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Euler Classes of Inner Product Modules*

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Ozeki [6] has defined the Chern class of a finitely generated projective module as an element of the de Rham cohomology of the ring. His resulting classes are stable invariants, but the functorial properties do not seem to be clear from the definition. On the other hand, he made no attempt to define Euler classes for projective modules.

According to the Chern–Weil theory of characteristic classes, for a given differentiable oriented bundle over a manifold, one can define its Euler class [2, 4, 5]. This is the cohomology class of a differential form constructed by taking the Pfaffian of the curvature form of a connection compatible with a Euclidean metric on the bundle. Since the set of differentiable cross sections to a bundle is a f.g. projective module over the ring of differentiable functions on the base manifold and the Euclidean metric defines a symmetric bilinear form on this module, it is only natural to ask whether the above construction can be generalized to any f.g. projective module.

Following R.G. Swan's ideas we have defined Euler classes for inner product modules. By considering the connection forms as Kähler differentials of the ring we can easily derive the functorial properties of these classes.

Let (P, h) be an inner product module, i.e., P is a f.g. projective module over a ring R, and h is a symmetric nondegenerate bilinear form on P. In Section 1, we prove the existence of a connection ∇ on P compatible with h. In Section 2, we define cohomology groups $H^*_{\mathrm{DR}}(P, h)$ associated to (P, h), by using a complex $\Lambda^{\cdot}\Omega_R \otimes \Lambda^n P$, where Ω_R is the module of Kähler differentials of R and $n =$ rank P [1; 3]. In the next section, we define the Euler class $e(P, h, \nabla)$ as the cohomology class of the Pfaffian of the curvature of ∇. In Section 4, we show that $e(P, h, \nabla)$ is independent of the connection ∇. In Section 5, we exhibit the functorial and multiplicative properties of these classes. We also establish the connec-

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tion with the differentiable Euler classes. Finally, in Section 6, we compute Euler classes for some inner product modules and, as an application, we show the nontriviality of the tangent bundle to the affine 2-sphere defined over an Archimedean ordered field.

1. Connections on Inner Product Modules

Let R be a commutative K-algebra. Let Ω^r denote the rth exterior product of the module $\Omega_{R/K}$ of Kähler differentials of R over K. Then we have a complex

$$K \longrightarrow R \xrightarrow{d} \Omega \xrightarrow{d} \Omega^2 \longrightarrow \cdots$$

whose cohomology groups $H^*_{\mathrm{DR}}(R/K)$ are called the de Rham groups of R/K.

If $\Phi: M \times N \to L$ is a bilinear map of R-modules, then there exists a unique bilinear map

$$\Lambda = \Lambda_\Phi: \Omega^p \otimes M \times \Omega^q \otimes N \to \Omega^{p+q} \otimes L$$

which satisfies $(\omega \otimes m) \wedge (\omega' \otimes n) = \omega \wedge \omega' \otimes \Phi(m, n)$, for all $\omega \in \Omega^p$, $\omega' \in \Omega^q$, $m \in M$, $n \in N$.

We consider the following examples:

Example 1. If $\Phi: R \times M \to M$ is the product $\Phi(r, m) = rm$ $(r \in R, m \in M)$, then we have the bilinear maps

$$\begin{aligned} \Lambda: \Omega^p \times \Omega^q \otimes M &\to \Omega^{p+q} \otimes M, \\ (\omega, \omega' \otimes m) &\mapsto \omega \wedge \omega' \otimes m. \end{aligned}$$

Example 2. If $\Phi: \Lambda^r M \times \Lambda^s M \to \Lambda^{r+s} M$ is the exterior product map $(x, y) \mapsto x \wedge y$, then we have the bilinear products

$$\Lambda: \Omega^p \otimes \Lambda^r M \times \Omega^q \otimes \Lambda^s M \to \Omega^{p+q} \otimes \Lambda^{r+s} M.$$

These products are associative and satisfy $\omega \wedge \omega' = (-1)^{pq+rs} \omega' \wedge \omega$, if $\omega \in \Omega^p \otimes \Lambda^r M$, $\omega' \in \Omega^q \otimes \Lambda^s M$.

Example 3. If $h: M \times M \to R$ is a bilinear form, then we have bilinear maps $\tilde{h} = \Lambda_h: \Omega^p \otimes M \times \Omega^q \otimes M \to \Omega^{p+q}$. If we assume that h is a symmetric map then $\tilde{h}(\omega, \omega') = (-1)^{pq} \tilde{h}(\omega', \omega)$, for $\omega \in \Omega^q \otimes M$, $\omega' \in \Omega^q \otimes M$.

For convenience, we write h instead of $\tilde{h}$.

Definition. A connection on an R-module M is a map $\nabla: M \to \Omega \otimes M$ such that

(1) ∇ is K-linear,

(2) $\nabla(rm) = dr \otimes m + r\nabla(m) \qquad (r \in R, m \in M)$.

In this case, we can define the K-linear maps $\nabla = \nabla^p \colon \Omega^p \otimes M \to \Omega^{p+1} \otimes M$ by

$$\nabla^p(\omega \otimes m) = d\omega \otimes m + (-1)^p \omega \wedge \nabla(m) \qquad (\omega \in \Omega^p, m \in M),$$

and the *curvature* K_∇ of ∇ by $\nabla^1\nabla \colon M \to \Omega^2 \otimes M$.

LEMMA 1.1. *If* $\omega \in \Omega^p$, $\eta \in \Omega^q \otimes M$, *then*

$$\nabla(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^p \omega \wedge \nabla\eta.$$

This is obvious.

COROLLARY 1.2. *For any connection* ∇ *on* M *we have* $K_\nabla \in \mathrm{Hom}_R(M, \Omega^2 \otimes M)$.

Proof. Let $r \in R$ and $m \in M$. Applying Lemma 1.1, we get

$$\begin{aligned} K_\nabla(rm) &= \nabla(\nabla(rm)) = \nabla(dr \otimes m + r\nabla m) \\ &= -dr \wedge \nabla m + dr \wedge \nabla m + r\nabla\nabla m = rK_\nabla(m). \end{aligned}$$

DEFINITION. Let $h \colon M \times M \to R$ be a symmetric bilinear form on M. A connection ∇ on M is said to be *compatible with h* in case $h(\nabla x, y) + h(x, \nabla y) = d(h(x, y))$ holds for all $x, y \in M$.

Recall that an inner product module over R is a finitely generated projective R-module together with a symmetric bilinear product which is nondegenerate.

THEOREM 1.3. *Suppose* $\frac{1}{2} \in K$. *Then, for any inner product module* (P, h) *over* R, *there exists a connection* ∇ *compatible with h.*

Proof. First we assume that P is a free module. Choose a basis $\{e_1, \ldots, e_n\}$ for P and let $C = (c_{ij}) \in M_n(R)$, $c_{ij} = h(e_i, e_j)$. Let $\omega_{ij} \in \Omega$ be defined by $(\omega_{ij}) = \frac{1}{2}C^{-1}d(C) \in M_n(\Omega)$. Define

$$\nabla\left(\sum_{i=1}^n r_i e_i\right) = \sum_{i=1}^n dr_i \otimes e_i + \sum_{i=1}^n \sum_{j=1}^n r_i \omega_{ji} \otimes e_j \qquad (r_i \in R).$$

Then, it is clear that ∇ is a connection on P compatible with h.

In the general case, we can choose an inner product module (Q, k) so that $P \oplus Q$ is a free module. In fact, if T is an R-module such that $P \oplus T$ is free, then using $P \approx P^*$, since h is nondegenerate, we can see that $P \oplus P \oplus T \oplus T^*$ is free; here $M^* = \mathrm{Hom}_R(M, R)$, for a module M. Now, if we take $Q = P \oplus T \oplus T^*$ and $k((p, t, t^*), (p_1, t_1, t_1^*)) = h(p, p_1) + t^*(t_1) + t_1^*(t)$, we obtain (Q, k) as we have required.

Next, let ∇' be a connection on $P \oplus Q$ compatible with $h \perp k$. Let $i: P \to P \oplus Q$ be the canonical injection and $\pi: P \oplus Q \to P$, the first projection map. Then

$$\nabla = (1 \otimes \pi) \circ \nabla' \circ i: P \xrightarrow{i} P \oplus Q \xrightarrow{\nabla'} \Omega \otimes (P \oplus Q) \xrightarrow{1 \otimes \pi} \Omega \oplus P,$$

is a connection on P compatible with h.

Remark. From now on, we make the assumption $\frac{1}{2} \in K$.

2. The De Rham Cohomology Groups $H^*_{\mathrm{DR}}(P, h)$

Definition. *If* $\nabla: P \to \Omega \otimes P$ is a connection on P, we define $\nabla_k: \Lambda^k P \to \Omega \otimes \Lambda^k P$ by

$$\nabla_k(x_1 \wedge \cdots \wedge x_k) = \sum_{i=1}^{k} x_1 \wedge \cdots \wedge \nabla x_i \wedge \cdots \wedge x_k ,$$

for $x_1, \ldots, x_k \in P$.

This is clearly well defined. In fact, we can see that

$$\nabla_k(rx_1 \wedge \cdots \wedge x_k) = dr \otimes x_1 \wedge \cdots \wedge x_k + r\nabla_k(x_1 \wedge \cdots \wedge x_k) \qquad (r \in R).$$

Proposition 2.1.

(1) *$\nabla_k: \Lambda^k P \to \Omega \otimes \Lambda^k P$ is a connection on $\Lambda^k P$.*

(2) *If $\omega \in \Omega^a \otimes \Lambda^k P$ and $\omega' \in \Omega^b \otimes \Lambda^l P$, then*

$$\nabla_{k+l}^{a+b}(\omega \wedge \omega') = \nabla_k{}^a(\omega) \wedge \omega' + (-1)^a \omega \wedge \nabla_l{}^b(\omega').$$

Remark. For brevity we often write ∇ instead of $\nabla_k{}^a$.

Proposition 2.2. *Let (P, h) be an inner product module and let h_k be the bilinear form induced on $\Lambda^k P$ so*

$$h_k(x_1 \wedge \cdots \wedge x_k, y_1 \wedge \cdots \wedge y_k) = \det(h(x_i, y_j)), \qquad x_i, y_j \in P.$$

If ∇ is a connection on P compatible with h, then ∇_k on $\Lambda^k P$ is compatible with h_k.

Proof. Let $x_i, y_j \in P$, $i, j = 1, \ldots, k$. Denote by m_{ij} the cofactor of $h(x_i, y_j)$ in $\det(h(x_i, y_j))$. Then, for $\xi = x_1 \wedge \cdots \wedge x_k$, $\eta = y_1 \wedge \cdots \wedge y_k$, we have

$$h_k(\nabla\xi, \eta) = \sum_{i,j} h(\nabla x_i, y_j) m_{ij},$$

$$h_k(\xi, \nabla\eta) = \sum_{i,j} h(x_i, \nabla y_j) m_{ij},$$

and

$$dh_k(\xi, \eta) = \sum_{i,j} dh(x_i, y_j)m_{ij}.$$

Hence, the compatibility condition of ∇ with respect to h gives the desired conclusion.

PROPOSITION 2.3. (1) *If ∇ is a connection compatible with h, then the curvature K_∇ satisfies $h(K_\nabla x, y) + h(x, K_\nabla y) = 0$ for all $x, y \in P$.*

(2) *Suppose that* rank $P = n$. *Then, for any $f \in \mathrm{Hom}_R(P, \Omega^a \otimes P)$ which satisfies $h(fx, y) + h(x, fy) = 0$ for all $x, y \in P$, it holds that $\sum_{i=1}^n x_1 \wedge \cdots \wedge f(x_i) \wedge \cdots \wedge x_n = 0$, for all $x_1, \ldots, x_n \in P$.*

In particular, if $n = 1$, then $f = 0$.

Proof. (1) Let $x, y \in P$. We claim that

$$h(K_\nabla x, y) = d(h(\nabla x, y)) + h(\nabla x, \nabla y).$$

To see this, let $\nabla x = \sum \omega_i \otimes x_i$, with $\omega_i \in \Omega$, $x_i \in P$. Then using the definition of K_∇ and substituting $h(\nabla x_i, y) = dh(x_i, y) - h(x_i, \nabla y)$ yields the claim.

Observing that $h(x, K_\nabla y) = h(K_\nabla y, x)$ and applying the above relation gives

$$h(K_\nabla x, y) + h(x, K_\nabla y) = dd(h(x, y)) = 0.$$

(2) After localizing we can assume that P is a free module. Take a basis $\{e_1, \ldots, e_n\}$ for P and write

$$f(e_i) = \sum \omega_{ji} \otimes e_j, \quad \omega_{ji} \in \Omega^a, \quad c_{ij} = h(e_i, e_j), \quad \omega = (\omega_{ij}), \quad \text{and} \quad C = (c_{ij}).$$

It is easy to see that $h(fx, y) + h(x, fy) = 0$ for all $x, y \in P$, is equivalent to $C\omega + (C\omega)^\perp = 0$. Since C is nonsingular, we get $\omega^\perp = -C\omega C^{-1}$ and so $\mathrm{Trace}\,\omega^\perp = -\mathrm{Trace}\,\omega$. Thus $\mathrm{Trace}\,\omega = 0$. On the other hand, it can be easily verified that

$$\sum_{i=1}^n x_1 \wedge \cdots \wedge f(x_i) \wedge \cdots \wedge x_n = (\mathrm{Trace}\,\omega) x_1 \wedge \cdots \wedge x_n.$$

THEOREM 2.4. *If (P, h) is an inner product module with* rank $P = 1$, *then there is a unique connection ∇ compatible with h and it has $K_\nabla = 0$. In particular, we have a complex*

$$P \to \Omega \otimes P \to \Omega^2 \otimes P \to \cdots,$$

i.e., $\nabla\nabla = 0$, and we define the cohomology groups

$$H^k_{\mathrm{DR}}(P, h) = \mathrm{Ker}\,\nabla^k / \mathrm{Im}\,\nabla^{k-1}.$$

Proof. Since $\nabla\nabla(\omega \otimes x) = \omega \wedge K_\nabla(x)$ holds for any $\omega \in \Omega^a$, $x \in P$, it is enough to prove the first statement. But this follows from Proposition 2.3 applied when $n = 1$.

DEFINITION. Let (P, h) be an inner product module with rank $P = n$. Define the kth *de Rham cohomology group of* (P, h) by $H^k_{\mathrm{DR}}(P, h) = H^k_{\mathrm{DR}}(\Lambda^n P, h_n)$, where the last group is the one defined in Theorem 2.4.

If ∇ is a connection compatible with h, then by Proposition 2.2 and Theorem 2.4, ∇_n is the unique connection compatible with h_n, so that by definition

$$H^k_{\mathrm{DR}}(P, h) = \operatorname{Ker} {\nabla_n}^k / \operatorname{Im} \nabla_n^{k-1}.$$

In particular, we observe that these groups are independent of the choice of the connection ∇.

Remarks. (1) If rank $P = 1$, then P has a nondegenerate bilinear form if and only if $P \approx P^*$. These are necessarily symmetric.

(2) If rank $P = 1$ and $du = 0$ for all units u of R, for example, if $U(R) = K^\cdot$, then all nondegenerate bilinear forms give the same connection. In fact, if h' and h are two such forms, then there exists a unit u in R such that $h' = uh$, and if ∇ is the connection compatible with h, using $du = 0$, we can see that ∇ is also compatible with h'.

3. THE EULER CLASS OF AN INNER PRODUCT MODULE

Let (P, h) be an inner product module. Define the following R-modules

(1) $L(P) = \operatorname{End}_R(P)$.

(2) $L(P, h) = \{f \in L(P) \mid h(fx, y) + h(x, fy) = 0, \text{ for all } x, y \in P\}$.

(3) $A^k(P) = \operatorname{Hom}_R(P, \Omega^k \otimes P)$.

(4) $A^k(P, h) = \{f \in A^k(P) \mid h(fx, y) + h(x, fy) = 0, \text{ for all } x, y \in P\}$.

Then $A^k(P, h)$ is a submodule of $A^k(P)$. Observe that $A^0(P) = L(P)$ and $A^0(P, h) = L(P, h)$.

We also define K-linear maps $d_\nabla\colon A^k(P) \to A^{k+1}(P)$, for a given connection ∇ on P, by the rule

$$d_\nabla(f) = \nabla^k \circ f - (I \otimes f) \circ \nabla \qquad (f \in A^k(P)),$$

where $I \otimes f$ denotes the composite map

$$\Omega \otimes P \xrightarrow{I \otimes f} \Omega \otimes (\Omega^k \otimes P) \longrightarrow \Omega^{k+1} \otimes P.$$

PROPOSITION 3.1.

(1) $d_\nabla(f) \in A^{k+1}(P)$ *for all* $f \in A^k(P)$.

(2) *If* ∇ *is compatible with* h, *then* d_∇ *maps* $A^k(P, h)$ *into* $A^{k+1}(P, h)$.

Proof. (1) Let $r \in R$, $x \in P$. Then we have

$$\begin{aligned} d_\nabla(f)(rx) &= \nabla(f(rx)) - (I \otimes f)(\nabla(rx)) \\ &= dr \wedge f(x) + r\nabla f(x) - (I \otimes f)(dr \otimes x + r\nabla x) \\ &= r\nabla f(x) - r(I \otimes f)\,\nabla x \\ &= rd_\nabla(f)(x). \end{aligned}$$

(2) First we prove the following relation

$$h(d_\nabla f(x), y) = dh(fx, y) + (-1)^{k+1}h(fx, \nabla y) + h(\nabla x, fy)$$

for $f \in A^k(P, h)$, $x, y \in P$. Write

$$f(x) = \sum \omega_i \otimes x_i\,, \qquad \omega_i \in \Omega^k, \quad x_i \in P$$

$$\nabla x = \sum \theta_j \otimes y_j\,, \qquad \theta_j \in \Omega, \quad y_j \in P.$$

From the definition of d we get

$$h(d_\nabla f(x), y) = \sum d\omega_i h(x_i\,, y) + (-1)^k \sum \omega_i \wedge h(\nabla x_i\,, y) - \sum \theta_j \wedge h(f(y_j), y);$$

then, substituting $h(\nabla x_i\,, y) = dh(x_i\,, y) - h(x_i\,, \nabla y)$ and $h(f(y_j), y) = -h(y_j\,, f(y))$ yields

$$\begin{aligned} h(d_\nabla f(x), y) &= \sum d\omega_i h(x_i\,, y) + (-1)^k \sum \omega_i \wedge dh(x_i\,, y) \\ &\quad + (-1)^{k+1} \sum \omega_i \wedge h(x_i\,, \nabla y) + \sum \theta_j \wedge h(y_j\,, f(y)) \\ &= dh(fx, y) + (-1)^{k+1}\, h(fx, \nabla y) + h(\nabla x, fy). \end{aligned}$$

Finally, we have $h(x, d_\nabla f(y)) = h(d_\nabla f(y), x)$ and using the above relation we obtain

$$\begin{aligned} &h(d_\nabla f(x), y) + h(x, d_\nabla f(y)) \\ &\quad = [dh(fx, y) + dh(fy, x)] \\ &\qquad + [(-1)^{k+1}h(fx, \nabla y) + h(\nabla y, fx)] + [(-1)^{k+1}h(fy, \nabla x) + h(\nabla x, fy)], \end{aligned}$$

which is 0 because each bracket is 0.

THEOREM 3.2. *Let* ∇ *be a connection compatible with* h. *Then we have*

(1) *The map* $\Lambda^2P \xrightarrow{u} L(P, h)$, *which sends* $a \wedge b$ *into* $u_{a\wedge b}\colon P \to P$, $u_{a\wedge b}(x) = h(a, x)b - h(b, x)a$, $x \in P$, *is an isomorphism.*

(2) *The map* $\Omega^k \otimes L(P) \xrightarrow{v} A^k(P)$, $\omega \otimes f \mapsto v_{\omega\otimes f}$, $v_{\omega\otimes f}(x) = w \otimes f(x)$, $x \in P$, *is an isomorphism. Moreover,* v *maps* $\Omega^k \otimes L(P, h)$ *isomorphically onto* $A^k(P, h)$.

(3) *In the diagram*

$$\begin{array}{ccccccc}
\Lambda^2P & \xrightarrow{\nabla_2^0} & \Omega \otimes \Lambda^2P & \xrightarrow{\nabla_2^1} & \Omega^2 \otimes \Lambda^2P & \longrightarrow & \cdots \\
\iota\downarrow u & & \iota\downarrow v\circ(I\otimes u) & & \iota\downarrow v\circ(I\otimes u) & & \\
A^0(P, h) & \xrightarrow{d_\nabla} & A^1(P, h) & \xrightarrow{d_\nabla} & A^2(P, h) & \longrightarrow & \cdots
\end{array}$$

all squares are commutative, and the vertical maps are isomorphisms over R.

Proof. (1) and (2). By localizing we can assume that P is a free module. Let $\{e_1, \ldots, e_n\}$ be a basis for P, and $C = (h(e_i, e_j))$. Identify $A^k(P)$ with the module $M_n(\Omega^k)$ of n-square matrices with entries in Ω^k by means of this basis, i.e., $f \in A^k(P)$ goes to $\omega = (\omega_{ij}) \in M_n(\Omega^k)$, where $f(e_i) = \sum_{j=1}^k \omega_{ji} \otimes e_j$. Then $A^k(P, h) = \{\omega \in M_n(\Omega^k) \mid C\omega + (C\omega)^\perp = 0\}$. Let S^k be the module of skew-symmetric matrices in $M_n(\Omega^k)$; $\alpha\colon A^k(P, h) \to S^k$, the isomorphism $\alpha(\omega) = C\omega$, and $\beta\colon \Omega^k \otimes S^0 \to S^k$, the isomorphism $\beta(\omega \otimes (a_{ij})) = (\omega a_{ij})$. Then we have a commutative diagram

$$\begin{array}{ccc}
\Omega^k \otimes L(P, h) & \xrightarrow{v} & A^k(P, h) \\
I\otimes\alpha \downarrow \iota & & \iota \downarrow \alpha \\
\Omega^k \otimes S^0 & \xrightarrow[\beta]{\sim} & S^k
\end{array}$$

from which it readily follows that v is an isomorphism.

Let $\{e_i^*\}$ be the dual basis of $\{e_i\}$ and let $E_{ij} \in L(P)$ for which $E_{ij}(e_k) = \delta_{jk}e_i$. Then $S^0 \underset{\sim}{\xrightarrow{\gamma}} \Lambda^2P$, $E_{ij} - E_{ji} \mapsto e_j^* \wedge e_i^*$, is an isomorphism, and the composite map

$$\Lambda^2P \xrightarrow{u} L(P, h) \underset{\sim}{\xrightarrow{\alpha}} S^0 \underset{\sim}{\xrightarrow{\gamma}} \Lambda^2P^*$$

is the isomorphism $h^* \wedge h^*$, where $h^*\colon P \xrightarrow{\sim} P^*$, $x \mapsto h(x, -)$. Whence u is an isomorphism.

(3) Let $\omega \in \Omega^k$, $a, b, x \in P$. Then by straight computations we see that the following relations hold

$$v \circ (I \otimes u)(\omega \wedge \nabla a \wedge b)(x) = \omega \wedge \{h(\nabla a, x) \otimes b - \nabla a h(x, b)\}; \qquad (3.1)$$

$$[I \otimes (v \circ (I \otimes u)(\omega \otimes a \wedge b))]\nabla x = (-1)^k \omega \wedge \{h(a, \nabla x)b - h(b, \nabla x)a\}; \qquad (3.2)$$

$$\nabla(u_{a\wedge b}(x)) = dh(a, x) \otimes b + h(x, a)\,\nabla b - dh(b, x) \otimes a - h(x, b)\,\nabla a. \qquad (3.3)$$

Now, by definition and (3.1) we have

$$\begin{aligned}&[v \circ (I \otimes u) \circ \nabla_2{}^k(\omega \otimes a \wedge b)](x)\\ &\quad = d\omega \otimes u_{a\wedge b}(x)\\ &\qquad + (-1)^k\omega \wedge \{h(\nabla a, x) \otimes b - \nabla a h(x, b) - h(\nabla b, x) \otimes a + \nabla b h(x, a)\}\end{aligned}$$

and substituting $h(\nabla a, x) = dh(a, x) - h(a, \nabla x)$, and similarly $h(\nabla b, x)$, yields

$$\begin{aligned}&= d\omega \otimes u_{a\wedge b}(x) - (-1)^k\omega \wedge \{h(a, \nabla x) \otimes b - h(b, \nabla x) \otimes a\}\\ &\quad + (-1)^k\omega \wedge \{dh(a, x) \otimes b + h(x, a)\, \nabla b - dh(b, x) \otimes a - h(x, b)\, \nabla a\}\end{aligned}$$

so that, by (3.2) and (3.3), this becomes

$$\begin{aligned}&= d\omega \otimes u_{a\wedge b}(x) - [I \otimes (v \circ (I \otimes u)(\omega \otimes a \wedge b))]\, \nabla x + (-1)^k\omega \wedge \nabla(u_{a\wedge b}(x))\\ &= d_\nabla(v \circ (I \otimes u)(\omega \otimes a \wedge b))(x).\end{aligned}$$

Lemma 3.3 (Bianchi's identity). *For any connection ∇ on a module P we have* $d_\nabla(K_\nabla) = 0$.

Proof. For $x \in P$ write $\nabla x = \sum \omega_i \otimes x_i$, $\omega_i \in \Omega$, $x_i \in P$. Then

$$\begin{aligned}\nabla^2 \circ K_\nabla(x) &= \nabla^2 \left(\sum d\omega_i \otimes x_i - \sum \omega_i \wedge \nabla x_i\right)\\ &= \sum d\omega_i \wedge \nabla x_i - \sum d\omega_i \wedge \nabla x_i + \sum \omega_i \wedge \nabla'\nabla x\\ &= \sum \omega_i \wedge K_\nabla(x_i) = (I \otimes K_\nabla)\left(\sum \omega_i \otimes x_i\right)\\ &= (I \otimes K_\nabla) \circ \nabla(x).\end{aligned}$$

Thus $d_\nabla(K_\nabla) = \nabla^2 \circ K_\nabla - (I \otimes K_\nabla) \circ \nabla = 0$.

Definition. Let (P, h) be an inner product module of rank $n = 2k$. Let ∇ be a connection on P compatible with h. Let $\theta_\nabla = (v \circ (I \otimes u))^{-1}(K_\nabla) \in \Omega^2 \otimes \Lambda^2 P$. By the Bianchi identity and Theorem 3.2(3), we know that $\nabla_2{}^2(\theta_\nabla) = 0$. Thus, by Proposition 2.1(2), $\Lambda^k\theta_\nabla \in \Omega^n \otimes \Lambda^n P$ is a cocycle. Define

$$e(P, h, \nabla) = \textit{the Euler class of } (P, h, \nabla) = \{(2k)!/k!\, \Lambda^k\theta \mathrm{P}\} \in H^n_{\mathrm{DR}}(P, h),$$

where $\{\eta\}$ denotes the cohomology class of η.

The coefficient $(2k)^q/k^q$ allows us to avoid the division by integers in the proof of the invariance theorem.

4. Invariance of the Euler Classes

We show that $e(P, h, \nabla)$ is independent of ∇.

We use the isomorphism $\Lambda^2 P \underset{\sim}{\overset{u}{\to}} L(P, h)$ to define an action of $\Omega^a \otimes \Lambda^2 P$ on $\Omega^b \otimes \Lambda^k P$. Let $[\ ,\ ]$ be the unique bilinear map $\Omega^a \otimes \Omega^2 P \times \Omega^b \otimes \Lambda^k P \to \Omega^{a+b} \otimes \Lambda^k P$ such that

$$[\omega \otimes c \wedge d, \omega' \otimes x_1 \wedge \cdots \wedge x_k] = \omega \wedge \omega' \otimes \sum_{i=1}^{k} x_1 \wedge \cdots \wedge u_{c\wedge d}(x_i) \wedge \cdots \wedge x_k ,$$

holds for $\omega \in \Omega^a$, $\omega' \in \Omega^b$, $c, d, x_i \in P$.

Proposition 4.1. *We have the following*

(1) *If* $\omega \in \Omega^a \otimes \Lambda^2 P$, $\omega_i \in \Omega^{a_i} \otimes \Lambda^{k_i} P$ $(i = 1, 2)$, *then*

$$[\omega, \omega_1 \wedge \omega_2] = [\omega, \omega_1] \wedge \omega_2 + (-1)^{aa_1} \omega_1 \wedge [\omega, \omega_2].$$

(2) *If* rank $P = n$, *then all maps*

$$[\ ,\ ]: \Omega^a \otimes \Lambda^2 P \times \Omega^b \otimes \Lambda^n P \to \Omega^{a+b} \otimes \Lambda^n P$$

are zero.

(3) *If* $k_1 + k_2 = n =$ rank P, and $\omega_i \in \Omega^{a_i} \otimes \Lambda^{k_i} P$ $(i = 1, 2)$, *then*

$$[\omega, \omega_1] \wedge \omega_2 = (-1)^{aa_1+1} \omega_1 \wedge [\omega, \omega_2]$$

for any $\omega \in \Omega^a \otimes \Lambda^2 P$.

(4) *If* $\omega \in \Omega \otimes \Lambda^2 P$, $\eta \in \Lambda^2 P$, *then*

$$\overline{[\omega, \eta]} = (-1)^a (I \otimes \bar{\omega}) \circ \bar{\eta} - (I \otimes \bar{\eta}) \circ \bar{\omega},$$

where $\bar{\eta}$ *denotes* $v \circ (I \otimes u)(\eta) \in A^a(P, h)$.

Proof. (1) This can be checked directly.

(2) If $c, d, x_1, \ldots, x_n \in P$, then Proposition 2.3 (2) gives $\sum x_1 \wedge \cdots \wedge u_{c\wedge d}(x_i) \wedge \cdots \wedge x_n = 0$, since $u_{c\wedge d} \in L(P, h)$ by Theorem 3.2(2).

(3) follows from (1) and (2).

(4) Let $\omega = \omega_1 \otimes c \wedge d$, $\eta = \eta_1 \otimes x \wedge y$, with $\omega_1 \in \Omega$, $\eta_1 \in \Omega^a$, and $c, d, x, y \in P$. Then, for any $p \in P$ we have

$$\begin{aligned}\overline{[\omega, \eta]}(p) &= v \circ (I \otimes u)(\omega_1 \wedge \eta_1 \otimes [c \wedge d, x \wedge y])(p) \\ &= \omega_1 \wedge \eta_1 \otimes u([c \wedge d, x \wedge y])(p) \qquad (4.1) \\ &= \omega_1 \wedge \eta_1 \otimes [u_{c\wedge d}, u_{x\wedge y}](p),\end{aligned}$$

since $u([c \wedge d, x \wedge y]) = [u_{c\wedge d}, u_{x\wedge y}]$, where the last bracket is the usual commutator in $L(P)$.

But we also have the following relations

$$(I \otimes \bar{\omega}) \circ \bar{\eta}(p) = (-1)^a \omega_1 \wedge \eta_1 \otimes u_{c\wedge d}(u_{x\wedge y}(p)), \tag{4.2}$$

$$(I \otimes \bar{\eta}) \circ \bar{\omega}(p) = \omega_1 \wedge \eta_1 \otimes u_{x\wedge y}(u_{u\wedge d}(p)). \tag{4.3}$$

From (4.1), (4.2), and (4.3) we get (4).

Proposition 4.2. *Let ∇ be a connection on P compatible with h. Then, for $\omega \in \Omega \otimes \Lambda^2 P$, we have*

(1) $\nabla[\omega, \omega] = -2[\omega, \nabla\omega]$,

(2) $\nabla \cdot \nabla(\omega) = -[\omega, \bar{\theta}_\nabla]$, *where* $\theta = K_\nabla$,

(3) $[\omega, [\omega, \omega]] = 0$.

Proof. We use the fact that the isomorphism $\overline{} = v \circ (I \otimes u)$ commutes with ∇ and d_∇ (Theorem 3.2(3) and Proposition 4.1(4)).

(1) We have

$$\begin{aligned} \nabla[\omega, \omega] = -2[\omega, \nabla\omega] &\Leftrightarrow \overline{\nabla[\omega, \omega]} = -2\overline{[\omega, \nabla\omega]} \Leftrightarrow d_\nabla[\omega, \omega] \\ &= -2\{(I \otimes \bar{\omega}) \circ \overline{\nabla\omega} - (I \otimes \overline{\nabla\omega}) \circ \bar{\omega}\} \Leftrightarrow -2d_\nabla((I \otimes \bar{\omega})\bar{\omega}) \\ &= -2\{(I \otimes \bar{\omega}) \circ d_\nabla\bar{\omega} - (I \otimes d_\nabla\bar{\omega}) \circ \bar{\omega}\}, \end{aligned}$$

since $[\omega, \omega] = -2(I \otimes \bar{\omega}) \circ \bar{\omega}$.

Thus, it suffices to show that

$$d_\nabla((I \otimes f) \circ f) = (I \otimes f) \circ d_\nabla f - (I \otimes d_\nabla f) \circ f, \qquad f \in A^1(P).$$

First, we show that

$$(I \otimes d_\nabla f) \circ f = (I \otimes f)\, \nabla f - \nabla \circ ((I \otimes f) \circ f). \tag{4.4}$$

Let $x \in P$ and write $f(x) = \sum \omega_i \otimes x_i$, $\omega_i \in \Omega$, $x_i \in P$. Then

$$(I \otimes f)\, \nabla f(x) = \sum_i d\omega_i \wedge f(x_i) - \sum_i \omega_i \wedge (I \otimes f)\, \nabla x_i$$

and

$$\nabla \circ ((I \otimes f) \circ f)(x) = \sum_i d\omega_i \wedge f(x_i) - \sum_i \omega_i \wedge \nabla f(x_i).$$

Subtracting these gives

$$\sum \omega_i \wedge d_\nabla f(x_i) = (I \otimes d_\nabla f) \left(\sum \omega_i \otimes x_i\right) = (I \otimes d_\nabla f)(f(x)).$$

Next, from (4.4) and the definition of d_∇, we have

$$(I \otimes f) \circ d_\nabla f - (I \otimes d_\nabla f) \circ f = \nabla \circ ((I \otimes f) \circ f) - (I \otimes f) \circ (I \otimes f) \circ \nabla. \quad (4.5)$$

But the right-hand side of (4.5) is precisely $d_\nabla((I \otimes f) \circ f)$, since it is easy to see that

$$\{I \otimes ((I \otimes f) \circ f)\} \circ \nabla = (I \otimes f) \circ (I \otimes f) \circ \nabla.$$

(2) Similarly, if $\bar{\theta}_\nabla = K_\nabla$, then

$$\nabla \circ \nabla(\omega) = -[\omega, \theta_\nabla] \Leftrightarrow d_\nabla \circ d_\nabla(\bar{\omega}) = -\{(I \otimes \bar{\omega}) \circ K_\nabla - (I \otimes K_\nabla) \circ \bar{\omega}\},$$

and so, it suffices to prove that

$$d_\nabla \circ d_\nabla(f) = -(I \otimes f) \circ K_\nabla + (I \otimes K_\nabla) \circ f, \qquad \text{for} \quad f \in A^1(P).$$

But the following relations can be checked directly from definition

$$\begin{aligned} d_\nabla \circ d_\nabla(f) &= \nabla \circ \nabla \circ f - \nabla \circ (I \otimes f) \circ \nabla - (I \otimes \nabla f) \circ \nabla \\ &\quad + (I \otimes ((I \otimes f) \circ \nabla)) \circ \nabla, \\ \nabla \circ \nabla \circ f &= (I \otimes \nabla\nabla) f, \end{aligned}$$

and

$$-(I \otimes f) \circ \nabla = -\nabla \circ (I \otimes f) - I \otimes \nabla f + I \otimes ((I \otimes f) \circ \nabla)).$$

(3) follows from a similar argument.

THEOREM 4.3. *Let* (P, h) *be an inner product module over* R *with* rank $P = n = 2k$. *Then, for any pair of connections* ∇ *and* ∇_1 *compatible with* h, $e(P, h, \nabla) = e(P, h, \nabla_1\nabla$ *in* $H^n_{\mathrm{DR}}(P, h)$.

Proof. Let $f = \nabla_1 - \nabla \in A^1(P, h)$ and write $\bar{\theta} = K_\nabla$, $\bar{\theta}_1 = K_{\nabla_1}$, $\bar{\omega} = f$, with $\omega \in \Omega \otimes \Lambda^2 P$, $\theta, \theta_1 \in \Omega^2 \Lambda^2 P$. Then, we have $K_{\nabla_1} = K_\nabla + d_\nabla f - (I \otimes f) \circ f$, and so $\theta_1 = \theta + \nabla\omega + \eta$, where $\eta = \frac{1}{2}[\omega, \omega]$. Thus

$$\theta_1{}^k = \sum_{i_1+i_2+i_3=k} (k!/i_1!\, i_2!\, i_3!)\, \theta^{i_1} (\nabla\omega)^{i_2} \eta^{i_3}, \quad (4.6)$$

where, for brevity, we are writing AB instead of $A \wedge B$.

Recall the following relations

$\nabla\eta = -[\omega, \nabla\omega]$,	by Proposition 4.2(1)	(4.7)
$\nabla\nabla\omega = -[\omega, \theta]$,	by Proposition 4.2(2).	(4.8)
$\nabla\theta = 0$,	by Bianchi's identity	(4.9)
$[\omega, \eta] = 0$,	by Proposition 4.2(3).	(4.10)

Define the following differential forms

$$\alpha_{i_1 i_2 i_3} = \theta^{i_1}(\nabla\omega)^{i_2}\eta^{i_3}, \qquad \text{if } i_1, i_2, i_3 \geqslant 0 \text{ and } i_1 + i_2 + i_3 \geqslant 1,$$

$$\beta_{i_1 i_2 i_3} = \omega\theta^{i_1}\nabla\{(\nabla\omega)^{i_2}\}\eta^{i_3}, \qquad \text{if } i_2 \geqslant 1 \text{ and } i_1, i_3 \geqslant 0,$$

$$\gamma_{i_1 i_2 i_3} = \omega\theta^{i_1}(\nabla\omega)^{i_2}\,\nabla(\eta^{i_3}), \qquad \text{if } i_3 \geqslant 1 \text{ and } i_1, i_2 \geqslant 0,$$

and set them equal to 0 for other values of $i_1, i_2, i_3 \geqslant 0$. Then $\alpha_{i_1 i_2 i_3} \in \Omega^{2r} \otimes \Lambda^{2r}P$, $\beta_{i_1 i_2 i_3} \in \Omega^{2(r+1)} \otimes \Lambda^{2(r+1)}P$, and $\gamma_{i_1 i_2 i_3} \in \Omega^{2(r+1)} \otimes \Lambda^{2(r+1)}P$, where $r = i_1 + i_2 + i_3$.

LEMMA 4.4. *Suppose that* $i_1 + i_2 + i_3 + 1 = k = (\operatorname{rank} P)/2$. *Then*

(1) $$(i_1 + 1)(i_3 + 1)\beta_{i_1 i_2 i_3} = -2i_2(i_3 + 1)\alpha_{i_1+1, i_2-1, i_3+1} - i_2(i_2 - 1)\gamma_{i_1+1, i_2-2, i_3+1};$$

(2) $$(i_2 + 1)(i_2 + 2)\gamma_{i_1 i_2 i_3} = -2i_3(i_2 + 2)\alpha_{i_1, i_2+1, i_3} - i_1 i_3 \beta_{i_1-1, i_2+2, i_3-1};$$

(3) $$\nabla(\omega\alpha_{i_1 i_2 i_3}) = \alpha_{i_1, i_2+1, i_3} - \beta_{i_1 i_2 i_3} - \gamma_{i_1 i_2 i_3}.$$

Proof. We use the distributive properties of both ∇ (Proposition 2.1(2)) and the bracket (Proposition 4.1(1) and (3)).

(1) If $i_2 = 0$, then both sides are zero. Assume that $i_2 \geqslant 1$. We have

$$\begin{aligned}\beta_{i_1 i_2 i_3} &= \omega\theta^{i_1}\nabla\{(\nabla\omega)^{i_2}\}\eta^{i_3} \\ &= i_2\omega\theta^{i_1}(\nabla\omega)^{i_2-1}\nabla\nabla\omega\eta^{i_3} \\ &= -i_2\nabla\nabla\omega\omega\theta^{i_1}(\nabla\omega)^{i_2-1}\eta^{i_3},\end{aligned}$$

since $\nabla\nabla\omega$ and $\omega\theta^{i_1}(\nabla\omega)^{i_2-1}$ both have odd degrees. Therefore by (4.8)

$$\beta_{i_1 i_2 i_3} = i_2[\omega, \theta]\,\omega\theta^{i_1}(\nabla\omega)^{i_2-1}\eta^{i_3}.$$

Now, we move ω to the right and distribute it:

$$\begin{aligned}\beta_{i_1 i_2 i_3} = {} & -i_2\theta[\omega, \omega]\,\theta^{i_1}(\nabla\omega)^{i_2-1}\eta^{i_3} + i_2\theta\omega[\omega, \theta^{i_1}](\nabla\omega)^{i_2-1}\eta^{i_3} \\ & + i_2\theta\omega\theta^{i_1}[\omega, (\nabla\omega)^{i_2-1}]\eta^{i_3} + i_2\theta\omega\theta^{i_1}(\nabla\omega)^{i_2-1}[\omega, \eta^{i_3}].\end{aligned}$$

By (4.10), the last summand is zero and using $[\omega, \omega] = 2\eta$, (4.7), and (4.8) we obtain

$$\begin{aligned}\beta_{i_1 i_2 i_3} = {} & -2i_2\theta^{i_1+1}(\nabla\omega)^{i_2-1}\eta^{i_3+1} - i_1 i_2\omega\theta^{i_1}(\nabla\nabla\omega)(\nabla\omega)^{i_2-1}\eta^{i_3} \\ & - (i_2 - 1)\,i_2\omega\theta^{i_1+1}(\nabla\omega)^{i_2-2}\nabla\eta\eta^{i_3}.\end{aligned}$$

Multiplying by $i_3 + 1$ yields

$$(i_3 + 1)\beta_{i_1 i_2 i_3} = -2_{i_2}(i_3 + 1)\alpha_{i_1+1, i_2-1, i_3+1} - i_1(i_3 + 1)\beta_{i_1 i_2 i_3} - (i_2 - 1)\, i_2 \gamma_{i_1+1, i_2-2, i_3+1}\,.$$

This gives (1).

(2) If $i_3 = 0$, then both sides are zero. Assume that $i_3 \geqslant 1$. We have

$$\begin{aligned}\gamma_{i_1 i_2 i_3} &= \omega\theta^{i_1}(\nabla\omega)^{i_2}\nabla(\eta^{i_3}) \\ &= i_3\omega\theta^{i_1}(\nabla\omega)^{i_2}\nabla\eta\eta^{i_3-1} \\ &= -i_3\nabla\eta\omega\theta^{i_1}(\nabla\omega)^{i_2}\eta^{i_3-1},\end{aligned}$$

since $\nabla\eta$ and $\omega\theta^{i_1}(\nabla\omega)^{i_2}$ both have odd degrees. Therefore, by (4.7) we have

$$\gamma_{i_1 i_2 i_3} = i_3[\omega, \nabla\omega]\, \omega\theta^{i_1}(\nabla\omega)^{i_2}\eta^{i_3-1}.$$

Again, we move ω to the right and distribute it:

$$\begin{aligned}\gamma_{i_1 i_2 i_3} = {} & -i_3\nabla\omega[\omega, \omega]\, \theta^{i_1}(\nabla\omega)^{i_2}\eta^{i_3-1} + i_3\nabla\omega\omega[\omega, \theta^{i_1}](\nabla\omega)^{i_2}\eta^{i_3-1} \\ & + i_3\nabla\omega\omega\theta^{i_1}[\omega, (\nabla\omega)^{i_2}]\eta^{i_3-1} + i_3\nabla\omega\omega\theta^{i_1}(\nabla\omega)^{i_2}[\omega, \eta^{i_3-1}].\end{aligned}$$

By (4.10), the last summand is zero and using $[\omega, \omega] = 2\eta$, (4.7), and (4.8), we obtain

$$\begin{aligned}\gamma_{i_1 i_2 i_3} = {} & -2i_3\theta^{i_1}(\nabla\omega)^{i_2+1}\eta^{i_3} - i_1 i_3\omega\theta^{i_1-1}\nabla\nabla\omega(\nabla\omega)^{i_2+1}\eta^{i_3-1} \\ & - i_2 i_3\omega\theta^{i_1}(\nabla\omega)^{i_2}\nabla\eta\eta^{i_3-1}.\end{aligned}$$

Multiplying by $i_2 + 2$ yields

$$(i_2 + 2)\gamma_{i_1 i_2 i_3} = -2i_3(i_2 + 2)\alpha_{i_1, i_2+1, i_3} - i_1 i_3\beta_{i_1-1, i_2+2, i_3-1} - i_2(i_2 + 2)\gamma_{i_1 i_2 i_3}\,.$$

This gives (2).

Finally, (3) follows immediately from Proposition 2.1(2) and thus we have completed the proof of Lemma 4.4.

Define the differential forms

$$\begin{aligned}\tilde{\alpha}_{i_1 i_2 i_3} &= ((2k)!/i_1!\, i_2!\, i_3!)\alpha_{i_1 i_2 i_3}\,, \\ \tilde{\beta}_{i_1 i_2 i_3} &= ((2k)!/(i_2 + 2i_3 + 1)\, i_1!\, i_2!\, i_3!)\beta_{i_1 i_2 i_3}\,, \\ \delta_{i_1 i_2 i_3} &= ((2k)!/(i_2 + 2i_3 + 1)\, i_1!\, i_2!\, i_3!)\omega\alpha_{i_1 i_2 i_3}\,.\end{aligned}$$

Remark. Using the fact that $(n_1 + n_2 + \cdots + n_p)!/(n_1!n_2! \cdots n_p!)$ is always an integer, for given nonnegative integers n_i, we can show that $(2k)!/(a!(b+1)!\, c!(b+2c))$ is an integer provided that $a + b + c = k$. This is easily seen by considering the following four cases: $b + 2c \leqslant k-1$, $b + 2c = k$, $b + 2c = k+1$, and $b + 2c \geqslant k+2$.

LEMMA 4.5. *If* $i_1 + i_2 + i_3 + 1 = k$, *then we have*

(1) $\nabla(\delta_{i_1 i_2 i_3}) = \tilde{\alpha}_{i_1, i_2+1, i_2} - \tilde{\beta}_{i_1 i_2 i_3} + \tilde{\beta}_{i_1-1, i_2+2, i_3-1}$,

(2) $\tilde{\alpha}_{i_1+1, 0, i_3+1} + \tilde{\beta}_{i_1, 1, i_3} = 0.$

Proof. From Lemma 4.4(3), we have

$$\nabla(\omega\alpha_{i_1 i_2 i_3}) = \alpha_{i_1, i_2+1, i_3} - \beta_{i_1 i_2 i_3} - \gamma_{i_1 i_2 i_3}.$$

Let $p = (2k)!/((i_2 + 2i_3 + 1)\, i_1! i_2! i_3!) = (i_2+1)(i_2+2)q$, where $q \in \mathbb{Z}$ by the preceding remark. Then, multiplying both sides by p and substituting $(i_2+1)(i_2+2)\,\gamma_{i_1 i_2 i_3}$ according to Lemma 4.4(2), gives the desired result.

(2) Applying Lemma 4.4(1), with $i_2 = 1$, we have

$$(i_1+1)(i_3+1)\beta_{i_1, 1, i_3} = -2(i_3+1)\alpha_{i_1+1, 0, i_3+1}.$$

Multiplying by the integer $(2k)!/((i_1+1)!1!(i_3+1)!(2+2i_3))$ we get

$$\tilde{\alpha}_{i_1+1, 0, i_3+1} + \tilde{\beta}_{i_1, 1, i_3} = 0.$$

LEMMA 4.6. *Let*

$$t = \sum_{i_1+i_2+i_3=k-1} \delta_{i_1 i_2 i_3} \in \Omega^{n-1} \otimes \Lambda^n P.$$

Then

$$((2k)!/k!)\, \theta_1{}^k = ((2k)!/k!)\, \theta^k + \nabla t.$$

With this lemma we conclude the proof of Theorem 4.3.

Proof of Lemma 4.6. By Lemma 4.5(1), we have

$$\nabla t = \sum_{\substack{i_1+i_2+i_3=k \\ i_2 \geqslant 1}} \tilde{\alpha}_{i_1, i_2, i_3} - \sum_{i_1+i_2+i_3=k-1} \tilde{\beta}_{i_1 i_2 i_3} + \sum_{\substack{i_1+i_2+i_3=k-1 \\ i_2 \geqslant 2}} \tilde{\beta}_{i_1 i_2 i_3}$$

and also by Lemma 4.5(2),

$$\sum_{\substack{i_1+i_3=k \\ i_1, i_3 \geqslant 1}} \tilde{\alpha}_{i_1, 0, i_3} + \sum_{i_1+i_3+2=k} \tilde{\beta}_{i_1, 1, i_3} = 0.$$

Adding this to ∇t gives

$$\nabla t = \sum_{i_1+i_2+i_3=k} \tilde{\alpha}_{i_1 i_2 i_3}$$

such that $i_2 \geqslant 1$ or, $i_2 = 0$ and $i_1, i_3 \geqslant 1$. Since

$$((2k)!/k!)\theta_1{}^k = \sum_{i_1+i_2+i_3=k} \tilde{\alpha}_{i_1, i_2 i_3} \qquad \text{by (4.6)},$$

and since $\tilde{\alpha}_{k,0,0} = ((2k)!/k!)$ and $\alpha_{0,0,k} = ((2k)!/k!)\,\eta^k$, we obtain

$$((2k)!/k!)\,\theta_1{}^k = ((2k)!/k!)\,\theta^k + \nabla t + ((2k)!/k!)\,\eta^k.$$

But $\eta^k = 0$. In fact, by Proposition 4.1 and (4.10)

$$\eta^k = \tfrac{1}{2}[\omega, \omega]\,\eta^{k-1} = ((k-1)/2)\,\omega\eta^{k-2}[\omega, \eta] = 0,$$

Thus we have obtained the desired relation.

Remark. We denote the Euler class $e(P, h, \nabla)$ simply by $e(P, h)$, since we have seen that $e(P, h, \nabla)$ is independent of the choice of the connection ∇ on P compatible with h.

5. Properties of Euler Classes Connections with the Differentiable Euler Classes

Let $\varphi\colon R \to R_1$ be a K-algebra homomorphism. Given an inner product module (P, h) over R define

(i) $P_1 = R_1 \otimes P$.

(ii) $h_1\colon P_1 \times P_1 \to R_1$ by $h_1(r_1 \otimes p, r_1' \otimes p') = r_1 r_1' \varphi(h(p, p'))$, $r_1, r_1' \in R_1$, $p, p' \in P$.

(iii) $\nabla_1\colon P_1 \to \Omega_{R_1} \otimes P_1$ by $\nabla_1(r_1 \otimes p) = dr_1 \otimes 1 \otimes p + \overline{r_1 \nabla(p)}$, $r_1 \in R_1$, $p \in P$, where $\overline{\nabla(p)}$ is the image of $\nabla(p)$ under the canonical map $\Omega_R \otimes P \to \Omega_{R_1} \otimes P_1$.

It is clear that if ∇ is a connection compatible with h, so is ∇_1 with respect to h_1. The next proposition is immediate.

Proposition 5.1. (1) *The following diagram is commutative*

$$\begin{tikzcd} P_1 = R_1 \otimes P \arrow[rr, leftarrow, "K_{\nabla_1}"] \arrow[dr, "I \otimes K_{\nabla}"'] & & \Omega^2_{R_1} \otimes P_1 \\ & R_1 \otimes (\Omega_R{}^2 \otimes P) \arrow[ur, "\eta"'] & \end{tikzcd}$$

where η *is the linear map which satisfies* $\eta(r_1 \otimes \omega \otimes p) = r_1\omega \otimes 1 \otimes p$, $r_1 \in R_1$, $\omega \in \Omega_R{}^2$, $p \in P$.

(2) *If* S *is a multiplicative closed subset of* R *and* $R \to R_S$ *is the natural map, then* $(K_\nabla)_S = K_{\nabla_S}$.

PROPOSITION 5.2. *Let* $\varphi: R \to R_1$ *be a* K*-algebra homomorphism. Let* (P, h) *be an inner product module over* R *of even rank, and let* (P_1, h_1) *be the one induced by* φ. *Then*

$$\varphi_*(e(P, h)) = e(P_1, h_1),$$

where φ_* *is the* K*-linear map* $H^*_{\mathrm{DR}}(P, h) \to H^*_{\mathrm{DR}}(P_1, h_1)$ *induced by the canonical map* $P \to P_1$.

Proof. Let ∇ be a connection on P compatible with h, and let ∇_1 be the one on P_1 induced by ∇, as in (iii) above. Let $\eta: R_1 \otimes (\Omega_R{}^2 \otimes P) \to \Omega^2_{R_1} \otimes P_1$ be the map of Proposition 5.1(1). Then $\psi: A^2(P, h) \to A^2(P_1, h_1)$, defined by $f \mapsto \eta \circ (I \otimes f)$, maps K_∇ into K_{∇_1}.

Now the result follows from the commutativity of all the following diagrams.

$$\begin{array}{ccc} \Omega_R{}^a \otimes \Lambda^k P & \xrightarrow{\nabla} & \Omega_R^{a+1} \otimes \Lambda^k P \\ \downarrow{\scriptstyle \varphi_*} & & \downarrow{\scriptstyle \varphi_*} \\ \Omega^a_{R_1} \otimes \Lambda^2 P_1 & \xrightarrow{\nabla_1} & \Omega^{a+1}_{R_1} \otimes \Lambda^k P_1 \end{array}$$

and

$$\begin{array}{ccc} \Omega_R{}^a \otimes \Lambda^2 P & \xrightarrow{v\circ(I\otimes u)} & A^2(P, h) \\ \downarrow{\scriptstyle \varphi_*} & & \downarrow{\scriptstyle \psi} \\ \Omega^a_{R_1} \otimes \Lambda^2 P_1 & \xrightarrow{v\circ(I\otimes u)} & A^2(P_1, h_1) \end{array}$$

THEOREM 5.3. *Suppose that for* $i = 1, 2$, (P_i, h_i) *is an inner product module over* R *of even rank* n_i, *and let* $(P, h) = (P_1, h_1) \perp (P_2, h_2)$. *Then*

$$e(P, h) = \binom{n_1 + n_2}{n_1} e(P_1, h_1) \wedge e(P_2, h_2)$$

in $H_{\mathrm{DR}}(P, h)$, *where* $\wedge$ *is the product induced in cohomology by the canonical bilinear map* $\Lambda^{n_1}P_1 \times \Lambda^{n_2}P_2 \to \Lambda^{n_1+n_2}P$.

Proof. Let ∇ be a connection on P_i compatible with h_i. Then $\nabla = \nabla_1 \oplus \nabla_2$ is a connection on P compatible with h. The identification $\Omega^a \otimes \Lambda^m P = \bigoplus_{i+j=m} (\Omega^a \otimes \Lambda^i P_1 \otimes \Lambda^j P_2)$ yields, when $a = 2$ and $m = 2$, the equation

$\theta = \theta_1 + \theta_2$ for the corresponding curvature forms. Thus, if $n_i = 2k_i$, we have

$$\theta^{k_1+k_2} = (\theta_1 + \theta_2)^{k_1+k_2} = \binom{k_1 + k_2}{k_1} \theta_1^{k_1} \wedge \theta_2^{k_2},$$

since $\theta_i^{\,p} = 0$ for $p > k_i$. Multiplying by $(2k_1 + 2k_2)!/(k_1 + k_2)!$ gives the result.

THEOREM 5.4. *Let $\alpha\colon (P, h) \to (Q, k)$ be an isometry of inner product modules of even rank n, i.e., an isomorphism $\alpha\colon P \xrightarrow{\sim} Q$ such that $h(x, y) = k(\alpha x, \alpha y)$ for all $x, y \in P$. Then $(\Lambda^n\alpha)_*(e(P, h)) = e(Q, k)$, where $(\Lambda^n\alpha)$ is the isomorphism $H^*_{\mathrm{DR}}(P, h) \xrightarrow{\sim} H^*_{\mathrm{DR}}(Q, k)$ induced by $\Lambda^n P \underset{\sim}{\xrightarrow{\Lambda^n\alpha}} \Lambda^n Q$.*

Proof. Consider a connection ∇ on P compatible with h. Then $\nabla_\alpha = (I \otimes \alpha) \circ \nabla \circ \alpha^{-1}$ is a connection on Q compatible with k.

Next, it is easy to check that the following diagrams are commutative

$$\begin{CD} P @>{K_\nabla}>> \Omega^2 \otimes P \\ @V{\alpha}VV @VV{I\otimes\alpha}V \\ Q @>{K_{\nabla_\alpha}}>> \Omega^2 \otimes Q \end{CD}$$

and

$$\begin{CD} \Omega^a \otimes \Lambda^2 P @>{v\circ(I\otimes u)}>{\sim}> A^a(P, h) \\ @V{I\otimes\alpha\wedge\alpha}VV @VV{\mu}V \\ \Omega^a \otimes \Lambda^2 Q @>{v\circ(I\otimes u)}>{\sim}> A^a(Q, k) \end{CD}$$

where $\mu(f) = (I \otimes \alpha) \circ f \circ \alpha^{-1}, f \in A^a(P, h)$.

Thus, if θ is the curvature of ∇ and θ_α that of ∇_α, we have $(I \otimes \alpha \wedge \alpha)(\theta) = \theta_\alpha$. Therefore, if $n = 2p$, we have $(I \otimes \Lambda^{2p}\alpha)(\Lambda^p\theta) = \Lambda^p((I \otimes \alpha \wedge \alpha)\theta) = \Lambda^p\theta_\alpha$. This shows that $(\Lambda^n\alpha)_*(e(P, h)) = e(Q, k)$.

COROLLARY 5.5. *If $\alpha\colon (P, h) \to (P, h)$ is an isometry and* rank $P = n$ *is even, then*

(1) $(\Lambda^n\alpha)_*(e(P, h)) = e(P, h)$.

(2) $(\Lambda^n\alpha) \circ (\Lambda^n\alpha) = I$.

In particular, if P is free then, for all $\alpha \in O(P, h)$ we have

(3) $\det \alpha \cdot e(P, h) = e(P, h)$.

(4) $(\det \alpha)^2 = 1$.

COROLLARY 5.6. *If (P, h) is an inner product module of even rank and $P = P_1 \perp P_2$, where P_1 has odd rank, then $e(P, h) = 0$.*

Proof. Define $\alpha \in O(P, h)$ by $\alpha(p_1, p_2) = (-p_1, p_2)$, $p_i \in P_i$. Then $\Lambda^n \alpha = -I$ and so $-e(P, h) = e(P, h)$. Thus $e(P, h) = 0$.

Let M be a C^∞-manifold and let E be an oriented vector bundle over M of even dimension $n = 2k$. Let R be the ring of C^∞-maps $M \to \mathbb{R}$, and $P = \Gamma(E)$, the R-module of C^∞-sections of E over M. Choose a Euclidean metric h on E. Then we have an inner product module (P, h).

PROPOSITION 5.7. *There exists an isomorphism $H^*_{\mathrm{DR}}(P, h) \xrightarrow[\sim]{\alpha} H^*(M)$ such that $\alpha(e(P, h)) = (2\pi)^{n/2} n! e(E)$, where $e(E)$ is the usual Euler class of E.*

Proof. Let $A^p(M) = \Gamma(\Lambda^p T^*)$ be the module of C^∞-differential forms of degree p on M; here T^* is the dual tangent bundle of M. Then we have an isomorphism of differential graded algebras $\Omega^{\cdot}_{R/\mathbb{R}} \approx A^*(M)$. Using the isomorphism $\Omega^a_{R/\mathbb{R}} \otimes \Lambda^\alpha P \approx \Gamma(\Lambda^a T^* \otimes \Lambda^\alpha E)$ we regard the elements of $\Omega^a_{R/\mathbb{R}} \otimes \Lambda^\alpha P$ as functions on M. Take a connection ∇ on P compatible with h. Then, ∇ is a local operator: if $\omega, \omega' \in \Omega^a \otimes \Lambda^\alpha P$ are such that $\omega = \omega'$ on an open set $U \subset M$, then $\nabla(\omega) = \nabla(\omega')$ on U.

Suppose now that $(e_1, \ldots, e_n)$ is an orthonormal basis for E over U. We can write

$$\nabla(e_i) = \sum_{j=1}^{n} \omega_{ji} \otimes e_j$$

$$K_\nabla(e_i) = \sum_{j=1}^{n} K_{ji} \otimes e_j$$

on U, with $\omega_{ij} \in \Omega_R$, $K_{ij} \in \Omega_R{}^2$. Since $(h(e_i, e_j))$ = identity matrix, the compatibility condition implies that (ω_{ij}) and (K_{ij}) are both skew-symmetric matrices.

If $\theta_\nabla = (v \circ (I \otimes u))^{-1}(K_\nabla)$, then $\theta_\nabla = -\frac{1}{2}\sum_{i,j} K_{ij} \otimes e_i \wedge e_j$. Thus

$$\Lambda^k \theta_\nabla = ((-1)^k/2^k)\left(\sum \epsilon_{i_1 i_2 \cdots i_{2k}} K_{i_1 i_2} \wedge \cdots \wedge K_{i_{2k-1}, i_{2k}}\right) \otimes e_1 \wedge \cdots \wedge e_{2k},$$

on U.

On the other hand, by [4], the Euler class of E is the cohomology class of the form γ, where γ can be expressed on U with respect to the orthonormal positive basis $(e_1, \ldots, e_n)$ by

$$\gamma = ((-1)^k/2^{2k}\pi^k k!)\left(\sum \epsilon_{i_1 i_2 \cdots i_{2k}} K_{i_1 i_2} \wedge \cdots \wedge K_{i_{2k-1}, i_{2k}}\right).$$

By choosing orthonormal positive local bases we define an isomorphism $\alpha: \Lambda^n P \oplus R$ as follows. Set

$$\alpha(s_1 \wedge \cdots \wedge s_n) = \det(a_{ij}) \text{ on } U$$

if $s_1, \ldots, s_n \in P$, $s_i = \sum a_{ji} e_j$ on U, and $(e_1, \ldots, e_n)$ is an orthonormal positive basis for E/U.

This extends to an isomorphism $\Omega^{\cdot} \otimes \Lambda^n P \to \Omega^*$ which sends $(2k)!/k!\, \Lambda^k \theta_\nabla$ into $(2\pi)^k (2k)!\gamma$.

It only remains to see that α commutes with the operators ∇ and d. This is equivalent to $\nabla(\omega_0) = 0$, where $\omega_0 = \alpha^{-1}(1)$. But, if $(e_1, \ldots, e_n)$ and U are as above, then $\omega_0/U = e_1 \wedge \cdots \wedge e_n$, and so

$$\begin{aligned} \nabla(\omega_0)/U &= \sum_{i=1}^{n} e_1 \wedge \cdots \wedge \nabla e_i \wedge \cdots \wedge e_n \\ &= (\text{Trace of } (\omega_{ij})) e_1 \wedge \cdots \wedge e_n \\ &= 0, \end{aligned}$$

since (ω_{ij}) is skew as we have observed earlier.

6. Applications

Euler Classes of Free Modules

Suppose that (F, h) is an inner product module with F free. Choose a basis $\{e_1, \ldots, e_n\}$ for F and write $c_{ij} = h(e_i, e_j)$, $C = (c_{ij})$, and $u = \det C$. Then, by the proof of Theorem 1.3, a connection ∇ on F compatible with h can be given by

$$\nabla(e_i) = \sum_j \omega_{ji} \otimes e_j$$

where $\omega = \frac{1}{2} C^{-1}\, dC$.

Let $\bar{C}: F \to F$ be the linear map given by $\bar{C}(e_i) = \sum_j c_{ji} e_j$. Then, it can be checked that

$$(I \otimes \bar{C} \wedge \bar{C})(\theta_\nabla) = \tfrac{1}{8} \sum_{i,j} \theta_{ij} \otimes e_i \wedge e_j,$$

where $(\theta_{ij}) = dC \wedge C^{-1}\, dC$, $I =$ identity map of Ω^2. Thus

$$\Lambda^k \theta_\nabla = (1/8^k u) \left\{ \sum \epsilon_{i_1 i_2 \cdots i_{2k}} \theta_{i_1 i_2} \wedge \cdots \wedge \theta_{i_{2k-1}, i_{2k}} \right\} \otimes e_1 \wedge \cdots \wedge e_{2k}.$$

Since h_n on $\Lambda^n F$ is multiplication by $u = \det C$, we see that $(\Omega^{\cdot} \otimes \Lambda^n F, \nabla) \approx (\Omega^{\cdot}, d_h)$, where $d_h: \Omega^p \to \Omega^{p+1}$ is the coboundary map given by $d_h(\eta) = d\eta + du/2u \wedge \eta$, $\eta \in \Omega^p$. Thus

$$e(F, h) = ((2k)!/k!\, 8^k u) \left\{ \sum \epsilon_{i_1 i_2 \cdots i_{2k}} \theta_{i_1 i_2} \wedge \cdots \wedge \theta_{i_{2k-1}, i_{2k}} \right\} \in H^{2k}_{\mathrm{DR}}(R, d_h).$$

The Euler Class of the Tangent Bundle to the n-Sphere

Let K be a commutative ring. For $n \geqslant 1$, let $R = R_n = K[X_1, ..., X_{n+1}]/(f)$, where $f = 1 - \sum_1^{n+1} X_i^2$. Let x_i be the image of X_i in R. Let $P = P_n = \{a \in R^{n+1} \mid h(a, u) = 0\}$, where $u = (x_1, ..., x_{n+1})$ and $h: R^{n+1} \times R^{n+1} \to R$ is the usual inner product $h(a, b) = \sum_i a_i b_i$.

If $\{e_1, ..., e_{n+1}\}$ is the standard basis for R^{n+1} and $e'_i = e_i - x_i u$, $i = 1, ..., n+1$, then $\{e'_1, ..., e'_{n+1}\}$ is a system of generators of P. A connection ∇ compatible with h on P can be given by $\nabla(a) = \sum_j da_j \otimes e_j'$, $a = (a_1, ..., a_{n+1}) \in P$. In particular, $\nabla(e'_i) = -\sum_j d(x_i x_j) \otimes e'_j$. Its corresponding curvature takes the form

$$K_\nabla(a) = \sum_{i<j} dx_i \wedge dx_j \otimes (a_i e'_j - a_j e'_i)$$

where a_i is the ith component of $a \in P$. Since $u_{e_i' \wedge e_i'}(a) = a_i e'_j - a_j e'_i$, $a \in P$, we have

$$\theta_\nabla = \tfrac{1}{2} \sum_{i,j} dx_i \wedge dx_j \otimes e'_i \wedge e'_j .$$

Let

$$\omega = \sum_{i=1}^{n+1} (-1)^{i+1} x_i \, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_{n+1} \in \Omega_R{}^n,$$

$$e = \sum_{i=1}^{n+1} (-1)^{i+1} x_i e'_1 \wedge \cdots \wedge \widehat{e'_i} \wedge \cdots \wedge e'_{n+1} \in \Lambda^n P.$$

It is immediate that

$$\begin{aligned} dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_{n+1} &= (-1)^{i+1} x_i \omega, \\ e'_1 \wedge \cdots \wedge \widehat{e'_i} \wedge \cdots \wedge e'_{n+1} &= (-1)^{i+1} x_i e. \end{aligned} \tag{6.1}$$

From these we readily obtain $\Omega_R{}^n = R\omega$ and $\Lambda^n P = Re$, and the map $\Omega_R{}^p \otimes \Lambda^n P \to \Omega_R{}^p$ defined by $\eta \otimes e \to \eta$ establishes an isomorphism of differential graded algebras.

Now, we assume that $n = 2k$. Then, by the above description of θ_∇ and (6.1), we see that

$$e(P, h) = ((2k)!^2/k!\, 2^k)\{\omega\} \qquad \text{in } H^n_{\mathrm{DR}}(R). \tag{6.2}$$

We wish to show that $e(P, h) = 0$ if char K is a prime $p \geqslant k+1$, and $e(P, h) \neq 0$ and is a generator of $H^n_{\mathrm{DR}}(R)$, if K contains the rational numbers.

If $\eta_1, \eta_2 \in \Omega_R{}^n$, we write $\eta_1 \equiv \eta_2$ in case η_1 and η_2 are cohomologous, and if $\alpha = (\alpha_1, ..., \alpha_{n+1})$, with nonnegative integers α_i, we set $x^\alpha = x_1^{\alpha_1} \cdots x_{n+1}^{\alpha_{n+1}}$ and $|\alpha| = \alpha_1 + \cdots + \alpha_{n+1}$.

Using (6.1), we obtain

$$d(x^\alpha x_i x_j \, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_{n+1})$$
$$= (-1)^{i+j}((\alpha_i + 1)\, x^\alpha x_j^2 - (\alpha_j + 1)\, x^\alpha x_i^2)\omega, \qquad i < j.$$

Thus

$$(\alpha_i + 1)\, x^\alpha x_j^2 \omega \equiv (\alpha_j + 1)\, x^\alpha x_i^2 \omega \qquad \text{for all} \quad \alpha, i, j.$$

Adding these relations over all $i = 1, 2,\ldots, n + 1$, and using

$$\sum_{i=1}^{n+1} x_i^2 = 1, \tag{6.3}$$

we get

$$(|\alpha| + n + 1)\, x^\alpha x_j^2 \omega \equiv (\alpha_j + 1)\, x^\alpha \omega \qquad \text{for all} \quad \alpha, j. \tag{6.4}$$

If char K is a prime $p \geqslant n + 1$, then (6.4) implies $x^\alpha \omega \equiv 0$ for all α such that $|\alpha| + n + 1 = p$. Next, using induction and (6.3) to lower degrees by 2 we arrive at $\omega \equiv 0$. If $k + 1 \leqslant p \leqslant 2k$ then $(2k)!/k! = 0$. Thus, $e(P, h) = 0$, as was claimed.

If K contains the rational numbers, then (6.4) can be written as

$$x^\alpha x_j^2 \omega \equiv (\alpha_j + 1)/(|\alpha| + n + 1)\, x^\alpha \omega$$

and from this and induction on $|\alpha|$, we can see that $H^n_{\mathrm{DR}}(R) = K\{\omega\}$.

Define a K-linear map φ: $K[X_1, \ldots, X_{n+1}] \to K$ as follows

$$\begin{aligned} \varphi(X_1^{\alpha_1} \cdots X_{n+1}^{\alpha_{n+1}}) &= 0 && \text{if some } \alpha_i \text{ is odd} \\ &= \left(\prod_{i=1}^{n+1} s_{\alpha_i - 1}\right) \Big/ s_{|\alpha|+n-1} && \text{if all } \alpha_i \text{ are even} \end{aligned} \tag{6.5}$$

where we set $s_{-1} = 1$ and $s_m = 1 \times 3 \times \cdots \times m$, if m is an odd number $\geqslant 1$.

It is easy to check that

$$\varphi(X^\alpha X_j^2) = ((\alpha_j + 1)/(|\alpha| + n + 1))\, \varphi(X^\alpha)$$

and so $\varphi(I) = \{0\}$, where $I = (1 - \sum_{i=1}^{n+1} X_i^2)$. Therefore φ induces a K-linear map

$$\varphi\colon \Omega_R{}^n = R\omega \approx R \to K, \qquad \omega \mapsto 1.$$

Furthermore, by checking on generators $d(x^\alpha dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_{n+1})$ we can see that φ maps $d\Omega_R^{n-1}$ into 0. Thus φ: $H^n_{\mathrm{DR}}(R) \approx K$ and $\varphi(\{\omega\}) = 1/s_{n-1}$.

PROPOSITION 6.1. *Suppose that K is an Archimedean ordered field. Then P_2 is not a free module.*

Proof. Let $S^2 = \{(a_1, a_2, a_3) \in K^3 \mid \sum_{i=1}^3 a_i^2 = 1\}$. We observe that any $f \in R = R_2$ defines a K-valued map on S^2 in an obvious manner.

Let $T = \{r \in R \mid r = 1 + h, h \in R \text{ and } h(a) \geqslant 0, \text{ for all } a \in S^2\}$. This is a multiplicative closed subset of R.

If $P = P_2$ were free, then $e(P_T, h_T) = 0$ in $H^2_{\mathrm{DR}}(P_T, h_T)$. In fact, choosing a basis $\{e_1, e_2\}$ for P and writing $h(e_1, e_1) = p$, $h(e_1, e_2) = q$, $h(e_2, e_2) = r$, then $pr - q^2 = u \in K^{\cdot}$, with $u > 0$. Since $p \cdot (r/u) = 1 + (q^2/u) \in T$, p becomes a unit in R_T, and so $P_T = R_T e_1 \perp (R_T e_1)^{\perp}$. Therefore, $e(P_T, h_T) = 0$, by Corollary 5.6.

However, we show that $e(P_T, h_T) \neq 0$.

DEFINITION. (1) Let f_n, f be functions on S^2 with values in K. We write $f_n \to f$ if $\forall \epsilon > 0\ \exists n_0$ such that $|f_n(a) - f(a)| \leqslant \epsilon\ \forall a \in S^2$, $n \leqslant n_0$.

(2) If $\eta_n, \eta \in \Omega_R^{\alpha}$, we write $\eta_n \to \eta$ if there exists $\omega_j \in \Omega_R^{\alpha}$, $f_j, f_{nj} \in R$, $j = 1, 2, \ldots, p$, $n = 1, 2, 3, \ldots$, such that

(i) $\eta_n = \sum_{j=1}^p f_{nj}\omega_j$ and $\eta = \sum_{j=1}^p f_j\omega_j$,

(ii) $f_{nj} \to f_j$, $j = 1, \ldots, p$.

Remarks. (a) Any $f \in R$ is bounded on S^2.

(b) If $f_n \to f$ and $g_n \to g$, then $f_n + g_n \to g + g$.

(c) If $f_n \to f$ and $g_n \to g$, and f and g are both bounded, then $f_n \cdot g_n \to f \cdot g$.

(d) If $\eta_n \to \eta$, $f_n \to f$, f bounded, then $f_n\eta_n \to f\eta$.

LEMMA 6.2. *Let $t \in T$. Then there exists a sequence (u_n), $u_n \in R$, such that*

(1) $u_n \to 1/t$,

(2) $du_n + u_n^2\, dt \to 0$.

Observe that (2) *means $du_n \to d(1/t)$.*

Proof. First of all, we observe that if $t \in T$, then we can write $t = 1 + mh$, where $h \in R$ and $0 \leqslant h \leqslant \frac{1}{2}$, and $m \in \mathbb{N}$. In fact, let $h' \in R$ such that $t = 1 + h'$ and $h'(a) \geqslant 0$, for all $a \in S^2$. Since h' is bounded on S^2 and K is Archimedean, we can find $m \in \mathbb{N}$ such that $2h' \leqslant m$. Then, $h = h'/m$ will do.

We proceed by induction on m. Let $r = 1 + (m-1)h$, so that $t = r + h$. Suppose we have already found (v_n), $v_n \in R$, such that

(1) $v_n \to 1/r$,

(2) $dv_n + v_n^2\, dr \to 0$.

Since $v_n h \to h/r$ and $h/r \leqslant \frac{1}{2}$ we can assume that

(3) $|v_n h| \leqslant \frac{2}{3}$ for all n.

Set $u_n = v_n \cdot \sum_0^n (-1)^k (v_n h)^k = v_n(f_n/g_n) \in R$, where $f_n = 1 + (-1)^n (v_n h)^{n+1}$, and $g_n = 1 + v_n h$. Then

(4) $f_n \to 1$, $g_n \to 1 + (h/r)$, and $|g_n| \leqslant \frac{5}{3}$, for all n. It is clear that $u_n \to (1/r)(1/(1 + (h/r))) = 1/(r + h) = 1/t$, in view of (1) and (4). Observe that $du_n + u_n^2\, dt \to 0$ is equivalent to $g_n^2(du_n + u_n^2\, dt) \to 0$, since $\frac{1}{3} \leqslant |g_n|$, for all n. From $g_n^2 d(f_n/g_n) = g_n df_n - f_n dg_n$, we obtain

$$g_n^2(du_n + u_n^2 dt) = g_n f_n dv_n + v_n g_n df_n - v_n f_n dg_n + v_n^2 f_n^2 dt.$$

Expressing this as a linear combination of dv_n, dh, and dr, yields

$$\begin{aligned} g_n^2(du_n + u_n^2\, dt) = {} & \{f_n g_n + (-1)^n (n+1)\, g_n \theta_n^{n+1} - f_n \theta\}\, dv_n \\ & + \{(-1)^n (n+1)\, g_n v_n^2 \theta_n^{\,n} - v_n^2 f_n + v_n^2 f_n^{\,2}\}\, dh + v_n^2 f_n^{\,2}\, dr, \end{aligned}$$

where $\theta_n = v_n h$. Writing $\beta_n = dv_n + v_n^2 dr$, we know that $\beta_n \to 0$, and substituting dv_n we get

$$g_n^2(du_n + u_n^2 dt) = A_n + B_n dr + C_n dh$$

where

$$\begin{aligned} A_n &= \{f_n g_n + (-1)^n (n+1)\, g_n \theta_n^{n+1} - f_n \theta_n\}\beta_n, \\ B_n &= -v_n^2 \{f_n g_n + (-1)^n (n+1)\, g_n \theta_n^{n+1} - f_n \theta_n - f_n^2\}, \\ C_n &= v_n^2 \{(-1)^n (n+1)\, g_n \theta_n^{\,n} - f_n + f_n^2\}. \end{aligned}$$

Using $f_n \to 1$, $\beta_n \to 0$, $g_n \to 1 + (h/r)$, $\theta_n \to (h/r)$, $(n+1)\theta_n^{\,n} \to 0$ (by (3)), $v_n \to (1/r)$, we get $A_n \to 0$, $B_n \to 0$, and $C_n \to 0$.

LEMMA 6.3. *Let* $\omega = x_1 dx_2 \wedge dx_3 + x_2 dx_3 \wedge dx_1 + x_2 dx_1 \wedge dx_3$. *Suppose that* $\omega = d(\eta/t)$ *for some* $\eta \in \Omega_R^{\,2}$ *and* $t \in T$, *Then there exist sequences of forms* (α_n), (β_n), $\alpha_n \in \Omega_R^{\,2}$, $\beta_n \in \Omega_R$, *such that* $\omega = \alpha_n + d\beta_n$, *and* $\alpha_n \to 0$.

Proof. We have

$$\begin{aligned} \omega &= d(\eta/t) = d\eta/t - (1/t^2)\, dt \wedge \eta, \\ t^2 \omega &= t d\eta - dt \wedge \eta. \end{aligned}$$

Take (u_n) as in Lemma 6.2. Then

$$\begin{aligned} u_n^2 t^2 \omega &= u_n^2 t d\eta - u_n^2 dt \wedge \eta \\ &= u_n^2 t d\eta - (du_n + u_n^2\, dt) \wedge \eta + du_n \wedge \eta \\ &= u_n(u_n t - 1)\, d\eta - (du_n + u_n^2\, dt) \wedge \eta + d(u_n \wedge \eta). \end{aligned}$$

Thus we have

$$u_n{}^2t^2\omega = \alpha_n' + d\beta_n\,, \qquad \text{with} \quad \alpha_n' \to 0.$$

Finally, $\omega = (1 - u_n{}^2t^2)\omega + \alpha_n' + d\beta_n = \alpha_n + d\beta_n$, where $\alpha_n = (1 - u_n{}^2t^2)\omega + \alpha_n' \to 0$.

LEMMA 6.4. *If $f \in R$ satisfies $f(a) \geqslant 0$, for all $a \in S^2$, then $\varphi(f) \geqslant 0$, where $\varphi\colon R \to K$ is the map defined in* (6.5), *which induces $H^2_{\mathrm{DR}}(R) \approx K$.*

Proof. We can assume $f = \sum \lambda_{\alpha\beta} x_1^{2\alpha} x_2^{2\beta}$, $\lambda_{\alpha\beta} \in K$. In fact, by definition of φ, $\varphi(f) = \varphi(\frac{1}{8}\sum f(\pm x_1, \pm x_2, \pm x_3))$, so that f can be replaced by a sum of monomials of even degree in x_1, x_2, and x_3. Then, using $1 = x_1{}^2 + x_2{}^2 + x_3{}^2$ we obtain the required expression.

Now, we prove two assertions:

Assertion 1. Let $N_\alpha = \sum_{p=0}^{\alpha} (-1)^p \binom{\alpha}{p} (1/(2p+1)) \in \mathbb{Q}$. Then there exists a sequence $\{(a_n, b_n)\}$, $a_n, b_n \in \mathbb{Q}$ such that

(i) $a_n{}^2 + b_n{}^2 = 1$, for all $n \in \mathbb{Z}$.

(ii) $\lim_{N\to\infty} (1/N) \sum_{n=0}^{N} a_n^{2\alpha} b_n^{2\beta} = \varphi(x_1^{2\alpha} x_2^{2\beta})/N_{\alpha+\beta}$, for all $\alpha, \beta \geqslant 0$.

Proof. Choose $\zeta = a + bi \in \mathbb{Q}(i)$, $i = (-1)^{1/2}$, such that $a^2 + b^2 = 1$ and $\zeta^\alpha \neq 1$ for all $\alpha \in \mathbb{Z}$, $\alpha \neq 0$. Define $a_n, b_n \in \mathbb{Q}$ by the equation $\zeta^n = a_n + b_n i$, $n \in \mathbb{Z}$. Then $a_n = a_{-n}$ and $b_n = -b_{-n}$.

Now, we contend that the following relations hold.

$$(1) \quad N_\alpha = \frac{2\alpha}{2\alpha+1} N_{\alpha-1}\,, \qquad \text{for all } \alpha \in \mathbb{Z}, \quad \alpha \geqslant 1.$$

$$(2) \quad \frac{\varphi(x_1^{2\alpha} x_2^{2\beta})}{N_{\alpha+\beta}} = \frac{\varphi(x_1^{2\alpha} x_2^{2\beta-2})}{N_{\alpha+\beta-1}} - \frac{\varphi(x_1^{2\alpha+2} x_2^{2\beta-2})}{N_{\alpha+\beta}}\,.$$

$$(3) \quad a_n^{2\alpha} b_n^{2\beta} = a_n^{2\alpha} b_n^{2\beta-2} - a_n^{2\alpha+2} b_n^{2\beta-2}, \qquad \text{for all } n \in \mathbb{Z}.$$

$$(4) \quad N_\alpha \cdot \binom{2\alpha}{\alpha} \cdot \frac{1}{2^{2\alpha}} = \varphi(x_1^{2\alpha}) = \frac{1}{2\alpha+1}\,.$$

$$(5) \quad \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N} a_n^{2\alpha} = \frac{1}{2^{2\alpha}} \binom{2\alpha}{\alpha}.$$

To prove (1), we write, for $0 \leqslant p \leqslant \alpha - 1$

$$\frac{2\alpha}{2\alpha+1} \binom{\alpha-1}{p} \frac{1}{2p+1} = \binom{\alpha}{p} \frac{2(\alpha-p)}{(2\alpha+1)(2p+1)} = \frac{1}{2p+1} \binom{\alpha}{p} - \frac{1}{2\alpha+1} \binom{\alpha}{p}.$$

Then

$$\frac{2\alpha}{2\alpha+1} N_{\alpha-1} = \frac{2\alpha}{2\alpha+1} \sum_0^{\alpha-1} (-1)^p \binom{\alpha-1}{p} \frac{1}{2p+1}$$

$$= \sum_0^{\alpha-1} (-1)^p \binom{\alpha}{p} \frac{1}{2p+1} - \frac{1}{2\alpha+1} \sum_0^{\alpha-1} (-1)^p \binom{\alpha}{p}$$

$$= N_\alpha - (-1)^\alpha \frac{1}{2\alpha+1} - \frac{1}{2\alpha+1} (-(-1)^\alpha) = N_\alpha .$$

Now, (2) follows from (1) and the definition of φ. (3) follows from $a_n{}^2+b_n{}^2 = 1$, and (4) is proved by induction using (1) and the definition of φ.

To prove (5), we observe that

$$\lim_{N\to\infty} (1/N) \sum_{n=0}^{N} \zeta^{n\alpha} = 1 \qquad \text{if} \quad \alpha = 0,$$

$$= 0 \qquad \text{if} \quad \alpha \neq 0.$$

In fact, if $\alpha = 0$, this is clear, while if $\alpha \neq 0$, then

$$\left\| \frac{1}{N} \sum_{n=0}^{N} \zeta^{n\alpha} \right\| = \left\| \frac{1}{N} \frac{1-\zeta^{(N+1)\alpha}}{1-\zeta^\alpha} \right\| \leqslant \frac{4}{N^2 \| 1-\zeta^\alpha \|},$$

where we are using $\| p+qi \| = p^2+q^2$, $\| u+v \| \leqslant 2(\| u \| + \| v \|)$. This gives (5) because

$$(1/N) \sum_{n=0}^{N} a_n^{2\alpha} = (1/N) \sum_{n=0}^{N} ((\zeta^n + \zeta^{-n})/2)^{2\alpha}$$

$$= (1/2^{2\alpha}) \sum_0^{2\alpha} \binom{2\alpha}{p} \left[(1/N) \sum_{n=0}^{N} \zeta^{(2\alpha-2p)n} \right]$$

implies

$$\lim_{N\to\infty} (1/N) \sum_{n=0}^{N} a_n^{2\alpha} = (1/2^{2\alpha}) \sum_0^{2\alpha} \binom{2\alpha}{p} \lim_{N\to\infty} (1/N) \sum_{n=0}^{N} \zeta^{(2\alpha-2p)n}$$

$$= (1/2^{2\alpha}) \binom{2\alpha}{\alpha}.$$

Finally, (4) and (5) show the assertion for $\beta = 0$, and using (2), (3), and induction on β, we complete the proof of Assertion 1.

Assertion 2. Given an integer $C \geqslant 0$ there exists a sequence $\{r_N = (r_{N,0}, \ldots, r_{N,N}), r_{N,i} \in \mathbb{Q}\}$ such that

(i) $0 < r_{N,j} \leqslant 1,\ 1 - r_{N,j}^2 = t_{N,j}^2$, with $t_{N,j} \in \mathbb{Q}$.

(ii) $\lim_{N\to\infty} (1/N) \sum_{j=0}^{N} r_{N,j}^{2\alpha} = N_\alpha$, for all $0 \leqslant \alpha \leqslant C$.

Proof. We first notice that, for all $\alpha \geqslant 0$, we have

$$\lim_{N\to\infty} (1/N) \sum_{j=0}^{N} (1 - (j^2/N^2))^\alpha = N_\alpha .$$

In fact,

$$(1 - (j^2/N^2))^\alpha = \sum_{p=0}^{\alpha} (-1)^p \binom{\alpha}{p} (j^{2p}/N^{2p})$$

and

$$\lim_{N\to\infty} (1/N^{2p+1}) \sum_{j=0}^{N} j^{2p} = (1/(2p+1)).$$

For each N, choose $\epsilon_N > 0$ small enough so that

$$(1 - (j^2/N^2) + \epsilon_N)^\alpha - (1 - (j^2/N^2))^\alpha \leqslant (1/N^2)$$

for all $1 \leqslant \alpha \leqslant C$ and $0 \leqslant j \leqslant N$. In particular, $\epsilon_N \leqslant (1/N^2)$. Now, choose $r_{N,j} \in \mathbb{Q}$ such that $r_{N,0} = 1$ and

$$1 - (j^2/N^2) \leqslant r_{N,j}^2 \leqslant 1 - (j^2/N^2) + \epsilon_N , \qquad 1 \leqslant j \leqslant N,$$

and such that $1 - r_{N,j}^2 = t_{N,j}^2$, with $t_{N,j} \in \mathbb{Q}$. For this, it is enough to choose $y \in \mathbb{Q}$ so that

$$1 - (j^2/N^2) \leqslant (2y/(1+y^2))^2 \leqslant 1 - (j^2/N^2) + \epsilon_N .$$

Then

$$(1 - (j^2/N^2))^\alpha \leqslant r_{N,j}^{2\alpha} \leqslant (1 - (j^2/N^2) + \epsilon_N)^\alpha \leqslant (1 - (j^2/N^2))^\alpha + (1/N^2)$$

for all $1 \leqslant \alpha \leqslant C$ and $0 \leqslant j \leqslant N$. It follows that

$$\lim_{N\to\infty} (1/N) \sum_{j=0}^{N} r_{N,j}^{2\alpha} = N_\alpha , \qquad \text{for all } 0 \leqslant \alpha \leqslant C.$$

We conclude the proof of Lemma 6.4. We have

$$\lim_{M,N\to\infty} (1/MN) \sum_{j=0}^{N} \sum_{k=0}^{M} (r_{N,j} a_k)^{2\alpha} (r_{N,j} b_k)^{2\beta} = \varphi(x_1^{2\alpha} x_2^{2\beta})$$

for all $\alpha, \beta \leqslant C =$ degree of f, by Assertions 1 and 2. This implies

$$\lim_{M,N\to\infty} (1/MN) \sum_{j=0}^{N} \sum_{k=0}^{M} f(r_{N,j}a_k , r_{N,j}b_k , t_{N,j}) = \varphi(f)$$

with

$$(r_{N,j}a_k)^2 + (r_{N,j}b_k)^2 + (t_{N,j})^2 = 1,$$

so that

$$f(r_{N,j}a_k , r_{N,j}b_k , t_{N,j}) \geqslant 0.$$

Hence $\varphi(f) \geqslant 0$.

From Lemma 6.4, it follows that if $\eta_n \to \eta$ in $\Omega_R{}^2$, then $\varphi(\eta_n) \to \varphi(\eta)$.

We are now ready to complete the proof of Proposition 6.1.

Suppose $e(P_T , h_T) = 0$. Then $\omega = d(\eta/t)$, $\eta \in \Omega_R$, and $t \in T$, since $e(P, h) = 2\{\omega\}$ by (6.2). Hence, by Lemma 6.3, there exist sequences of forms (α_n), (β_n), such that $\omega = \alpha_n + d(\beta_n)$, for all n, and $\alpha_n \to 0$. Thus $1 = \varphi(\omega) = \varphi(\alpha_n)$ and $\varphi(\alpha_n) \to 0$ by the above consequence of Lemma 6.4. This is a contradiction.

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