(u) e^{-t\Phi(u)} = tu \left(\int_0^\infty \varphi^2(s) e^{-u\varphi(s)} \mathrm{d}s \right) e^{-t \int_0^\infty (1-e^{-u\varphi(s)}) \mathrm{d}s}. \tag{53}$$

We have in this notation,

$$E R_t(T) = \int_0^\infty f_{T,t}(u) \, \mathrm{d}u. \tag{54}$$

Case 1: $\beta \in [0, 1)$. In this case we must first show that as $T \to \infty$

$$E R_t(T) = \int_0^\infty f_{T,t}(u) \, \mathrm{d}u \to \int_0^\infty f_{(t)}(u) \, \mathrm{d}u, \tag{55}$$

which by (51) implies

$$\int_0^\infty f_{(t)}(u) \, \mathrm{d}u = E R_t. \tag{56}$$

It turns out to be surprisingly tricky to justify the passing-to-the-limit in (55). Lemma 9 and Proposition 7 in Appendix handle this problem, and imply that expression (56) is valid for $E R_t$. After this identity is established, the proof is completed by an easy modification of that of Proposition 5.2 in [12], which is based on Tauberian theorems. Therefore we omit it.

Case 2: $\beta = 1$. In this case, it is not necessary to verify (55). Note that we have that by (47) with $\beta = 1$

$$E R_t \to 0, \quad \text{as } t \searrow 0 \ (t \to \infty).$$

Therefore since

$$ER_t(T) \to ER_t \geq \int_0^\infty f_{(t)}(u)\,\mathrm{d}u,$$

we can conclude that as $t \searrow 0$ ($t \to \infty$),

$$-t\int_0^\infty u\Phi''(u)e^{-t\Phi(u)}\mathrm{d}u = \int_0^\infty f_{(t)}(u)\,\mathrm{d}u \to 0, \tag{57}$$

which is all we need for the following argument to work for $\beta = 1$. Applying Lemma 1, we get

$$\Phi(u) = \int_0^\infty (1 - e^{-ux})\Lambda(\mathrm{d}x),$$

which by integrating by parts and using (2) is equal to

$$\Phi(u) = u\int_0^\infty \overline{\Lambda}(y)e^{-uy}\mathrm{d}y.$$

Let $q(y)$ denote the inverse function of Φ. From the expression for $f_{(t)}(u)$ in (53) and (57) we obtain

$$t^{-1}\int_0^\infty f_{(t)}(u)\,\mathrm{d}u = -\int_0^\infty e^{-ty}q(y)\Phi''(q(y))q(\mathrm{d}y) \sim o\left(t^{-1}\right),$$

as $t \to 0$ ($t \to \infty$). Using Theorem 1.7.1 (Theorem 1.7.1') in Bingham et al. [2] we obtain

$$-\int_0^x q(y)\Phi''(q(y))q(\mathrm{d}y) \sim o(x),$$

as $x \to \infty$ ($x \to 0$). Changing the variables and putting $x = \Phi(v)$ we have

$$-\int_0^v u\Phi''(u)\mathrm{d}u = o(\Phi(v)),$$

as $v \to \infty$ ($v \to 0$). Integrating by parts we get

$$-\int_0^v u\Phi''(u)\mathrm{d}u = -v\Phi'(v) + \Phi(v) = o(\Phi(v)),$$

which gives

$$\frac{v\Phi'(v)}{\Phi(v)} \to 1,$$

as $v \to \infty$ ($v \to 0$). This last limit readily implies that

$$v^{-1}\Phi(v) = \int_0^\infty \overline{\Lambda}(y)e^{-vy}\mathrm{d}y$$

is slowly varying at infinity (zero). By Theorem 1.7.1' (Theorem 1.7.1) in [2] we obtain that $\int_0^x \overline{\Lambda}(y)\mathrm{d}y$ is slowly varying at zero (infinity), which by Theorem 1.7.2.b (Theorem 1.7.2) in [2] implies that $\overline{\Lambda}$ is regularly varying at zero with index -1 (at infinity). $\square$

4.3. Proof of Theorem 2

Before we proceed with the proofs it will be helpful to first cite some results from Maller and Mason [13–15].

Let Y_t be a Lévy process with Lévy triplet (σ^2, γ, ν), i.e. Y_1 has $\mathrm{id}(\sigma^2, \gamma, \nu)$ distribution. Theorem 1 in Maller and Mason [13] states that Y_t belongs to the Feller class at infinity, if and only if

$$\limsup_{x\to\infty} \frac{x^2\nu\{(-\infty,-x)\cup(x,\infty)\}}{\sigma^2+\int_{|y|\le x} y^2\nu(\mathrm{d}y)} < \infty, \tag{58}$$

and furthermore Y_t belongs to the *centered Feller class* at infinity if and only if

$$\limsup_{x\to\infty} \frac{x^2\nu\{(-\infty,-x)\cup(x,\infty)\} + x\left|\gamma+\int_{1<|y|\le x} y\nu(\mathrm{d}y)\right|}{\sigma^2+\int_{|y|\le x} y^2\nu(\mathrm{d}y)} < \infty. \tag{59}$$

For the corresponding equivalences of *Feller class* at zero and *centered Feller class* at zero replace $x \to \infty$ by $x \searrow 0$, respectively; see Theorems 2.1 and 2.3 in [14].

It turns out by using the assumption that V_t is a subordinator and by arguing as in the proof of Proposition 1 or of Proposition 5.1 in [12] we get that

$$\sqrt{R_t^{-1}} = \frac{\sum_{i=1}^{\infty}\varphi\left(\frac{S_i}{t}\right)}{\sqrt{\sum_{i=1}^{\infty}\varphi^2\left(\frac{S_i}{t}\right)}} \overset{\mathrm{D}}{=} \frac{V_t}{\sqrt{\sum_{0\le s\le t}(\Delta V_t)^2}}.$$

From this distributional equality one can conclude that $\sqrt{R_t^{-1}}$ is stochastically bounded as $t \searrow 0\,(t \to \infty)$ if and only if

$$\limsup_{t\searrow 0\ (t\to\infty)} \frac{t\int_0^t x\Lambda(\mathrm{d}x)}{\int_0^t x^2\Lambda(\mathrm{d}x) + t^2\overline{\Lambda}(t)} < \infty \tag{60}$$

by applying Theorem 3.1 in [15] in the case $t \to \infty$, and Proposition 5.1 in [14] (with $a(t) \equiv 0$ there, and a small modification) when $t \searrow 0$. The partial sum version of this result was proved by Griffin [7].

Proof of Proposition 3. We first assume that X is nondegenerate and $EX = 0$, which implies that there is an $a \ge 1$ such that

$$F(a) - F(0) > 0 \quad \text{and} \quad F(0) - F(-a) > 0. \tag{61}$$

We need the following lemma.

Lemma 5. *Whenever (26) holds and X is nondegenerate and $EX = 0$, there exist $0 < \kappa < 1$ and $d > 0$ such that with $a \ge 1$ as in (61), if $2a(|\theta_1| \vee |\theta_2|) \ge 1$, then*

$$\mathfrak{Re}\left\{\int_{(0,\infty)}\int_{\mathbb{R}}\left(e^{\mathrm{i}(\theta_1 x+\theta_2 v)}-1\right)F\left(\frac{\mathrm{d}x}{v}\right)\Lambda(\mathrm{d}v)\right\} \le -d\left(|\theta_1|^\kappa + |\theta_2|^\kappa\right). \tag{62}$$

Proof. Notice that

$$\begin{aligned}
&\mathfrak{Re} \int_{(0,\infty)} \int_{\mathbb{R}} \left(e^{\mathrm{i}(\theta_1 x + \theta_2 v)} - 1 \right) F\left(\frac{\mathrm{d}x}{v}\right) \Lambda(\mathrm{d}v) \\
&\quad = \int_{(0,\infty)} \int_{\mathbb{R}} (\cos(\theta_1 x + \theta_2 v) - 1)\, F\left(\frac{\mathrm{d}x}{v}\right) \Lambda(\mathrm{d}v) \\
&\quad \le \int_{0 \le v \le 1/(2a(|\theta_1| \vee |\theta_2|))} \int_{|x| \le va} (\cos(\theta_1 x + \theta_2 v) - 1)\, F\left(\frac{\mathrm{d}x}{v}\right) \Lambda(\mathrm{d}v).
\end{aligned}$$

Observe that whenever $|x| \le av$ with $a \ge 1$ and $0 \le v \le 1/\left(2a\left(|\theta_1| \vee |\theta_2|\right)\right)$,

$$|\theta_1 x| + |\theta_2 v| \le (|a\theta_1| + |\theta_2|)\, v \le 1.$$

For some $c > 0$,

$$\sup_{0 \le |u| \le 1} \frac{\cos u - 1}{u^2} \le -c,$$

thus

$$\begin{aligned}
&\int_{0 \le v \le 1/(2a(|\theta_1| \vee |\theta_2|))} \int_{|x| \le va} (\cos(\theta_1 x + \theta_2 v) - 1)\, F\left(\frac{\mathrm{d}x}{v}\right) \Lambda(\mathrm{d}v) \\
&\quad \le -c \int_{0 \le v \le 1/(2a(|\theta_1| \vee |\theta_2|))} \int_{|x| \le av} (\theta_1 x + \theta_2 v)^2\, F\left(\frac{\mathrm{d}x}{v}\right) \Lambda(\mathrm{d}v).
\end{aligned}$$

Now when $\theta_1\theta_2 \ge 0$ we have $\theta_1\theta_2 \int_{0 \le x \le va} x F\left(\frac{\mathrm{d}x}{v}\right) \ge 0$, and we get that the last bound is

$$\le -c \int_{0 \le v \le 1/(2a(|\theta_1| \vee |\theta_2|))} \int_{0 \le x \le av} \left(\theta_1^2 x^2 + \theta_2^2 v^2\right) F\left(\frac{\mathrm{d}x}{v}\right) \Lambda(\mathrm{d}v),$$

and when $\theta_1\theta_2 < 0$ we have $\theta_1\theta_2 \int_{-va \le x \le 0} x F\left(\frac{\mathrm{d}x}{v}\right) \ge 0$, which gives

$$\begin{aligned}
&\int_{0 \le v \le 1/(2a(|\theta_1| \vee |\theta_2|))} \int_{|x| \le va} (\cos(\theta_1 x + \theta_2 v) - 1)\, F\left(\frac{\mathrm{d}x}{v}\right) \Lambda(\mathrm{d}v) \\
&\quad \le -c \int_{0 \le v \le 1/(2a(|\theta_1| \vee |\theta_2|))} \int_{-av \le x \le 0} \left(\theta_1^2 x^2 + \theta_2^2 v^2\right) F\left(\frac{\mathrm{d}x}{v}\right) \Lambda(\mathrm{d}v).
\end{aligned}$$

Notice that

$$\begin{aligned}
&c \int_{0 \le v \le 1/(2a(|\theta_1| \vee |\theta_2|))} \int_{0 \le x \le av} \theta_2^2 v^2 F\left(\frac{\mathrm{d}x}{v}\right) \Lambda(\mathrm{d}v) \\
&\quad = c\,(F(a) - F(0)) \int_{0 \le v \le 1/(2a(|\theta_1| \vee |\theta_2|))} \theta_2^2 v^2 \Lambda(\mathrm{d}v).
\end{aligned}$$

We get by arguing as on the top of page 968 in Pruitt [18] or in the remark after the proof of Proposition 6.1 in Buchmann et al. [4], that for some $c_1 > 0$ and $0 < \kappa < 1$, whenever $2a\left(|\theta_1| \vee |\theta_2|\right) \ge 1$

$$\begin{aligned}
&-c\,(F(a) - F(0))\, \theta_2^2 \int_{0 \le v \le 1/(2a(|\theta_1| \vee |\theta_2|))} v^2 \Lambda(\mathrm{d}v) \\
&\quad \le -\frac{c_1 \theta_2^2}{4a^2\left(|\theta_1| \vee |\theta_2|\right)^2} \left(2a\left(|\theta_1| \vee |\theta_2|\right)\right)^{\kappa}.
\end{aligned}$$

Next,

$$-c\int_{0\le v\le 1/(2a(|\theta_1|\vee|\theta_2|))}\int_{0\le x\le av}\theta_1^2x^2F\left(\frac{\mathrm{d}x}{v}\right)\Lambda(\mathrm{d}v)$$
$$=-c\theta_1^2\int_{0\le x\le a}u^2F\,(\mathrm{d}u)\int_{0\le v\le 1/(2a(|\theta_1|\vee|\theta_2|))}v^2\Lambda(\mathrm{d}v),$$

which by the previous argument is for some $c_2>0$, for $2a\,(|\theta_1|\vee|\theta_2|)\ge 1$

$$\le -\frac{c_2\theta_1^2}{(2a\,(|\theta_1|\vee|\theta_2|))^2}\,(2a\,(|\theta_1|\vee|\theta_2|))^{\kappa}\,.$$

Thus with $c_3=c_1\wedge c_2$,

$$-c\int_{0\le v\le 1/(2a(|\theta_1|\vee|\theta_2|))}\int_{0\le x\le av}(\theta_1^2x^2+\theta_2^2v^2)F\left(\frac{\mathrm{d}x}{v}\right)\Lambda(\mathrm{d}v)$$
$$\le -c_3\left(\frac{\theta_1^2}{4a^2\,(|\theta_1|\vee|\theta_2|)^2}+\frac{\theta_2^2}{4a^2\,(|\theta_1|\vee|\theta_2|)^2}\right)(2a\,(|\theta_1|\vee|\theta_2|))^{\kappa}\,.$$

Notice that

$$\frac{\theta_1^2}{4a^2\,(|\theta_1|\vee|\theta_2|)^2}+\frac{\theta_2^2}{4a^2\,(|\theta_1|\vee|\theta_2|)^2}\ge\frac{1}{4a^2}.$$

Hence when $\theta_1\theta_2>0$ and $2a\,(|\theta_1|\vee|\theta_2|)\ge 1$ for some $c_4>0$,

$$-\,c\int_{0\le v\le 1/(2a(|\theta_1|\vee|\theta_2|))}\int_{0\le x\le av}(\theta_1^2x^2+\theta_2^2v^2)F\left(\frac{\mathrm{d}x}{v}\right)\Lambda(\mathrm{d}v)$$
$$\le -c_4\,(|\theta_1|\vee|\theta_2|)^{\kappa}\,. \tag{63}$$

The analogous inequality holds when $\theta_1\theta_2\le 0$ and $2a\,(|\theta_1|\vee|\theta_2|)\ge 1$, namely for some $c_5>0$,

$$\int_{0\le v\le 1/(2a(|\theta_1|\vee|\theta_2|))}\int_{|x|\le va}(\cos(\theta_1x+\theta_2v)-1)\,F\left(\frac{\mathrm{d}x}{v}\right)\Lambda(\mathrm{d}v)$$
$$\le -c\int_{0\le v\le 1/(2a(|\theta_1|\vee|\theta_2|))}\int_{-av\le x\le 0}\left(\theta_1^2x^2+\theta_2^2v^2\right)F\left(\frac{\mathrm{d}x}{v}\right)\Lambda(\mathrm{d}v)$$
$$\le -c_5\,(|\theta_1|\vee|\theta_2|)^{\kappa}\,. \tag{64}$$

Note that since $0<\kappa<1$ the function $\rho\,(u)=|u|^{\kappa}$ is concave on $(0,\infty)$, and thus

$$(|\theta_1|\vee|\theta_2|)^{\kappa}\ge\left|\frac{|\theta_1|+|\theta_2|}{2}\right|^{\kappa}\ge\frac{|\theta_1|^{\kappa}+|\theta_2|^{\kappa}}{2},$$

which, in combination with (63) and (64), gives for some $d>0$, whenever $2a\,(|\theta_1|\vee|\theta_2|)\ge 1$,

$$\int_{0\le v\le 1/(2a(|\theta_1|\vee|\theta_2|))}\int_{|x|\le va}(\cos(\theta_1x+\theta_2v)-1)\,F\left(\frac{\mathrm{d}x}{v}\right)\Lambda(\mathrm{d}v)$$
$$\le -d\left(|\theta_1|^{\kappa}+|\theta_2|^{\kappa}\right).\quad\square$$

The lemma implies that whenever $2a\,(|\theta_1|\vee|\theta_2|)\ge 1$, then for some $d>0$ and $0<\kappa<1$,

$$\left|Ee^{\mathrm{i}(\theta_1U+\theta_2V)}\right|\le\exp\left(-d\left(|\theta_1|^{\kappa}+|\theta_2|^{\kappa}\right)\right).$$

As in [18] this allows us to apply the inversion formula for densities and shows that it may be repeatedly differentiated, from which we readily infer that (U, V) has a C^{∞} density when $EX = 0$. If $EX = \mu \neq 0$, the same argument applied to $(U', V) = (U - \mu V, V)$ shows that (U', V) has a C^{∞} density, which by a simple transformation implies that (U, V) does too. $\square$

Proof of Corollary 2. Note that each $V_{t_k}/B(t_k)$ is an infinitely divisible random variable without a normal component with Lévy measure concentrated on $(0, \infty)$ given by $t_k \Lambda(\cdot B(t_k))$ with characteristic function

$$\Psi_k(\theta) = \exp\left\{ i\theta b_k + \int_0^{\infty} \left(e^{i\theta x} - 1 - i\theta x \mathbf{1}_{\{0<x\le 1\}} \right) t_k \Lambda(B(t_k)\mathrm{d}x) \right\},$$

where

$$b_k = \int_0^1 x t_k \Lambda(B(t_k)\mathrm{d}x).$$

Since $V_{t_k}/B(t_k) \xrightarrow{\mathrm{D}} W_2$, by Proposition 7.8 of Sato [21], W_2 is infinitely divisible. Since W_2 is necessarily non-negative, it does not have a normal component and has a Lévy measure Λ_0 concentrated on $(0, \infty)$. Now by Theorem 3 and its proof, necessarily $\int_0^1 x\Lambda_0(\mathrm{d}x) < \infty$ and W_2 has characteristic function

$$\Psi_0(\theta) = \exp\left\{ i\theta b + \int_0^{\infty} \left(e^{i\theta x} - 1 \right) \Lambda_0(\mathrm{d}x) \right\},$$

where $b \ge 0$. By (30) and (31) in the proof of Theorem 3 for any continuity point $v > 0$ of $\overline{\Lambda}_0$,

$$t_k \overline{\Lambda}(vB(t_k)) \to \overline{\Lambda}_0(v), \quad \text{as } k \to \infty, \tag{65}$$

and

$$\int_0^v x t_k \Lambda(B(t_k)\mathrm{d}x) \to \int_0^v x\Lambda_0(\mathrm{d}x) + b, \quad \text{as } k \to \infty. \tag{66}$$

From (66) we easily get that for any continuity point $v > 0$ of $\overline{\Lambda}_0$,

$$\int_0^v x^2 t_k \Lambda(B(t_k)\mathrm{d}x) \to \int_0^v x^2 \Lambda_0(\mathrm{d}x) = V_{0,2}(v), \quad \text{as } k \to \infty. \tag{67}$$

(Recall the notation (25).) Now, since V_t is in the centered Feller class, (59) implies that for some $K > 0$

$$\limsup_{k\to\infty} \frac{v^2 B^2(t_k)\overline{\Lambda}(vB(t_k))}{V_2(vB(t_k))} \le K. \tag{68}$$

Note that

$$\frac{v^2 B^2(t_k)\overline{\Lambda}(vB(t_k))}{V_2(vB(t_k))} = \frac{v^2 t_k \overline{\Lambda}(vB(t_k))}{\int_0^v x^2 t_k \Lambda(B(t_k)\mathrm{d}x)},$$

which by (65) and (67) converges to $v^2\overline{\Lambda}_0(v)/V_{0,2}(v)$ for each continuity point $v > 0$ of $\overline{\Lambda}_0$. This with (68) implies that

$$\sup_{v>0} \frac{v^2\overline{\Lambda}_0(v)}{\int_0^v x^2\Lambda_0(\mathrm{d}x)} \le K,$$

so Proposition 3 applies. $\square$

Proof of Proposition 4. The proof is a simple adaptation of the proof of Theorem 4 in [11], so we only sketch it here. Putting

$$B_t(k) = \left\{ \frac{\left|\sum_{i=2}^{\infty} X_i \varphi(S_i/t)\right|}{\sum_{i=1}^{\infty} \varphi(S_i/t)} \le \frac{E|X|}{\sqrt{k}} \right\},$$

and recalling definition (27), the conditional version of Chebyshev's inequality implies that $P\{B_t(k)|A_t(k^{-1})\} \ge 1 - 1/\sqrt{k}$. Noticing that on the set $B_t(k) \cap A_t(k^{-1})$

$$\Delta_t \le \frac{|X_1|}{k} + \frac{E|X|}{\sqrt{k}},$$

a tightness argument finishes the proof. □

Now we are ready to prove Theorem 2.

Choose any cdf F in the class $\mathcal{X}$. Corollary 2 says whenever V_t is in the centered Feller class at 0 (∞) then every subsequential limit law of U_t/V_t, as $t \searrow 0$, (as $t \to \infty$) has a Lebesgue density on $\mathbb{R}$ and hence is continuous.

Suppose V_t is in the Feller class at 0 (∞), but not in the centered Feller class at 0 (∞). In this case Corollary 3 implies that one of the subsequential limits is the constant EX and thus not continuous. Next Proposition 5.5 in [14] in the case $t \searrow 0$ and Proposition 3.2 in [15] in the case $t \to \infty$ show that whenever V_t is not in the Feller class at 0 (∞), that is

$$\limsup_{t \searrow 0\ (t \to \infty)} \frac{t^2 \overline{\Lambda}(t)}{\int_0^t y^2 \Lambda(\mathrm{d}y)} = \infty,$$

and (60) holds, then there exists a subsequence $t_k \searrow 0$ ($t_k \to \infty$), such that (28) holds, which by Corollary 4 for any X such that $P\{X = x_0\} > 0$ for some x_0, there exists a subsequence $t_k \searrow 0$ ($t_k \to \infty$) such that $U_{t_k}/V_{t_k} \xrightarrow{\mathrm{D}} T$, with $P\{T = x_0\} > 0$, that is, T is not continuous. Finally, assume that (60) does not hold, then by Proposition 5 there exists a subsequence $t_k \searrow 0$ or $t_k \to \infty$ such that $U_{t_k}/V_{t_k} \xrightarrow{\mathrm{D}} T$, with $P\{T = EX\} > 0$, and again T is not continuous. This completes the proof of Theorem 2.

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Appendix

To finish the proofs of Proposition 6 and thus Theorem 1 we shall require the following technical result.

Proposition 7. *Assume that*

$$\liminf_{s \searrow 0} \frac{s\overline{\Lambda}(s)}{\int_0^s \overline{\Lambda}(x)\mathrm{d}x} > 0, \tag{69}$$

then

$$ER_t = \int_0^\infty f_{(t)}(u)\,\mathrm{d}u = -t\int_0^\infty u\,\Phi''(u)\,e^{-t\,\Phi(u)}\mathrm{d}u. \tag{70}$$

Proof. Clearly for each $u > 0$, $f_{T,t}(u) \to f_{(t)}(u)$, as $T \to \infty$. Therefore by Fatou's lemma

$$\int_0^\infty f_{(t)}(u)\,\mathrm{d}u \le \liminf_{T\to\infty}\int_0^\infty f_{T,t}(u)\,\mathrm{d}u = \liminf_{T\to\infty} ER_t(T) \le 1. \tag{71}$$

Keeping in mind (51) and (54), this implies that

$$\int_0^\infty f_{(t)}(u)\,\mathrm{d}u \le ER_t \le 1.$$

Therefore on account of (51) to prove (70) it suffices to establish (55), as $T \to \infty$. One can readily check using (11) that for some constants $C_1 > 0$ and $C_2 > 0$ and all $u > 0$

$$0 \le -tu\,\Phi''(u) \le t\left(C_1 + u^{-1}C_2\right).$$

To see this note that for each $u > 0$

$$\begin{aligned}-u\,\Phi''(u) &= u\int_0^\infty x^2 e^{-ux}\Lambda(\mathrm{d}x)\\ &= \int_0^1 x^2 u e^{-ux}\Lambda(\mathrm{d}x) + u^{-1}\int_1^\infty u^2x^2e^{-ux}\Lambda(\mathrm{d}x),\\ &\le \max_{0\le y} ye^{-y}\int_0^1 x\Lambda(\mathrm{d}x) + u^{-1}\overline{\Lambda}(1)\max_{0\le y} y^2e^{-y} =: C_1 + u^{-1}C_2.\end{aligned}$$

Thus since

$$f_{T,t}(u) \le -ut\,\Phi_T''(u) \le -ut\,\Phi''(u),$$

we get by the bounded convergence theorem that for all $D > \delta > 0$

$$\lim_{T\to\infty}\int_\delta^D f_{T,t}(u)\,\mathrm{d}u = \int_\delta^D f_{(t)}(u)\,\mathrm{d}u.$$

Notice that since

$$\int_0^\infty f_{(t)}(u)\,\mathrm{d}u \le 1,$$

we have

$$\lim_{\delta\to 0}\int_0^\delta f_{(t)}(u)\,\mathrm{d}u = 0 \quad \text{and} \quad \lim_{D\to\infty}\int_D^\infty f_{(t)}(u)\mathrm{d}u = 0.$$

We see now that to complete the verification of (55) we have to show that

$$\lim_{\delta\to 0}\limsup_{T\to\infty}\int_0^\delta f_{T,t}(u)\,\mathrm{d}u = 0 \tag{72}$$

and

$$\lim_{D\to\infty} \limsup_{T\to\infty} \int_D^\infty f_{T,t}(u)\mathrm{d}u = 0. \tag{73}$$

The first condition (72) is easy to show. Recalling (52), notice that

$$f_{T,t}(u) \le tu \int_0^\infty \varphi^2(s) e^{-u\varphi(s)} \mathrm{d}s,$$

and so by Fubini

$$\begin{aligned} \int_0^\delta f_{T,t}(u)\mathrm{d}u &\le t \int_0^\infty \varphi^2(s)\mathrm{d}s \int_0^\delta u e^{-u\varphi(s)}\mathrm{d}u \\ &= t\int_0^\infty \left[-\varphi(s)\delta e^{-\delta\varphi(s)} + (1-e^{-\delta\varphi(s)})\right]\mathrm{d}s \\ &= t\left(\Phi(\delta) - \delta\Phi'(\delta)\right) \le t\Phi(\delta), \end{aligned}$$

which goes to 0 as $\delta \to 0$ and thus (72) holds.

For the second condition (73), choose $D > 0$. We see that for all large enough $T > 0$

$$\int_D^\infty f_{T,t}(u)\mathrm{d}u = \int_D^{1/\varphi(T)} f_{T,t}(u)\mathrm{d}u + \int_{1/\varphi(T)}^\infty f_{T,t}(u)\mathrm{d}u. \tag{74}$$

Recall that

$$f_{T,t}(u) = tu\int_0^T \varphi^2(s)e^{-u\varphi(s)}\mathrm{d}s \exp\left\{-t\int_0^T \left(1-e^{-u\varphi(s)}\right)\mathrm{d}s\right\}. \tag{75}$$

We shall first bound the second integral on the right side of (74). For $u\varphi(T) \ge 1$ and keeping in mind that $\varphi(s) \ge \varphi(T)$ for $0 < s \le T$, we have

$$\exp\left\{-t\int_0^T \left(1-e^{-u\varphi(s)}\right)\mathrm{d}s\right\} \le e^{-t(1-e^{-1})T}$$

and so

$$\int_{1/\varphi(T)}^\infty f_{T,t}(u)\mathrm{d}u \le te^{-t(1-e^{-1})T}\int_{1/\varphi(T)}^\infty u\int_0^T \varphi^2(s)e^{-u\varphi(s)}\mathrm{d}s\mathrm{d}u.$$

Using Fubini's theorem the last integral is easy to calculate. We get

$$\begin{aligned} \int_{1/\varphi(T)}^\infty u\int_0^T \varphi^2(s)e^{-u\varphi(s)}\mathrm{d}s\mathrm{d}u &= \int_0^T \varphi^2(s)\mathrm{d}s\int_{1/\varphi(T)}^\infty ue^{-u\varphi(s)}\mathrm{d}u \\ &= \int_0^T\left(e^{-\varphi(s)/\varphi(T)} + \frac{\varphi(s)}{\varphi(T)}e^{-\varphi(s)/\varphi(T)}\right)\mathrm{d}s \\ &\le T\left(1+\max_{y\ge 0} ye^{-y}\right) \le 2T. \end{aligned}$$

So we obtain

$$\int_{1/\varphi(T)}^\infty f_{T,t}(u)\mathrm{d}u \le 2Tte^{-t(1-e^{-1})T}, \tag{76}$$

which tends to 0 as $T \to \infty$.

Therefore to complete the verification that (73) holds and thus (55) we must prove that

$$\lim_{D\to\infty} \limsup_{T\to\infty} \int_D^{1/\varphi(T)} f_{T,t}(u)\mathrm{d}u = 0. \tag{77}$$

We shall bound $f_{T,t}(u)$ in the integral (77). Since $1/u \geq \varphi(T)$, and thus $\overline{\Lambda}(1/u) \leq \overline{\Lambda}(\varphi(T)) \leq T$, we get that the second factor of $f_{T,t}(u)$ given in (75) is

$$\begin{aligned}\exp\left\{-t\int_0^T \left(1-e^{-u\varphi(s)}\right)\mathrm{d}s\right\} &\leq \exp\left\{-t\int_0^{\overline{\Lambda}(1/u)} \left(1-e^{-u\varphi(s)}\right)\mathrm{d}s\right\}\\ &\leq e^{-t(1-e^{-1})\overline{\Lambda}(1/u)}.\end{aligned}$$

While for the first factor in $f_{T,t}(u)$ given in (75) we use the simple bound

$$tu\int_0^T \varphi^2(s)e^{-u\varphi(s)}\mathrm{d}s \leq tu\int_0^\infty \varphi^2(s)e^{-u\varphi(s)}\mathrm{d}s =: t\psi_\Lambda(u).$$

We see that

$$\begin{aligned}\int_D^{1/\varphi(T)} f_{T,t}(u)\mathrm{d}u &\leq t\int_D^{1/\varphi(T)} \psi_\Lambda(u)\, e^{-t(1-e^{-1})\overline{\Lambda}(1/u)}\mathrm{d}u\\ &\leq t\int_D^\infty \psi_\Lambda(u)\, e^{-t(1-e^{-1})\overline{\Lambda}(1/u)}\mathrm{d}u.\end{aligned}$$

Clearly (73) holds whenever for all $\gamma > 0$,

$$\int_1^\infty \psi_\Lambda(u)\, e^{-\gamma\overline{\Lambda}(1/u)}\mathrm{d}u < \infty. \tag{78}$$

Lemma 6. *Whenever (69) is satisfied, then for all $\gamma > 0$, (78) holds.*

Proof. Recall the definition (53). Notice that by (71) for all $t > 0$

$$\int_0^\infty f_{(t)}(u)\,\mathrm{d}u < \infty. \tag{79}$$

Write

$$\int_0^\infty (1-e^{-u\varphi(s)})\mathrm{d}s = \int_0^{1/u}(1-e^{-ux})\Lambda(\mathrm{d}x) + \int_{1/u}^\infty (1-e^{-ux})\Lambda(\mathrm{d}x).$$

We see that

$$\int_{1/u}^\infty (1-e^{-ux})\Lambda(\mathrm{d}x) \leq \overline{\Lambda}(1/u)$$

and

$$\begin{aligned}\int_0^{1/u}(1-e^{-ux})\Lambda(\mathrm{d}x) &= -\left(1-e^{-1}\right)\overline{\Lambda}(1/u) + \int_0^{1/u} u\overline{\Lambda}(x)\,e^{-ux}\mathrm{d}x\\ &\leq \int_0^{1/u} u\overline{\Lambda}(x)\,e^{-ux}\mathrm{d}x \leq u\int_0^{1/u}\overline{\Lambda}(x)\,\mathrm{d}x.\end{aligned}$$

By assumption (69) for some $\eta > 0$ for all u large

$$u \int_0^{1/u} \overline{\Lambda}(x)\,\mathrm{d}x \le \eta \overline{\Lambda}(1/u). \tag{80}$$

This implies that

$$t \int_0^\infty (1 - e^{-u\varphi(s)})\mathrm{d}s \le (1+\eta)\, t\overline{\Lambda}(1/u).$$

Thus for all large enough $D > 0$ and all $t > 0$

$$\int_D^\infty f_{(t)}(u)\,\mathrm{d}u \ge \int_D^\infty t\psi_\Lambda(u) \exp\left\{-(1+\eta)\, t\overline{\Lambda}(1/u)\right\}\mathrm{d}u,$$

and hence since (79) holds for all $t > 0$, we get that for all $\gamma > 0$, (78) is satisfied. □

We see from Lemma 6 that (78) holds whenever assumption (69) is satisfied and thus by the arguments preceding the lemma the limit (55) is valid. This completes the proof of Proposition 7. □

A.1. Return to the proofs of Proposition 6 and Theorem 1

We shall now finish the proof of Proposition 6. To do this we shall need three more lemmas. Let X_t be a subordinator with canonical measure Λ. Assume that X_t is without drift. Define

$$I(x) = \int_0^x \overline{\Lambda}(y)\mathrm{d}y.$$

We give a criterion for subsequential relative stability of X at 0.

Lemma 7. *Let X be a driftless subordinator with $\overline{\Lambda}(0+) > 0$. There are nonstochastic sequences $t_k \downarrow 0$ and $B_k > 0$, such that, as $k \to \infty$,*

$$\frac{X(t_k)}{B_k} \xrightarrow{\mathrm{P}} 1 \tag{81}$$

if and only if

$$\liminf_{x \downarrow 0} \frac{x\overline{\Lambda}(x)}{I(x)} = 0. \tag{82}$$

Proof. From the convergence criteria for subordinators, e.g. part (ii) of Theorem 15.14 of [10], p. 295, (81) is equivalent to

$$\lim_{t_k \to 0} t_k \overline{\Lambda}(xB_k) = 0 \quad \text{for every } x > 0 \quad \text{and} \quad \lim_{t_k \to 0} t_k \int_0^1 x\Lambda(\mathrm{d}B_k x) = 1. \tag{83}$$

Noting that $I(x) = \int_0^x y\Lambda(\mathrm{d}y) + x\overline{\Lambda}(x)$, we see that (83) implies

$$t_k B_k^{-1} I(B_k) = t_k B_k^{-1} \int_0^{B_k} x\Lambda(\mathrm{d}x) + t_k\overline{\Lambda}(B_k) \to 1, \tag{84}$$

and clearly (84) and $t_k\overline{\Lambda}(B_k) \to 0$ imply (82). (Note that necessarily $B_k \to 0$.)

Conversely, let (82) hold and choose a subsequence $c_k \to 0$ as $k \to \infty$ such that

$$\lim_{k\to\infty} \frac{c_k \overline{\Lambda}(c_k)}{I(c_k)} = 0.$$

Define

$$t_k := \sqrt{\frac{c_k}{\overline{\Lambda}(c_k) I(c_k)}}.$$

Then

$$\lim_{k\to\infty} t_k \overline{\Lambda}(c_k) = \lim_{k\to\infty} \sqrt{\frac{c_k \overline{\Lambda}(c_k)}{I(c_k)}} = 0,$$

and

$$\lim_{k\to\infty} \frac{t_k I(c_k)}{c_k} = \lim_{k\to\infty} \sqrt{\frac{I(c_k)}{c_k \overline{\Lambda}(c_k)}} = \infty.$$

Then set $B_k := t_k I(c_k)$, so $\lim_{k\to\infty} B_k = 0$ and $\lim_{k\to\infty} B_k/c_k = \infty$. Given $x > 0$ choose k so large that $x B_k \ge c_k$. Then

$$t_k \overline{\Lambda}(x B_k) \le t_k \overline{\Lambda}(c_k) \to 0. \tag{85}$$

Furthermore, by writing

$$\frac{t_k I(B_k)}{B_k} = \frac{t_k I(c_k)}{B_k} + \frac{t_k\,(I(B_k) - I(c_k))}{B_k} = 1 + \frac{t_k\,(I(B_k) - I(c_k))}{B_k}$$

and noting that for all large k

$$0 \le \frac{t_k\,(I(B_k) - I(c_k))}{B_k} \le \frac{B_k t_k \overline{\Lambda}(c_k)}{B_k} \to 0,$$

we also have $t_k B_k^{-1} I(B_k) \to 1$ and thus by (85) and the identity in (84)

$$\lim_{t_k\to 0} t_k \int_0^1 x \Lambda(\mathrm{d} B_k x) = 1$$

which in combination with (85) implies (81), by (83). □

To continue we need the following lemma from [12].

Lemma 8. *Let Ψ be a non-negative measurable real valued function defined on* $(0, \infty)$ *satisfying*

$$\int_0^\infty \Psi(y)\,\mathrm{d}y < \infty.$$

Then

$$E\left(\sum_{i=1}^{\infty} \Psi(S_i)\right) = \int_0^\infty \Psi(y)\,\mathrm{d}y \tag{86}$$

and $\lim_{n\to\infty} E\left(\sum_{i=n}^{\infty} \Psi(S_i)\right) = 0.$

Lemma 9. *(i) Assume that (47) holds as $t \searrow 0$ with $\beta < 1$. Then (69) holds.*

(ii) Assume that (47) holds as $t \to \infty$ with $\beta < 1$. Then without loss of generality we can assume that (69) holds.

Proof. (i) We shall show that (47) implies (69). Assume on the contrary that (69) does not hold. Then, since V_t is a driftless subordinator by Lemma 7 for some sequences $B_k > 0$, $t_k \downarrow 0$, $V_{t_k}/B_k \xrightarrow{\mathrm{P}} 1$. By Proposition 1 the infinite sum $\sum_{i=1}^{\infty} \varphi\left(\frac{S_i}{t}\right)$ is equal in distribution to V_t and $\sum_{i=1}^{\infty} \varphi^2\left(\frac{S_i}{t}\right)$ is equal in distribution to the subordinator W_t with Lévy measure Λ_2 on $(0, \infty)$ defined by

$$\overline{\Lambda}_2(x) = \overline{\Lambda}\left(\sqrt{x}\right).$$

From (83) in the proof of Lemma 7 above

$$t_k \overline{\Lambda}(x B_k) \to 0 \quad \text{and} \quad \int_0^1 t_k \overline{\Lambda}(x B_k)\, \mathrm{d}x \to 1, \tag{87}$$

with $t_k \to 0$ and $B_k \to 0$. Thus we easily see that

$$t_k \overline{\Lambda}_2\left(x B_k^2\right) = t_k \overline{\Lambda}\left(\sqrt{x} B_k\right) \to 0$$

and

$$\int_0^1 t_k \overline{\Lambda}_2\left(x B_k^2\right) \mathrm{d}x = \int_0^1 t_k \overline{\Lambda}\left(\sqrt{x} B_k\right) \mathrm{d}x = 2 \int_0^1 y t_k \overline{\Lambda}(y B_k)\, \mathrm{d}y,$$

which for any $0 < \delta < 1$ is

$$\le 2\delta \int_0^1 t_k \overline{\Lambda}(x B_k)\, \mathrm{d}x + 2 \int_\delta^1 t_k \overline{\Lambda}(x B_k)\, \mathrm{d}x.$$

Clearly by (87)

$$\limsup_{k \to \infty} \left(2\delta \int_0^1 t_k \overline{\Lambda}(x B_k)\, \mathrm{d}x + 2 \int_\delta^1 t_k \overline{\Lambda}(x B_k)\, \mathrm{d}x \right) = 2\delta.$$

Thus since $0 < \delta < 1$ can be made arbitrarily small we get

$$\lim_{k \to \infty} \int_0^1 t_k \overline{\Lambda}_2\left(x B_k^2\right) \mathrm{d}x = 0.$$

Hence applying Theorem 15.14 on page 295 of [10], we get $W_{t_k}/B_k^2 \xrightarrow{\mathrm{P}} 0$ and thus

$$R_{t_k} \stackrel{\mathrm{D}}{=} W_{t_k} / \left(V_{t_k}\right)^2 \xrightarrow{\mathrm{P}} 0,$$

which since $R_{t_k} \le 1$ implies $E R_{t_k} \to 0$, as $t_k \downarrow 0$, which clearly contradicts to (47). So we have (69) in this case.

(ii) We shall first assume that

$$\int_0^\infty \varphi(u)\, \mathrm{d}u = \infty, \tag{88}$$

which by (11) implies

$$\int_0^1 \varphi(u)\,\mathrm{d}u = \infty. \tag{89}$$

Set

$$V(t) := \sum_{i=1}^{\infty} \varphi\left(\frac{S_i}{t}\right) \mathbf{1}\left\{\frac{S_i}{t} \le 1\right\} \text{ and } \overline{V}(t) := \sum_{i=1}^{\infty} \varphi\left(\frac{S_i}{t}\right) \mathbf{1}\left\{\frac{S_i}{t} > 1\right\}.$$

We see that

$$V(t) \ge \sum_{k=1}^{\infty} \varphi\left(2^{-k+1}\right) \sum_{i=1}^{\infty} \mathbf{1}\left\{2^{-k} < \frac{S_i}{t} \le 2^{-k+1}\right\}.$$

Now for each fixed $L \ge 1$, as $t \to \infty$,

$$\begin{aligned} t^{-1} \sum_{k=2}^{L+1} \left(\varphi\left(2^{-k+1}\right) \sum_{i=1}^{\infty} \mathbf{1}\left\{2^{-k} < \frac{S_i}{t} \le 2^{-k+1}\right\} \right) &\xrightarrow{\mathrm{P}} \sum_{k=1}^{L} \varphi\left(2^{-k}\right) 2^{-k-1} \\ &\ge 2^{-1} \int_{2^{-L}}^{1} \varphi(u)\mathrm{d}u. \end{aligned}$$

Thus since $L \ge 1$ can be made arbitrarily large, on account of (89),

$$t^{-1} V(t) \xrightarrow{\mathrm{P}} \infty, \quad \text{as } t \to \infty. \tag{90}$$

Next, using (86), we get

$$t^{-1} E\overline{V}(t) = t^{-1} \int_t^{\infty} \varphi(y/t)\mathrm{d}y = \int_1^{\infty} \varphi(u)\mathrm{d}u < \infty,$$

which implies that

$$t^{-1}\overline{V}(t) = O_P(1), \quad \text{as } t \to \infty. \tag{91}$$

Hence by (90) and (91)

$$\overline{V}(t) / V_t \xrightarrow{\mathrm{P}} 0, \text{ as } t \to \infty. \tag{92}$$

We get then that

$$V_t = \sum_{i=1}^{\infty} \varphi\left(\frac{S_i}{t}\right) = V(t)(1 + o(1)), \quad \text{as } t \to \infty. \tag{93}$$

Now set

$$W_t := \sum_{i=1}^{\infty} \varphi^2\left(\frac{S_i}{t}\right), \qquad W(t) := \sum_{i=1}^{\infty} \varphi^2\left(\frac{S_i}{t}\right) \mathbf{1}\left\{\frac{S_i}{t} \le 1\right\}$$

$$\text{and} \quad \overline{W}(t) := \sum_{i=1}^{\infty} \varphi^2\left(\frac{S_i}{t}\right) \mathbf{1}\left\{\frac{S_i}{t} > 1\right\}.$$

Clearly

$$t^{-1} E\overline{W}(t) = t^{-1} \int_t^{\infty} \varphi^2(y/t)\mathrm{d}y = \int_1^{\infty} \varphi^2(u)\mathrm{d}u < \infty,$$

which says that $t^{-1}\overline{W}(t) = O_P(1)$ as $t \to \infty$. Hence by (92), $\overline{W}(t)/V_t \xrightarrow{\mathrm{P}} 0$ as $t \to \infty$, which when combined with (93) gives

$$R_t = \frac{W_t}{V_t^2} = \frac{W(t)}{V^2(t)} + o_P(1), \quad \text{as } t \to \infty. \tag{94}$$

Notice that $V(t)$ is a Lévy process with canonical measure Λ_1 defined via

$$\overline{\Lambda}_1(x) = \overline{\Lambda}(x), \quad \text{for } x \ge \varphi(1), \quad \text{and} \quad \overline{\Lambda}_1(x) = \overline{\Lambda}(\varphi(1)) \quad \text{for } 0 < x < \varphi(1).$$

Set $\varphi_1(s) = \varphi(s)\mathbf{1}\{s < 1\}$. Note that we have

$$\varphi_1(s) = \sup\left\{y : \overline{\Lambda}_1(y) > s\right\}, \quad s > 0,$$

where the supremum of the empty set is taken as 0. Let $R_t^{(1)}$ be defined as R_t with φ replaced by φ_1, that is,

$$R_t^{(1)} = \frac{W(t)}{(V(t))^2} = \frac{\sum_{i=1}^{\infty} \varphi_1^2\left(\frac{S_i}{t}\right)}{\left(\sum_{i=1}^{\infty} \varphi_1\left(\frac{S_i}{t}\right)\right)^2}.$$

Since $R_t(1) = R_t^{(1)}$, we see by formula (54) that

$$ER_t^{(1)} = \int_0^\infty f_{1,t}(u)\,\mathrm{d}u. \tag{95}$$

Next from (94), we get $R_t^{(1)} - R_t \xrightarrow{\mathrm{P}} 0$, as $t \to \infty$, which implies that

$$\lim_{t\to\infty} ER_t = \lim_{t\to\infty} ER_t^{(1)}.$$

Clearly the tail behavior conclusions about $\Lambda_1(x)$, as $x \to \infty$, will be identical to those for $\Lambda(x)$, as $x \to \infty$. Moreover, since $\overline{\Lambda}_1(0+)$ is finite (69) trivially holds for Λ_1. Therefore in our proof in the case $t \to \infty$ we can without loss of generality assume that (69) is satisfied.

The case $\mu := \int_0^\infty \varphi(u)\,\mathrm{d}u < \infty$ cannot occur when $\beta < 1$ in (47). In this case it is easily checked that

$$t\overline{\Lambda}(x\mu t) \to 0 \quad \text{for all } x > 0 \quad \text{and} \quad \int_0^1 t\overline{\Lambda}(x\mu t)\,\mathrm{d}x \to 1.$$

Therefore by proceeding exactly as above we get that $ER_t \to 0$ as $t \to \infty$, which forces $\beta = 1$. □

Returning to the proof of Proposition 6, in the case $t \searrow 0$, Lemma 9 shows that the assumption of Proposition 7 holds and, in the case $t \to \infty$, it says that we can assume without loss of generality that it is satisfied. This completes the proof of Proposition 6 and hence that of Theorem 1. □

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