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An Ordering for Linear Differential Systems and a Characterization of Exponential Separation in Terms of Reducibility

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1. INTRODUCTION

Let $A(t)$, $B(t)$ be (real or complex) $n \times n$ matrix functions, bounded and continuous on $[0, \infty)$. The systems of linear differential equations

$$\dot{x} = A(t)x \tag{1}$$

and

$$\dot{y} = B(t)y \tag{2}$$

are said to be *kinematically similar* if there exists a continuously differentiable invertible matrix function $S(t)$ (called a *kinematic similarity*) such that $S(t)$ and $S^{-1}(t)$ are bounded and such that the transformation $x = S(t)y$ takes the solutions of (1) onto the solutions of (2). Since $A(t)$ and $B(t)$ are assumed to be bounded, $\dot{S}(t)$ will also be bounded for every kinematic similarity $S(t)$.

(1) is said to be *reducible* (cf. Coppel [3, p. 38]) if it is kinematically similar to a system (2) whose coefficient matrix has the diagonal block form

$$\begin{bmatrix} B_1(t) & 0 \\ 0 & B_2(t) \end{bmatrix},$$

$B_1(t)$ and $B_2(t)$ being matrices of lower order than $B(t)$. In Lemma 2 in [3, p. 40] Coppel shows that (1) is reducible if and only if there is a projection $P \neq 0, I$ such that $X(t)PX^{-1}(t)$ is bounded. Here $X(t)$ is the fundamental matrix for (1) with $X(0) = I$. In this case, we also say that (1) is reducible with respect to the decomposition $V_1 \oplus V_2$ of n-dimensional Euclidean space E^n, where V_1 is the range of P and V_2 its kernel. By induction using Coppel's result, it is easy to show (cf. Daleckiĭ and Kreĭn [4]) that (1) is kinematically

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similar to a system (2) whose coefficient matrix has the block diagonal form $\operatorname{diag}(B_1(t),..., B_k(t))$, where $B_i(t)$ has order n_i ($n_i \geqslant 1$, $\sum_{i=1}^k n_i = n$), if and only if there exist supplementary projections $P_1,..., P_k$ of respective ranks $n_1,..., n_k$ such that $X(t) P_i X^{-1}(t)$ is bounded for $i = 1,..., k$. In this case we say that (1) is *$(n_1,..., n_k)$-reducible* with respect to the decomposition $V_1 \oplus \cdots \oplus V_k$ of E^n, where V_i is the range of P_i. If a system is not reducible, we say it is *irreducible.*

The ordered pair V_1, V_2 of subspaces of E^n is said to be *exponentially separated* (Bylov *et al.* [2], Palmer [8], Bronshtein and Chernii [1]) with respect to the system (1) if $\dim V_i \geqslant 1$, $V_1 \cap V_2 = \{0\}$ and there exist constants $K \geqslant 1$, $\alpha > 0$ such that for $s \leqslant t$

$$\frac{|x_1(t)|\,|x_2(s)|}{|x_1(s)|\,|x_2(t)|} \leqslant K e^{-\alpha(t-s)},$$

whenever $x_i(t)$ is a solution of (1) with $x_i(0) \neq 0$ in V_i. (Throughout this paper $|\cdot|$ denotes the Euclidean norm when the argument is a vector and the corresponding operator norm when the argument is a matrix). If $V_1 \oplus V_2 = E^n$ we say simply that (1) is *exponentially separated.* If $k \geqslant 2$ and $n_1,..., n_k$ are positive integers such that $\sum_{i=1}^k n_i = n$, system (1) is said to be *$(n_1,..., n_k)$-exponentially separated* if E^n can be decomposed as a direct sum $V_1 \oplus \cdots \oplus V_k$ with $\dim V_i = n_i$ such that V_i, V_{i+1} are exponentially separated with respect to (1) for $i = 1,..., k-1$. In this case we also say that $V_1,..., V_k$ are *exponentially separated* with respect to (1). If no V_i can be expressed as the direct sum of proper subspaces which are exponentially separated with respect to (1), then $V_1 \oplus \cdots \oplus V_k$ is said to be a *minimal decomposition* [8] for (1).

The main aim of this paper is to show that *a system* (1) *is exponentially separated if and only if all neighboring systems are reducible.* Our only concern here is the sufficiency of the latter condition since its necessity follows from the roughness of exponential separation [2, 1; 8, Corollary 2] and the reducibility of exponentially separated systems [2; 8, Lemma 1].

In Sections 2 and 3 we derive some preliminary results on bounded solutions and exponential dichotomies of matrix systems of the form

$$\dot{Z} = A(t)Z - ZB(t), \tag{3}$$

where $A(t)$, $B(t)$ are $m \times m$, $n \times n$ matrix functions bounded and continuous on $[0, \infty)$.

To prove our main result we need an ordering (strictly, preordering) between systems of the form (1). This is defined in Section 4. If the letters A and B denote the systems (1) and (2), respectively (B need *not* have the same order as A), then $B \geqslant A$ means that whenever (1) is kinematically similar to a block upper triangular system

$$\dot{y} = \begin{bmatrix} A_1(t) & A_{12}(t) \\ 0 & A_2(t) \end{bmatrix} y, \tag{4}$$

then the matrix system

$$\dot{Z} = A_2(t)\, Z - ZB(t)$$

has a nontrivial bounded solution. It turns out that among other properties this ordering is reflexive and transitive.

Sections 5 and 6 arose from consideration of the question of whether a reducible system (1) is kinematically similar to a block upper triangular system

$$\dot{x} = \begin{bmatrix} A_1(t) & A_{12}(t) & \cdots & A_{1k}(t) \\ 0 & A_2(t) & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & 0 & A_k(t) \end{bmatrix} x, \tag{5}$$

where each A_i is irreducible and

$$A_k \geqslant A_{k-1} \geqslant \cdots \geqslant A_2 \geqslant A_1. \tag{6}$$

It turns out that there are systems for which this is not the case. These include the *prime* systems. System (1) is *prime* means that whenever it is kinematically similar to a block upper triangular system of the form (4), then it is not true that $A_2 \geqslant A_1$. One of two other equivalent definitions is that the matrix system

$$\dot{Z} = A(t)Z - ZA(t) \tag{7}$$

have no nonzero bounded solution with square zero. (Compare this with Coppel's result that system (1) is irreducible if and only if (7) has no bounded idempotent solution apart from $Z = 0, I$.) In Section 5 a nontrivial example of a prime system is given and it is shown that prime systems can be reducible but cannot be exponentially separated. On the other hand an irreducible system need not be prime.

In Section 6 it is shown that any system (1) is kinematically similar to a block upper triangular system (5) where each A_i is prime and (6) holds. In Section 7 we consider the special case of block upper triangular systems of the form (5), where each A_i is irreducible and for $i > j$ the matrix system

$$\dot{Z} = A_i(t)Z - ZA_j(t)$$

has no nontrivial bounded solution, and show that if all neighboring systems are reducible, then system (5) is exponentially separated. An important role here is played by the functional analytic characterization of exponential dichotomy [3, 5].

In Section 8 the general case is treated. It is first shown that if A is prime and $A \geqslant B$ the matrix system

$$\dot{Z} = (A(t) + a(t)I)Z - ZB(t)$$

has no nontrivial bounded solution whenever $a(t)$ is a bounded real continuous function such that $\int_0^t a(s)\,ds$ is bounded below but not above. Using this and the result of Section 6 the general case is reduced to the special case considered in Section 7. A corollary of this our main result is that *if* $V_1 \oplus \cdots \oplus V_k$ *is a minimal decomposition for* (1) *then all neighboring systems are* $(m_1,..., m_l)$*-reducible if and only if there is a partition* $J_1 \cup J_2 \cup \cdots \cup J_l$ of $\{1,..., k\}$ *such that for* $i = 1,..., l$,

$$\sum_{j \in J_i} \dim V_j = m_i.$$

This generalizes theorems of Millionščikov [6] and Palmer [7]. Finally in Section 9 the same problem is discussed for systems on $(-\infty, \infty)$.

2. Bounded Solutions of Matrix Systems

In this and the next section we consider matrix systems of the form (3) where $A(t)$, $B(t)$ are $m \times m$, $n \times n$ matrix functions, bounded and continuous on $[0, \infty)$. The solutions of (3) are $m \times n$ matrix functions $Z(t)$. If $X(t)$ is the fundamental matrix for (1) with $X(0) = I$ and $Y(t)$ is the fundamental matrix for (2) with $Y(0) = I$, then

$$Z(t) = X(t)\, MY^{-1}(t)$$

is the solution of (3) such that $Z(0) = M$. Note that rank $Z(t) =$ rank M for all t so that we may speak of the *rank* of a solution of (3).

If $x = S(t)u$ and $y = T(t)v$ are kinematic similarities taking (1) and (2) into the systems

$$\dot{u} = C(t)u$$

and

$$\dot{v} = D(t)v,$$

respectively, the transformation

$$Z = S(t)\, WT^{-1}(t)$$

takes (3) into

$$\dot{W} = C(t)W - WD(t) \tag{8}$$

and clearly (3) has a nontrivial bounded solution if and only if (8) has; moreover corresponding solutions have the same ranks.

LEMMA 1. *Let $A(t)$, $B(t)$ be $m \times m$, $n \times n$ matrix functions, bounded and continuous on* $[0, \infty)$.

(i) *Suppose the kinematic similarity $S(t)$ takes* (1) *into a block upper triangular system of the form* (4). *System* (4) *has a fundamental matrix of the form*

$$\begin{bmatrix} X_1(t) & X_{12}(t) \\ 0 & X_2(t) \end{bmatrix}.$$

Then if the matrix system

$$\dot{Z} = A_1(t)Z - ZB(t) \tag{9}$$

has a bounded solution $X_1(t)MY^{-1}(t)$,

$$S(t)\begin{bmatrix} X_1(t)MY^{-1}(t) \\ 0 \end{bmatrix} \tag{10}$$

is a bounded solution of (3) *with the same rank. Conversely, if the matrix system* (3) *has a bounded solution of rank k there is a kinematic similarity $S(t)$ taking* (1) *into a system* (4) *with $A_1(t)$ $k \times k$ such that the bounded solution has the form* (10).

(ii) *Suppose the kinematic similarity $T(t)$ takes* (2) *into a block upper triangular system*

$$\dot{y} = \begin{bmatrix} B_1(t) & B_{12}(t) \\ 0 & B_2(t) \end{bmatrix} y. \tag{11}$$

This has a fundamental matrix of the form

$$\begin{bmatrix} Y_1(t) & Y_{12}(t) \\ 0 & Y_2(t) \end{bmatrix}.$$

Then if the matrix system

$$\dot{Z} = A(t)Z - ZB_2(t) \tag{12}$$

has a bounded solution $X(t)\,MY_2^{-1}(t)$,

$$[0 \quad X(t)\,MY_2^{-1}(t)]\,T^{-1}(t) \tag{13}$$

is a bounded solution of (3) *with the same rank. Conversely, if* (3) *has a bounded solution of rank k there is a kinematic similarity* $T(t)$ *taking* (2) *into a system* (11) *with* $B_2(t)$ $k \times k$ *such that the bounded solution has the form* (13).

Proof. The first part of (i) follows from the observation that

$$\begin{bmatrix} X_1(t)\,MY^{-1}(t) \\ 0 \end{bmatrix} = \begin{bmatrix} X_1(t) & X_{12}(t) \\ 0 & X_2(t) \end{bmatrix} \begin{bmatrix} M \\ 0 \end{bmatrix} Y^{-1}(t)$$

is a bounded solution of

$$\dot{Z} = \begin{bmatrix} A_1(t) & A_{12}(t) \\ 0 & A_2(t) \end{bmatrix} Z - ZB(t).$$

For the converse let $X(t)\,MY^{-1}(t)$ be a bounded solution of (3) with rank $M = k$. We may write $M = M_1 M_2$, where M_1 is $m \times k$, M_2 is $k \times n$ and both have rank k. Choose N_1 so that the partitioned $m \times m$ matrix $[M_1 \;\; N_1]$ is nonsingular and apply the Gram–Schmidt orthonormalization process to the columns of $X(t)[M_1\, N_1]$ to get

$$X(t)[M_1 \;\; N_1] = S(t) \begin{bmatrix} X_1(t) & X_{12}(t) \\ 0 & X_2(t) \end{bmatrix},$$

where $S(t)$ is unitary and the matrix on its right is upper triangular with $X_1(t)$ $k \times k$, etc. This means that the kinematic similarity $S(t)$ takes (1) into a system (4), where $A_1(t)$ is $k \times k$ and $X_1(t)$ is a fundamental matrix for the system

$$\dot{x} = A_1(t)x.$$

Finally a simple calculation shows that $X(t)\,MY^{-1}(t)$ equals the expression in (10), thus completing the proof of (i).

To prove the first part of (ii) note that

$$[0 \;\; X(t)\,MY_2^{-1}(t)] = X(t)[0\; M] \begin{bmatrix} Y_1(t) & Y_{12}(t) \\ 0 & Y_2(t) \end{bmatrix}$$

is a bounded solution of

$$\dot{Z} = A(t)Z - Z \begin{bmatrix} B_1(t) & B_{12}(t) \\ 0 & B_2(t) \end{bmatrix}.$$

For the converse: if $X(t)MY^{-1}(t)$ is a bounded solution of rank k of (3) write $M = M_1 M_2$ as in the proof of part (i) and let N_2 be a matrix such that the $n \times n$ matrix $\left[\begin{smallmatrix} N_2 \\ M_2 \end{smallmatrix}\right]$ is nonsingular and apply Gram–Schmidt to get

$$Y(t)\begin{bmatrix} N_2 \\ M_2 \end{bmatrix}^{-1} = S(t)\begin{bmatrix} Y_1(t) & Y_{12}(t) \\ 0 & Y_2(t) \end{bmatrix},$$

where $S(t)$ is unitary and the matrix on its right is upper triangular with $Y_2(t)$ $k \times k$, etc. Now the kinematic similarity $S(t)$ takes (2) into a system (11), where $B_2(t)$ is $k \times k$ and $Y_2(t)$ is a fundamental matrix for the system

$$\dot{y} = B_2(t)\, y.$$

The proof is completed by observing that $X(t)MY^{-1}(t)$ equals the expression in (13).

(Note: (ii) could also be deduced from (i) by looking at the system

$$\dot{Z} = -B^*(t)Z + ZA^*(t),$$

where * denotes adjoint.)

3. Exponential Dichotomies for Matrix Systems

Let $\mathscr{E}$ be a finite-dimensional (real or complex) vector space and $\mathscr{A}(t)$ a function defined on $[0, \infty)$ with values in $\mathrm{Hom}(\mathscr{E}, \mathscr{E})$, the space of linear mappings of $\mathscr{E}$ into itself. We suppose that $\sup |\mathscr{A}(t)| < \infty$, where $|\cdot|$ here denotes the operator norm, and that $\mathscr{A}(t)$ is continuous with respect to the topology this norm defines. Now consider the differential equation

$$\dot{x} = \mathscr{A}(t)x \qquad (x \in \mathscr{E}) \tag{14}$$

and let $\mathscr{L}(t)$ be a *fundamental operator solution* (cf. [4, 5]).

Equation (14) is said to have an exponential dichotomy [3–5] if there is a projection $\mathscr{P}$ of $\mathscr{E}$ into itself and constants $K \geqslant 1$, $\alpha > 0$ such that

$$|\mathscr{L}(t)\mathscr{P}\mathscr{L}^{-1}(s)| \leqslant Ke^{-\alpha(t-s)} \qquad (s \leqslant t),$$
$$|\mathscr{L}(t)(I - \mathscr{P})\mathscr{L}^{-1}(s)| \leqslant Ke^{-\alpha(s-t)} \qquad (s \geqslant t).$$

The range of $\mathscr{P}$ is $\{\zeta \in \mathscr{E} : \mathscr{L}(t)\zeta \to 0 \text{ as } t \to \infty\}$ and is called the *stable subspace*; the kernel of $\mathscr{P}$ is called the *unstable subspace*. If the stable subspace is the whole of $\mathscr{E}$, (14) is said to be *uniformly asymptotically stable*.

Usually $\mathscr{E}$ is E^n and $\mathscr{A}(t)$ is an $n \times n$ matrix function. However, in this section $\mathscr{E}$ is the space of $m \times n$ matrices and the operator-valued function $\mathscr{A}(t)$ is defined by

$$\mathscr{A}(t)(Z) = A(t)Z - ZB(t),$$

where Z is an $m \times n$ matrix and $A(t)$, $B(t)$ are $m \times m$, $n \times n$ matrix functions bounded and continuous on $[0, \infty)$. A fundamental operator solution for the corresponding differential equation (3) has the form

$$\mathscr{L}(t)(Z) = X(t)\, ZY^{-1}(t), \tag{15}$$

where $X(t)$, $Y(t)$ are fundamental matrices for (1), (2), respectively.

The following lemma collects together a few facts concerning systems (3) which are uniformly asymptotically stable.

LEMMA 2. *Let $A(t)$, $B(t)$ be $m \times m$, $n \times n$ matrix functions bounded and continuous on $[0, \infty)$. Then*

(i) (3) *is uniformly asymptotically stable if and only if there exist constants $K \geqslant 1$, $\alpha > 0$ such that for $s \leqslant t$*

$$|X(t)\, X^{-1}(s)|\, |Y(s)\, Y^{-1}(t)| \leqslant Ke^{-\alpha(t-s)}. \tag{16}$$

(ii) *Let $C(t)$ be a $p \times p$ matrix function, bounded and continuous on $[0, \infty)$. Then if* (3) *and the system*

$$\dot{Z} = B(t)Z - ZC(t)$$

are uniformly asymptotically stable, so also is the system

$$\dot{Z} = A(t)Z - ZC(t).$$

(iii) *Let $C(t)$ be an $m \times n$ matrix function bounded and continuous on $[0, \infty)$. Then if* (3) *is uniformly asymptotically stable the block upper triangular system*

$$\dot{x} = \begin{bmatrix} A(t) & C(t) \\ 0 & B(t) \end{bmatrix} x \tag{17}$$

is exponentially separated.

Proof. (i) (3) is uniformly asymptotically stable if there exist constants $K \geqslant 1$, $\alpha > 0$ such that for $s \leqslant t$,

$$|\mathscr{L}(t)\, \mathscr{L}^{-1}(s)| \leqslant Ke^{-\alpha(t-s)},$$

where $\mathscr{L}(t)$ is defined in (15). This means that for all $m \times n$ matrices Z and $s \leqslant t$,

$$|X(t)\,X^{-1}(s)\,ZY(s)\,Y^{-1}(t)| \leqslant Ke^{-\alpha(t-s)}\,|Z|. \tag{18}$$

(18) is clearly implied by (16) and if (18) holds so also does (16), but with a new K which depends only on the old K and m and n (since if A and B are $m \times m$, $n \times n$ matrices such that

$$|AZB| \leqslant K\,|Z|$$

for all $m \times n$ matrices Z there exists K_1 depending only on m and n such that

$$|A|\,|B| \leqslant K_1 K).$$

(ii) If $W(t)$ is a fundamental matrix for the system

$$\dot{x} = C(t)x,$$

then the assertion follows from part (i) and the inequality,

$$\begin{aligned}&|X(t)\,X^{-1}(s)|\,|W(s)\,W^{-1}(t)|\\ &\quad\leqslant |X(t)\,X^{-1}(s)|\,|Y(s)\,Y^{-1}(t)|\,|Y(t)\,Y^{-1}(s)|\,|W(s)\,W^{-1}(t)|.\end{aligned}$$

(iii) If (3) is uniformly asymptotically stable, then (16) holds for $s \leqslant t$. It follows from Theorem 1 in [8] that the block diagonal system

$$\dot{x} = \begin{bmatrix} A(t) & 0 \\ 0 & B(t) \end{bmatrix} x$$

is (m, n)-exponentially separated with projection $P_1 = \operatorname{diag}(I_m, 0)$, I_m being the $m \times m$ identity matrix. Then it follows from Corollary 4 in [8] and the remark after it that the block upper triangular system (17) is also (m, n)-exponentially separated.

Next we consider systems (3) where $A(t)$ and $B(t)$ are both block upper triangular and show that whether or not (3) has an exponential dichotomy depends only on the diagonal blocks in $A(t)$ and $B(t)$.

LEMMA 3. *Let $A(t)$ and $B(t)$ be square matrix functions (not necessarily of the same order), bounded and continuous on $[0, \infty)$. If system (1) is block upper triangular of the form (5) and (2) is block upper triangular of the form*

$$\dot{y} = \begin{bmatrix} B_1(t) & B_{12}(t) & \cdots & B_{1l}(t) \\ 0 & & \ddots & \vdots \\ \vdots & & & \\ 0 & \cdots & 0 & \cdots\ B_l(t) \end{bmatrix} y,$$

the matrix system (3) *has an exponential dichotomy if and only if the system*

$$\dot{Z} = A_i(t)Z - ZB_j(t) \tag{19}$$

has one for all i and j. Moreover the dimension of the stable subspace for (3) *is the sum of the dimensions of the stable subspaces for* (19).

Proof. Partitioning Z as $[Z_{ij}]$, where Z_{ij} has the same number of rows as $A_i(t)$ and the same number of columns as $B_j(t)$, system (3) becomes

$$\dot{Z}_{ij} = A_i(t)\, Z_{ij} - Z_{ij} B_j(t) + \sum_{m>i} A_{im}(t)\, Z_{mj} - \sum_{m<j} Z_{im} B_{mj}(t),$$
$$(i = 1,..., k;\ j = 1,..., l).$$

If we order the Z_{ij} as

$$Z_{1l},..., Z_{11}, Z_{2l},..., Z_{21},..., Z_{kl},..., Z_{k1}$$

the system is block upper triangular with

$$\dot{Z}_{ij} = A_i(t)\, Z_{ij} - Z_{ij} B_j(t) \qquad (i = 1,..., k;\ j = 1,..., l)$$

as the corresponding block diagonal system. Now the lemma follows from Corollary 2 in [8] and the remark after it (cf. also Sacker and Sell [10]).

Finally we prove a corollary of Lemma 3.

LEMMA 4. *Let* $A(t)$, $B(t)$ *be square matrix functions bounded and continuous on* $[0, \infty)$. *Then if* (3) *has an exponential dichotomy so also has the system*

$$\dot{Z} = B(t)Z - ZA(t) \tag{20}$$

and the stable subspace for (3) *and the unstable subspace for* (20) *have the same dimension.*

Proof. Since both systems (1) and (2) are kinematically similar to systems with upper triangular coefficient matrices (cf. [3, p. 87]), we may assume without loss of generality that $A(t) = [a_{ij}(t)]$ and $B(t) = [b_{ij}(t)]$ are both upper triangular. Then the assertion follows from Lemma 3 and the facts that if the scalar equation

$$\dot{x} = (a_{ii}(t) - b_{jj}(t))x$$

has an exponential dichotomy so also has

$$\dot{x} = (b_{jj}(t) - a_{ii}(t))x,$$

and the former has nontrivial bounded solutions if and only if the latter has none.

4. An Ordering for Linear Differential Systems

In this section we define an ordering for systems of the form (1) and prove some properties of it. Throughout $A(t)$, $B(t)$, $C(t)$ denote square matrix functions (not necessarily of the same order), bounded and continuous on $[0, \infty)$. For the sake of brevity we shall often just use the letters A, B, etc., to denote the systems (1), (2), etc. Also the notation

$$A \sim (A_1, A_2, \ldots, A_k)$$

means that the system (1) is kinematically similar to a block upper triangular system (5).

If the matrix system (3) has no nontrivial bounded solution we write $A > B$. Otherwise we write $A \not> B$. In the first proposition we present some elementary properties of this relation.

Proposition 1. *Suppose $A \sim (A_1, A_2)$. Then*

(i) *if $A_1 > B$ and $A_2 > B$, $A > B$;*

(ii) *if $B > A_1$ and $B > A_2$, $B > A$.*

Proof. (i) If we partition $Z = \mathrm{col}(Z_1, Z_2)$ the matrix system

$$\dot{Z} = \begin{bmatrix} A_1(t) & A_{12}(t) \\ 0 & A_2(t) \end{bmatrix} Z - ZB(t) \tag{21}$$

becomes

$$\dot{Z}_1 = A_1(t) Z_1 - Z_1 B(t) + A_{12}(t) Z_2,$$
$$\dot{Z}_2 = A_2(t) Z_2 - Z_2 B(t).$$

It follows that (21) has no nontrivial bounded solution and hence that (3) also has none.

(ii) If we partition $Z = [Z_1 Z_2]$ the system

$$\dot{Z} = B(t) Z - Z \begin{bmatrix} A_1(t) & A_{12}(t) \\ 0 & A_2(t) \end{bmatrix}$$

becomes

$$\dot{Z}_1 = B(t) Z_1 - Z_1 A_1(t),$$
$$\dot{Z}_2 = B(t) Z_2 - Z_2 A_2(t) - Z_1 A_{12}(t).$$

As in (i) the result follows at once.

Remark. The converses to (i) and (ii) are valid in case $A(t) = \operatorname{diag}(A_1(t), A_2(t))$.

DEFINITION. We say that

$$A \geqslant B$$

if whenever $B \sim (B_1, B_2)$ (including the case $B \sim B_2$), $B_2 \not> A$.

EXAMPLE. Suppose the matrix system

$$\dot{Z} = B(t)Z - ZA(t) \tag{22}$$

has a nontrivial bounded solution of rank equal to the order of $B(t)$ but none of lower rank. Then $A \geqslant B$. For suppose $B \sim (B_1, B_2)$. Now $B \not> A$ so that by Proposition 1 either $B_1 \not> A$ or $B_2 \not> A$. But if $B_1 \not> A$ it would follow from Lemma 1 that (22) had a nontrivial bounded solution of rank less than the order of $B(t)$. So we must have $B_2 \not> A$ so that $A \geqslant B$, as asserted.

The advantage of the notion $\geqslant$ defined above is that it is *transitive* (note that $>$ is not transitive, in general).

PROPOSITION 2. (i) *If $A > B$ and $B \geqslant C$, then $A > C$;*

(ii) *if $A \geqslant B$ and $B \geqslant C$, then $A \geqslant C$.*

Proof. (i) Suppose $A \not> C$. Then by Lemma 1, $C \sim (C_1, C_2)$ such that the system

$$\dot{Z} = A(t)Z - ZC_2(t)$$

has a bounded solution $X(t) MW_2^{-1}(t)$ with the rank of M equal to the order of $C_2(t)$. Here $X(t)$ is a fundamental matrix for (1) and $W_2(t)$ one for the system

$$\dot{x} = C_2(t)x.$$

Now since $B \geqslant C$, $C_2 \not> B$ so that system

$$\dot{Z} = C_2(t)Z - ZB(t)$$

has a bounded solution $W_2(t) NY^{-1}(t)$ with $N \neq 0$. Here $Y(t)$ is a fundamental matrix for (2).

But then

$$X(t) MNY^{-1}(t) = X(t) MW_2^{-1}(t) \cdot W_2(t) NY^{-1}(t)$$

is a bounded solution of (3) with $MN \neq 0$ since $N \neq 0$ and M has full column rank. This contradicts $A > B$. Hence $A > C$.

(ii) Let $C \sim (C_1, C_2)$. Suppose $C_2 > A$. Then since $A \geqslant B$ it follows by (i) that $C_2 > B$. This contradicts $B \geqslant C$. So $C_2 \not> A$ and hence $A \geqslant C$.

We now prove a proposition which does for $\geqslant$ what Proposition 1 did for $>$.

PROPOSITION 3. *Suppose $A \sim (A_1, A_2)$ (including the case $A \sim A_2$). Then*

(i) $A \geqslant A_2$;

(ii) *if $A_2 \geqslant B$, then $A \geqslant B$;*

(iii) *if $B \geqslant A_1$ and $B \geqslant A_2$, then $B \geqslant A$.*

Proof. (i) Suppose $A_2 \sim (A_1', A_2')$. Then

$$A \sim (A_1, A_1', A_2')$$

since if $S(t)$ is the kinematic similarity taking system A_2 into one with block diagonal (A_1', A_2'), diag$(I, S(t))$, where I is an identity matrix with order equal to that of $A_1(t)$, is a kinematic similarity taking the system A into one with block diagonal (A_1, A_1', A_2'). Now the system

$$\dot{Z} = A_2'(t) Z - Z A_2'(t)$$

has the constant solution $Z =$ the identity matrix. It follows by Lemma 1 that the system

$$\dot{Z} = A_2'(t) Z - Z A(t)$$

has a bounded solution of the same rank. That is, $A_2' \not> A$ and so $A \geqslant A_2$.

(ii) Since $A \geqslant A_2$ and $A_2 \geqslant B$, it follows by Proposition 2 that $A \geqslant B$.

(iii) Let $A \sim (A_1', A_2')$ and suppose $A_2' > B$. Then since $B \geqslant A_1$ and $B \geqslant A_2$ it follows by Proposition 2 that $A_2' > A_1$ and $A_2' > A_2$. But then Proposition 1 implies that $A_2' > A$, which contradicts $A \geqslant A_2'$(part (i) of this proposition). So $A_2' \not> B$ and hence $B \geqslant A$.

The following lemma is a tool we shall need in the next section.

LEMMA 5. *Let A, B be two systems. Then B is kinematically similar to an upper triangular system with block diagonal (B_1, B_2) (possibly $B = B_1$ or $B = B_2$) such that $B_2 > A$ and $A \geqslant B_1$.*

Proof. The proof is by induction on the order n of B. If B is scalar, then either $B > A$ or $B \not> A$. By the example before Proposition 2 the latter implies $A \geqslant B$. So the lemma does hold in the scalar case.

Now we assume the lemma holds for $1, 2, \ldots, n-1$ and prove it for n. So let $B(t)$ be $n \times n$. If $A \geqslant B$, we are finished. Otherwise $B \sim (B_1, B_2)$ with

$B_2 > A$. Applying the induction hypothesis to B_1, we have $B_1 \sim (B_1', B_2')$ with $B_2' > A$, $A \geqslant B_1'$. Reasoning as at the beginning of the proof of Proposition 3, it follows that $B \sim (B_1', B_2'')$, where B_2'' is a block upper triangular system with block diagonal (B_2', B_2). It follows from Proposition 1 that $B_2'' > A$ and we already know that $A \geqslant B_1'$. So the induction proof is complete.

5. Prime Systems

In this section we use the same notations as in the previous section. We begin with a lemma about matrices.

Lemma 6. *Let M be an $n \times n$ matrix. Then*

$$M^2 = 0$$

if and only if M is similar to a matrix of the form

$$\begin{bmatrix} 0 & N \\ 0 & 0 \end{bmatrix}, \tag{23}$$

where N is $k \times (n-k)$ with $1 \leqslant k < n$.

Proof. The sufficiency is clear. To prove the necessity, we assume that $M^2 = 0$. Let k be the rank of M. For E^n choose a basis such that the first k vectors span the range of M. Then M is similar to its representation with respect to this basis. This has the form

$$M' = \begin{bmatrix} Q & N \\ 0 & 0 \end{bmatrix},$$

where N is a $k \times (n-k)$ matrix. Since $M'^2 = 0$, M' annihilates its own range. Therefore $Q = 0$.

Proposition 4. *Let A be any system* (1). *Then the following three statements are equivalent*:

(i) *the matrix system*

$$\dot{Z} = A(t)Z - ZA(t) \tag{24}$$

has a nontrivial bounded solution with square zero;

(ii) $A \sim (A_1, A_2)$ *such that* $A_1 \not> A_2$;

(iii) $A \sim (A_1, A_2)$ *such that* $A_2 \geqslant A_1$.

Proof. First we show that (ii) implies (i). Suppose the kinematic similarity $S(t)$ takes system A into one with block diagonal (A_1, A_2) with $A_1 \not\succ A_2$. So the matrix system

$$\dot{Z} = A_1(t)Z - ZA_2(t) \tag{25}$$

has a nontrivial bounded solution $Z(t)$. It follows from part (i) of Lemma 1 that the system

$$\dot{Z} = A(t)Z - ZA_2(t) \tag{26}$$

has the nontrivial bouded solution

$$S(t)\begin{bmatrix} Z(t) \\ 0 \end{bmatrix}$$

and then from part (ii) of the same lemma that (24) has the bounded solution

$$S(t)\begin{bmatrix} 0 & Z(t) \\ 0 & 0 \end{bmatrix} S^{-1}(t),$$

which has square zero. Thus (ii) implies (i).

Now we show that (i) implies (ii). Let $X(t)$ be a fundamental matrix for (1) and suppose $X(t)MX^{-1}(t)$ is a nontrivial solution of (24) with square zero. Then $M^2 = 0$ and so by Lemma 6 there exists a nonsingular matrix L such that $L^{-1}ML$ has the form (23), where N is $k \times (n-k)$ with k the rank of M. Using the Gram–Schmidt process we may write

$$X(t)L = U(t)\begin{bmatrix} X_1(t) & X_{12}(t) \\ 0 & X_2(t) \end{bmatrix},$$

where $U(t)$ is unitary and the matrix on its right is upper triangular with $X_1(t)$ $k \times k$, etc. The transformation $U(t)$ takes (1) into a system (4), where for $i = 1, 2$, $X_i(t)$ is a fundamental matrix of

$$\dot{x} = A_i(t)x. \tag{27}$$

That is, $A \sim (A_1, A_2)$ and a simple calculation shows

$$\begin{bmatrix} 0 & X_1(t)NX_2^{-1}(t) \\ 0 & 0 \end{bmatrix} = U^{-1}(t)X(t)MX^{-1}(t)U(t)$$

so that (25) has the nontrivial bounded solution $X_1(t)NX_2^{-1}(t)$. This means that $A_1 \not\succ A_2$, Thus (i) implies (ii).

That (iii) implies (ii) is trivial. To prove that (ii) implies (iii) suppose that $A \sim (A_1, A_2)$ with $A_1 \not> A_2$. By Lemma 5, $A_1 \sim (A_1', A_2')$ with $A_2' > A_2$, $A_2 \geqslant A_1'$. A_2' has a lower order than A_1 since $A_1 \not> A_2$. Then reasoning as at the beginning of the proof of Proposition 3, $A \sim (A_1', A_2'')$ where A_2'' is a block upper triangular system with block diagonal (A_1', A_2). Since $A_2 \geqslant A_1'$ it follows by Proposition 3 that $A_2'' \geqslant A_1'$. Thus (ii) implies (iii).

DEFINITION. System A is said to be *prime* if whenever $A \sim (A_1, A_2)$, $A_2 \not\geqslant A_1$. By Proposition 4 this is equivalent to the statement that whenever $A \sim (A_1, A_2)$, $A_1 > A_2$ and also to the condition that the matrix system (24) has no nontrivial bounded solution with square zero.

EXAMPLE. Trivially any scalar system is prime. A nontrivial example of a prime system is

$$\dot{x} = A(t)x = \begin{bmatrix} a(t) & b(t) \\ 0 & 0 \end{bmatrix} x,$$

where $a(t)$, $b(t)$ are bounded continuous real functions satisfying

(i) $\int_0^t a(s)\, ds$ is unbounded above and below,

(ii) the scalar inhomogeneous system

$$\dot{x} = a(t)x + b(t) \tag{28}$$

has no bounded solution. (If $a(t)$ satisfies (i) the homogeneous equation

$$\dot{x} = a(t)x \tag{29}$$

cannot have an exponential dichotomy and so by Proposition 3 in [3, p. 22] it is always possible to find a bounded continuous real $b(t)$ such that (28) does not have a bounded solution.)

We show that (24), with $A(t)$ as above, has no rank 1 bounded solution. Let $Z(t)$ be a nontrivial bounded solution of (24) and partition it as $[z_1(t)\, z_2(t)]$. Then (24) becomes

$$\dot{z}_1 = (A(t) - a(t))\, z_1, \tag{30}$$

$$\dot{z}_2 = A(t)\, z_2 - b(t)\, z_1. \tag{31}$$

Since clearly the system

$$\dot{z}_2 = A(t)\, z_2$$

has no nontrivial bounded solution, $z_1(t)$ cannot be zero. But the only bounded solutions of (30) are constants $\operatorname{col}(\alpha, 0)$ so that $z_1(t) = \operatorname{col}(\alpha, 0)$

with $\alpha \neq 0$. Now suppose $Z(t)$ has rank 1. Then there is a scalar function $\lambda(t)$ such that $z_2(t) = -\lambda(t)\, z_1(t) = \mathrm{col}(-\lambda(t)\alpha, 0)$ for all t. It follows that $\lambda(t)$ is continuously differentiable and bounded and, substituting in (31), we find it is a solution of (28). This is a contradiction. Hence (24) has no rank 1 bounded solution. So all nontrivial bounded solutions of (24) must be nonsingular and hence cannot have square zero. Thus the system is prime.

Remarks (a). *An irreducible system need not be prime.* For example, let $a(t)$ be a bounded continuous real function such that $\int_0^t a(s)\, ds$ is bounded above but (29) does not have an exponential dichotomy. Then the system

$$\dot{x} = \begin{bmatrix} a(t) & 1 \\ 0 & 0 \end{bmatrix} x$$

is not prime but it is irreducible by Theorem 3 in [7].

Remark (b). *A prime system need not be irreducible.* For consider a system (1), where $A(t) = \mathrm{diag}(A_1(t), A_2(t))$. Suppose each system (27) is prime and that $A_1 > A_2$, $A_2 > A_1$. Then A is prime for if $X_i(t)$ is a fundamental matrix for (27) any bounded solution of the matrix system (24) has the form

$$\begin{bmatrix} X_1(t) & 0 \\ 0 & X_2(t) \end{bmatrix} \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} X_1^{-1}(t) & 0 \\ 0 & X_2^{-1}(t) \end{bmatrix}$$
$$= \begin{bmatrix} X_1(t)\, M_{11} X_1^{-1}(t) & X_1(t)\, M_{12} X_2^{-1}(t) \\ X_2(t)\, M_{21} X_1^{-1}(t) & X_2(t)\, M_{22} X_2^{-1}(t) \end{bmatrix}.$$

Since $A_1 > A_2$, $A_2 > A_1$ the off-diagonal blocks are zero. So if the solution has square zero, $M_{11}^2 = 0$ and $M_{22}^2 = 0$. But since both systems (27) are prime, this means M_{11} and M_{22} are both zero. So the solution is zero and A is prime, as asserted.

Conversely if A is prime, it follows from one of the equivalent definitions that $A_1 > A_2$, $A_2 > A_1$, and reversing the above argument that each system (27) is prime.

Remark (c). *A prime system cannot be exponentially separated.* For suppose the system (1) is exponentially separated. Then by Lemma 1 in [8] it is kinematically similar to a block diagonal system

$$\dot{y}_1 = B_1(t)\, y_1,$$
$$\dot{y}_2 = B_2(t)\, y_2,$$

with respect to which the subspaces $y_2 = 0$ and $y_1 = 0$ are exponentially separated. It follows from Theorem 1 in [8] and its proof that there exist constants $K \geqslant 1$, $\alpha > 0$ such that for $s \leqslant t$,

$$|Y_1(t)\,Y_1^{-1}(s)|\,|Y_2(s)\,Y_2^{-1}(t)| \leqslant Ke^{-\alpha(t-s)},$$

where $Y_i(t)$ is a fundamental matrix for the system

$$\dot{y}_i = B_i(t)\,y_i.$$

By Lemma 2 this implies that the matrix system

$$\dot{Z} = B_1(t)Z - ZB_2(t)$$

is uniformly asymptotically stable so that all its solutions are bounded. Hence $B_1 \not\geqslant B_2$ and so A cannot be prime.

6. A Block Upper Triangular Form with Ordered Prime Diagonal Blocks

Our aim in this section is to establish the following theorem, which states that any system (1) has a block upper triangular form in which the diagonal blocks are prime and are linearly ordered with respect to $\geqslant$. The notations are the same as those used in Sections 4 and 5.

THEOREM 1. *Let $A(t)$ be an $n \times n$ matrix function, bounded and continuous on $[0, \infty)$. Then system (1) is kinematically similar to a block upper triangular system (5) where each system A_i is prime and*

$$A_k \geqslant A_{k-1} \geqslant \cdots \geqslant A_2 \geqslant A_1.$$

Proof. We first assert that either A is prime or $A \sim (A_1, A_2)$ with A_2 prime and $A_2 \geqslant A_1$. The proof is by induction on the order n of A. If A is scalar it is prime and we are finished. Suppose the assertion holds for $1, 2, \ldots, n-1$. We prove it for n. So let $A(t)$ be $n \times n$. If A is prime, we are finished. Otherwise $A \sim (A_1, A_2)$ with $A_2 \geqslant A_1$. Applying the induction hypothesis to A_2, $A_2 \sim (A_1', A_2')$ with A_2' prime and $A_2' \geqslant A_1'$. Then reasoning as at the beginning of the proof of Proposition 3, $A \sim (A_1'', A_2')$ where A_1'' is a block upper triangular system with block diagonal (A_1, A_1').

Since $A_2' \geqslant A_1'$ and $A_2' \geqslant A_2'$ by Proposition 3(i), it follows by Proposition 3(iii) that $A_2' \geqslant A_2$. But $A_2 \geqslant A_1$ and so by Proposition 2(ii), $A_2' \geqslant A_1$. Then Proposition 3(iii) again implies that $A_2' \geqslant A_1''$, since $A_2' \geqslant A_1$ and $A_2' \geqslant A_1'$. This completes the induction proof so that the assertion is established.

We now prove the theorem by induction again on the order n of A. If A is scalar it is prime and we are finished. Suppose the theorem holds for $1, 2, \ldots, n-1$. We prove it for n. So let $A(t)$ be $n \times n$. Then by the first part either A is prime and we are finished or $A \sim (A_1, A_2)$ with A_2 prime and $A_2 \geqslant A_1$. We can apply the induction hypothesis to A_1 to get

$$A_1 \sim (A_1', A_2', \ldots, A_l')$$

with the A_i' prime and

$$A_l' \geqslant A_{l-1}' \geqslant \cdots \geqslant A_2' \geqslant A_1'.$$

Then as at the beginning of the proof of Proposition 3,

$$A \sim (A_1', A_2', \ldots, A_l', A_2)$$

and all that remains to complete the induction proof is to show that $A_2 \geqslant A_l'$. But this follows from $A_2 \geqslant A_1$, Proposition 3(i) and 2(ii).

7. Determination of Those Systems All Sufficiently Small Perturbations of Which Are Reducible: A Special Case

It is known [2; 8, Corollary 2] that if system (1) is exponentially separated sufficiently small perturbations of it are also exponentially separated and hence reducible by Lemma 1 in [8]. Our aim in the next two sections is to show that the converse of this statement is true. In this section we treat a special case.

Proposition 5. *Let the coefficient matrix of the block upper triangular system*

$$\dot{x} = A(t)x = \begin{bmatrix} A_1(t) & A_{12}(t) & \cdots & A_{1k}(t) \\ 0 & \ddots & & \vdots \\ \vdots & & & \\ 0 & \cdots & 0 & A_k(t) \end{bmatrix} x \tag{32}$$

be bounded and continuous on $[0, \infty)$, *where for* $i = 1, \ldots, k$ *the system* (27) *is irreducible and for* $i > j$ *the matrix system*

$$\dot{Z} = A_i(t)Z - ZA_j(t) \tag{33}$$

has no nontrivial bounded solution. Then if there exists $\delta > 0$ *such that the perturbed system*

$$\dot{x} = (A(t) + B(t))x \tag{34}$$

is reducible for all continuous matrix functions $B(t)$ *of the form*

$$\begin{bmatrix} 0 & B_{12}(t) & \cdots & B_{1k}(t) \\ 0 & & \ddots & \vdots \\ \vdots & & & B_{k-1,k}(t) \\ 0 & & \cdots & 0 \end{bmatrix}$$

satisfying $\sup_{t\geqslant 0}|B(t)|<\delta$, *there exists* l $(1\leqslant l<k)$ *such that the matrix system* (33) *is uniformly asymptotically stable for* $1\leqslant i\leqslant l$, $l+1\leqslant j\leqslant k$.

Proof. The proof is by induction on the number k of diagonal blocks. First, we consider the case $k=2$. Let $X(t)$ be a fundamental matrix for system (34) of the form

$$\begin{bmatrix} X_1(t) & X_{12}(t) \\ 0 & X_2(t) \end{bmatrix}.$$

Since (34) is reducible there is a projection

$$P=\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$$

such that $X(t)PX^{-1}(t)$ is bounded. Now the block in the left-hand bottom corner of $X(t)PX^{-1}(t)$ is $X_2(t)P_{21}X_1^{-1}(t)$, which is a bounded solution of (33) for $i=2$, $j=1$. Hence $P_{21}=0$ so that P_{11}, P_{22} are both projections and the diagonal blocks in $X(t)PX^{-1}(t)$ are $X_1(t)P_{11}X_1^{-1}(t)$ and $X_2(t)P_{22}X_2^{-1}(t)$. Since both systems (27) are irreducible, each P_{ii} is either 0 or the identity. Since P is not zero or the identity, either P_{11} is the identity and P_{22} is 0 or vice versa. But because $X(t)(I-P)X^{-1}(t)$ is also bounded, we may assume without loss of generality that P_{11} is the identity and P_{22} is 0.

Then it turns out that the top right-hand block in $X(t)PX^{-1}(t)$ is

$$Z(t)=-X_{12}(t)X_2^{-1}(t)+X_1(t)P_{12}X_2^{-1}(t)$$

and one verifies by differentiation that this is a solution of the matrix system

$$\dot{Z}=A_1(t)Z-ZA_2(t)-(A_{12}(t)+B(t)). \tag{35}$$

Thus (35) has a bounded solution for all $B(t)$ with $\sup_{t\geqslant 0}|B(t)|<\delta$. By subtracting the bounded solution of (35) for $B(t)=0$ and using homogeneity, it follows that the system

$$\dot{Z}=A_1(t)Z-ZA_2(t)+B(t)$$

has a bounded solution for all bounded continuous $k \times (n-k)$ matrix functions $B(t)$. Hence by Proposition 3 in [3, p. 22] system (25) has an exponential dichotomy. By Lemma 4 we know that the system

$$\dot{Z} = A_2(t)Z - ZA_1(t)$$

also has an exponential dichotomy and since it has no nontrivial bounded solution the stable subspace must be $\{0\}$. Then by Lemma 4 again, (25) must be uniformly asymptotically stable.

This establishes the proposition for $k = 2$. Assuming it holds for $k-1$ we prove it for k. So let us assume small perturbations of system (32) of the given type are reducible.

Suppose, first, that for some m the matrix system

$$\dot{Z} = A_m(t)Z - ZA_{m+1}(t) \tag{36}$$

is not uniformly asymptotically stable. Then by the $k = 2$ case there exists a continuous matrix function $B_{m,m+1}(t)$ with arbitrarily small (supremum) norm such that the perturbed system

$$\dot{x} = \begin{bmatrix} A_m(t) & A_{m,m+1}(t) + B_{m,m+1}(t) \\ 0 & A_{m+1}(t) \end{bmatrix} x \tag{37}$$

is irreducible. Now consider the block upper triangular system got from (32) by combining the blocks $A_m(t)$, $A_{m+1}(t)$ into the coefficient matrix of (37). Using Proposition 1 we see that this system is in the required form and small perturbations of it of the given type are reducible provided $\sup_{t\geqslant 0} |B_{m,m+1}(t)|$ is small enough. But it has only $k-1$ diagonal blocks so that by the induction hypothesis and using Lemma 3, there exists l $(1 \leqslant l \leqslant k,\ l \neq m)$ such that (33) is uniformly asymptotically stable for $1 \leqslant i \leqslant l$, $l+1 \leqslant j \leqslant k$.

So we are left with the case where (36) is uniformly asymptotically stable for $m = 1,..., k-1$. But then it follows by Lemma 2 that the conclusion of the proposition holds for all l, $1 \leqslant l < k$. So the induction proof is complete.

Remark. If the conclusion of the proposition holds it follows by Lemma 3 that the matrix system

$$\dot{Z} = \mathscr{A}(t)Z - Z\mathscr{B}(t)$$

is uniformly asymptotically stable, where

$$\mathscr{A}(t) = \begin{bmatrix} A_1(t) & A_{12}(t) & \cdots \\ 0 & \ddots & \\ \vdots & & \\ 0 & 0 & A_l(t) \end{bmatrix}, \qquad \mathscr{B}(t) = \begin{bmatrix} A_{l+1}(t) & \cdots \\ 0 & \ddots \\ \vdots & \\ 0 \cdots 0 & A_k(t) \end{bmatrix}.$$

But then it follows by Lemma 2 that (32) is exponentially separated.

8. The General Case

Our aim in this section is to show that all sufficiently small perturbations of a system (1) are reducible if and only if it is exponentially separated. The method is to begin with the block upper triangular form of Theorem 1 and modify it so that Proposition 5 can be applied.

Lemma 7. *Suppose $A(t)$, $B(t)$ are $n\times n$ matrix functions bounded and continuous on $[0,\infty)$ such that the matrix system*

$$\dot{Z}=B(t)Z-ZA(t) \tag{38}$$

has a nonsingular bounded solution but no nontrivial bounded solution of lower rank. Then if $a(t)$ is a bounded continuous real function such that $\int_0^t a(s)\,ds$ is bounded below but not above, the matrix system

$$\dot{Z}=(A(t)+a(t)I)Z-ZB(t) \tag{39}$$

has no nontrivial bounded solution.

Proof. First, we show that a nontrivial bounded solution of the matrix system (3) must be nonsingular. Let $X(t)$, $Y(t)$ be fundamental matrices for systems (1) and (2), respectively, and let $Y(t)MX^{-1}(t)$ be the given nonsingular bounded solution of (38). Then if $X(t)NY^{-1}(t)$ is a bounded solution of (3), $Y(t)\,MNMX^{-1}(t)=Y(t)\,MX^{-1}(t)\cdot X(t)\,NY^{-1}(t)\cdot Y(t)MX^{-1}(t)$ is a bounded solution of (38). So either $MNM=0$ or is nonsingular. This means that $N=0$ or N is nonsingular, as required.

Now write $\phi(t)=e^{\int_0^t a(s)ds}$. Let

$$Z(t)=\phi(t)\,X(t)\,NY^{-1}(t)$$

be a bounded solution of the system (39). Then

$$X(t)\,NY^{-1}(t)=\phi^{-1}(t)\,Z(t)$$

is a bounded solution of (3) so that $N=0$ or is nonsingular. If N is nonsingular then

$$\phi^n(t)[\det Y(t)]^{-1}\det X(t)=[\det N]^{-1}\det Z(t)$$

is bounded. But

$$[\det X(t)]^{-1}\det Y(t)=[\det M]^{-1}\det Y(t)\,MX^{-1}(t)$$

is also bounded. Multiplying we see that $\phi^n(t)$ is bounded too. This is a contradiction and so N must be 0.

PROPOSITION 6. *Let $A(t)$, $B(t)$ be $m \times m$, $n \times n$ matrix functions bounded and continuous on $[0, \infty)$ such that system (1) is prime and $A \geqslant B$. Then if $a(t)$ is a bounded continuous real function such that $\int_0^t a(s)\,ds$ is bounded below but not above, the matrix system (39) has no nontrivial bounded solution.*

Proof. First suppose the matrix system (38) has a nontrivial bounded solution of rank equal to the order of $B(t)$ but none of lower rank. Then by Lemma 1 system (1) is kinematically similar to a block upper triangular system (4) such that the system

$$\dot{Z} = B(t)Z - ZA_2(t)$$

has a nonsingular bounded solution but no nontrivial bounded solution of lower rank. By the example before Proposition 2, $A_2 \geqslant B$. But since A is prime $A_1 > A_2$ and so $A_1 > B$ using Proposition 2. It is still true that the system

$$\dot{Z} = (A_1(t) + a(t)I)Z - ZB(t)$$

has no nontrivial bounded solution. Also by Lemma 7 the system,

$$\dot{Z} = (A_2(t) + a(t)I)Z - ZB(t)$$

has the same property. Then by Proposition 1 so has the system (39) too.

Now we prove the proposition by induction on the order n of $B(t)$. If $B(t)$ is scalar it follows by the previous argument, since $A \geqslant B$ implies that (38) has a nontrivial bounded solution. Now we assume the proposition true for $1, \ldots, n-1$ and prove it for n. So let $B(t)$ be $n \times n$. Since $A \geqslant B$ the system (38) has a nontrivial bounded solution. If it has one of rank n but none of lower rank the previous argument applies and we are done. Otherwise take a nontrivial bounded solution of lowest rank. Then by Lemma 1 system (2) is kinematically similar to a block upper triangular system (11) such that the system

$$\dot{Z} = B_1(t)Z - ZA(t)$$

has a nontrivial bounded solution of rank equal to the order of $B_1(t)$ but none of lower rank. It follows by the example before Proposition 2 that $A \geqslant B_1$. But $A \geqslant B_2$ also by Proposition 2(ii), since $A \geqslant B$ and $B \geqslant B_2$ by Proposition 3(i). Then by the induction hypothesis both the systems,

$$\dot{Z} = (A(t) + a(t)I)Z - ZB_1(t)$$

and

$$\dot{Z} = (A(t) + a(t)I)Z - ZB_2(t),$$

have no nontrivial bounded solution. By Proposition 1(ii) this means that the system (39) has none too and so the induction proof is complete.

Remark. This proposition is not true without the assumption of primeness. For example, take $B(t)$ as the scalar function 0 and $A(t)$ as the 2×2 diagonal matrix with elements $-a(t)$ and $a(t)$, where $a(t)$ is a function as in the proposition.

THEOREM 2. *Let $A(t)$ be an $n \times n$ matrix function bounded and continuous on $[0, \infty)$. Then there exists $\delta > 0$ such that the perturbed system*

$$\dot{x} = (A(t) + B(t))x$$

is reducible for all continuous matrix functions $B(t)$ with $\sup_{t \geq 0} |B(t)| < \delta$ if and only if the system (1) *is exponentially separated.*

Proof. The sufficiency follows from the roughness and reducibility of exponentially separated systems.

We begin the proof of the necessity with a remark about bounded continuous real functions $a(t)$ with *zero spectrum*; this means that the corresponding scalar equation (29) has $\{0\}$ as its Sacker–Sell *spectrum* (cf. Sacker and Sell [9], Palmer [8]) so that for all $\varepsilon > 0$ there exists $T(\varepsilon)$ such that

$$\left| (t-s)^{-1} \int_s^t a(u)\, du \right| < \varepsilon \qquad \text{when} \quad t - s > T(\varepsilon).$$

It is clear that the sum of two functions with zero spectrum also has this property.

Now consider a system (14) as in Section 3. If $a(t)$ is a function with zero spectrum the perturbed system

$$\dot{x} = (\mathscr{A}(t) + a(t)I)x \tag{40}$$

has fundamental operator solution $e^{\int_0^t a(s)\,ds}\mathscr{L}(t)$. It is clear that (40) has an exponential dichotomy if and only if (14) has; moreover both systems have the same stable subspace. In particular this remark can be applied to a matrix system (3) to show that it has an exponential dichotomy if and only if the system

$$\begin{aligned} \dot{Z} &= (A(t) + a(t)I)Z - Z(B(t) + b(t)I) \\ &= A(t)Z - ZB(t) + (a(t) - b(t))Z \end{aligned}$$

has, whenever $a(t)$ and $b(t)$ are functions with zero spectrum. Moreover, the systems have the same stable subspace.

Now to prove the necessity, since both properties are preserved by kinematic similarity, we may assume by Theorem 1 that the system has the block upper triangular form (5) where for all i system (27) is prime and

$$A_k \geqslant A_{k-1} \geqslant \cdots \geqslant A_2 \geqslant A_1.$$

Furthermore we may assume each $A_i(t)$ is a block diagonal matrix each block $B_j(t)$ being such that the system

$$\dot{x} = B_j(t)x$$

is irreducible. As in Remark (b) in Section 5, $B_j > B_l$ if $j \neq l$ and $B_j(t)$, $B_l(t)$ are blocks in the same $A_i(t)$.

Now let $a(t)$ be a bounded continuous real function with zero spectrum such that $\int_0^t a(s)\,ds$ is bounded below but not above. Then we replace $A_i(t)$ by

$$A_i'(t) = A_i(t) + (i-1)\,a(t)I.$$

Since the ordering $\geqslant$ is not affected by adding the same scalar function to the diagonals of both matrix functions, the system with coefficient matrix $A_j(t) + (i-1)\,a(t)I$ is $\geqslant$ the system with coefficient matrix $A_i(t) + (i-1)\,a(t)I$ when $j > i$. But then since both these systems are still prime (clearly, primeness also is not affected by the addition of a scalar function to the diagonal), it follows by Proposition 6 that the matrix system

$$\dot{Z} = A_j'(t)Z - ZA_i'(t)$$

has no nontrivial bounded solution when $j > i$. Consequently if $B_m'(t)$ is one of the diagonal blocks in $A_j'(t)$ and $B_l'(t)$ one in $A_i'(t)$ the matrix system

$$\dot{Z} = B_m'(t)Z - ZB_l'(t)$$

also has no nontrivial bounded solution. From the remarks in the previous paragraph this is also true if $B_m'(t)$ and $B_l'(t)$ belong to the same $A_i(t)$.

The modified system with diagonal blocks $B_i'(t)$ now has the properties of the system considered in Proposition 5 in Section 7, the only difference from the unmodified system being that scalar functions with zero spectrum have been added to the $B_i(t)$. Multiplying these scalar functions by a suitably small positive number we can ensure that the modified system still has the property that small perturbations of it are reducible. Then it follows by Proposition 5 that there is a positive integer l such that the matrix system

$$\dot{Z} = B_i'(t)Z - ZB_j'(t) \tag{41}$$

is uniformly asymptotically stable if $i \leqslant l$, $j \geqslant l+1$. But now using the remarks at the beginning of the proof about scalar functions with zero spectrum (41) with the unmodified $B_i(t)$ is still uniformly asymptotically stable. As in the remark after Proposition 5, it follows that the original system is exponentially separated. So the theorem is proved.

In the following two corollaries we consider a system (1), where $A(t)$ is an $n \times n$ matrix function bounded and continuous on $[0, \infty)$. Let

$$V_1 \oplus \cdots \oplus V_k$$

be a minimal decomposition (cf. the Introduction) of E^n with respect to (1) with $\dim V_i = n_i$. By Lemma 1 in [8] and its proof, (1) is kinematically similar to a block diagonal system

$$\dot{x}_i = A_i(t)\, x_i \qquad (i = 1,..., k) \tag{42}$$

such that $A_i(t)$ is $n_i \times n_i$ and (42) is exponentially separated with the ith subspace in the corresponding splitting of E^n being those x of the form $\mathrm{col}(0,..., 0, x_i, 0,..., 0)$. It follows by Theorem 1 in [8] and Lemma 2(i) that the matrix system

$$\dot{Z} = A_i(t) Z - Z A_{i+1}(t) \tag{43}$$

is uniformly asymptotically stable for $i = 1,..., k-1$. Also since $V_1 \oplus \cdots \oplus V_k$ is a minimal decomposition no subsystem in (42) is exponentially separated.

We say that the subspace V_i is *irreducible* with respect to the system (1) if the ith subsystem in (42) is irreducible. This is equivalent to the geometrical condition that V_i cannot be expressed as the direct sum of two proper subspaces such that there is a number $\gamma > 0$ with the property that if $x(t)$ is a solution beginning in one subspace and $y(t)$ a solution beginning in the other the angle between $x(t)$ and $y(t)$ is bounded below by γ (cf. [4, 5]).

COROLLARY 1. *Let $A(t)$ be a bounded continuous $n \times n$ matrix function defined on $[0, \infty)$ such that $V_1 \oplus \cdots \oplus V_k$ is a minimal decompositon of E^n with respect to system* (1). *Then given $\delta > 0$ there is a continuous matrix function $B(t)$ with $\sup_{t \geqslant 0} |B(t)| < \delta$ such that $V_1 \oplus \cdots \oplus V_k$ is also a minimal decomposition of E^n with respect to the perturbed system*

$$\dot{x} = (A(t) + B(t))x \tag{44}$$

and such that each V_i is irreducible with respect to (44).

Proof. Without loss of generality we can restrict attention to a block diagonal system (42) such that no subsystem is exponentially separated and for $i = 1,..., k-1$ the matrix system (43) is uniformly asymptotically stable. This system has a minimal decomposition $V_1 \oplus \cdots \oplus V_k$, where V_i consists of all vectors of the form $\mathrm{col}(0,..., 0, x_i, 0,..., 0)$.

By Theorem 2 there is, for each i, a continuous matrix function $B_i(t)$ such that $\sup_{t\geqslant 0} |B_i(t)| < \delta$ and the system

$$\dot{x}_i = (A_i(t) + B_i(t))\, x_i \tag{45}$$

is irreducible. If $\sup_{t\geqslant 0} |B_i(t)|$ is sufficiently small it follows from the roughness property of exponential dichotomy that the matrix system

$$\dot{Z} = (A_i(t) + B_i(t))Z - Z(A_{i+1}(t) + B_{i+1}(t))$$

is uniformly asymptotically stable for $i = 1,..., k-1$. By Lemma 2 this means that $V_1 \oplus \cdots \oplus V_k$ is exponentially separated with respect to the system

$$\dot{x}_i = (A_i(t) + B_i(t))\, x_i \qquad (i = 1,..., k). \tag{46}$$

Also no V_i can be exponentially separated with respect to (46) since (45) is irreducible. Hence $V_1 \oplus \cdots \oplus V_k$ is a minimal decomposition with respect to (46) such that each V_i is irreducible.

Systems, such as in Corollary 1, which have a minimal decomposition in which each subspace is irreducible, have an interesting property. We state this as a lemma.

LEMMA 8. *Let $A(t)$ be a bounded continuous $n \times n$ matrix function on $[0, \infty)$ such that the system* (1) *has a minimal decomposition $V_1 \oplus \cdots \oplus V_k$ in which each subspace V_i is irreducible. Then if $V_1' \oplus \cdots \oplus V_k'$ is any other minimal decomposition each V_i' is irreducible also. Moreover if* (1) *is reducible with respect to a decomposition $W_1 \oplus W_2$ of E^n, there is a minimal decomposition $V_1' \oplus \cdots \oplus V_k'$ and a subset J of $\{1,..., k\}$such that*

$$W_1 = \bigoplus_{i\in J} V_i', \qquad W_2 = \bigoplus_{i\notin J} V_i'.$$

Proof. We demonstrate the second assertion first. Without loss of generality, we may assume that the system has the form (42) such that each subsystem is irreducible and the matrix system (43) is uniformly asymptotically stable for $i = 1,..., k-1$. Let $X(t) = \operatorname{diag}(X_1(t),..., X_k(t))$ be the fundamental matrix such that $X(0) = I$. Then if P is the projection with range W_1 and kernel W_2, $X(t)\, PX^{-1}(t)$ must be bounded. If we partition P as

$$\begin{bmatrix} P_{11} & \cdots & P_{1k} \\ \vdots & \ddots & \vdots \\ P_{k1} & \cdots & P_{kk} \end{bmatrix},$$

the (i, j)th block in $X(t)PX^{-1}(t)$ turns out to be $X_i(t)P_{ij}X_j^{-1}(t)$, which is a bounded solution of the matrix system (33). Since if $i > j$ this matrix system has by Lemmas 2(ii) and 4 an exponential dichotomy with stable subspace $\{0\}$ we must have $P_{ij} = 0$ if $i > j$. (This means that $V_1 \oplus \cdots \oplus V_i$ is an invariant subspace of P for all i—this is true even when some V_i is reducible.) Then each P_{ii} is a projection such that $X_i(t)P_{ii}X_i^{-1}(t)$ is bounded. Since each subsystem in (42) is irreducible it follows that for each i, P_{ii} is zero or the identity.

It is easy then to prove by induction on k that there is an invertible matrix L of the form

$$\begin{bmatrix} I_{n_1} & L_{12} & \cdots & L_{1k} \\ 0 & \ddots & & \vdots \\ \vdots & & & \\ 0 & \cdots & 0 & I_{n_k} \end{bmatrix},$$

where I_{n_i} is the $n_i \times n_i$ identity matrix, such that

$$L^{-1}PL = \operatorname{diag}(P_{11}, P_{22}, \dots, P_{kk}).$$

Define $J = \{i = 1, \dots, k : P_{ii} \neq 0\}$. Then we see that the range of P is $\bigoplus_{i \in J} V_i'$ and the kernel $\bigoplus_{i \notin J} V_i'$, where V_i' is the image under L of the subspace V_i of vectors of the form $(0, \dots, 0, x_i, 0, \dots, 0)$. But because of L's special form

$$V_1' \oplus \cdots \oplus V_i' = V_1 \oplus \cdots \oplus V_i$$

for all i so that by Theorem 3 in [8], $V_1' \oplus \cdots \oplus V_k'$ is also a minimal decomposition for (42). This completes the proof of the second assertion.

Now let $V_1' \oplus \cdots \oplus V_k'$ be a minimal decomposition such that V_j' is reducible. If we block diagonalize (1) with respect to this decomposition we get a system

$$\dot{x}_i = A_i'(t)\, x_i \qquad (i = 1, \dots, k), \tag{47}$$

in which the jth equation is reducible. Let Q be the corresponding projection. Then the whole system (47) is reducible with corresponding projection $\operatorname{diag}(0, \dots, 0, Q, 0, \dots, 0)$. That is, the original system (1) is reducible and the range W of the corresponding projection is a proper subspace of V_j'. It follows from the first part that there is another minimal decomposition $V_1'' \oplus \cdots \oplus V_k''$ and a subset J of $\{1, \dots, k\}$ such that $W = \bigoplus_{i \in J} V_i''$. But

$$W \cap (V_1'' \oplus \cdots \oplus V_{j-1}'') = W \cap (V_1' \oplus \cdots \oplus V_{j-1}') = \{0\}$$

and

$$W \subset V_1' \oplus \cdots \oplus V_j' = V_1'' \oplus \cdots \oplus V_j''.$$

The only possibility is that $W = V_j''$. This is a contradiction since $\dim W < \dim V_j' = \dim V_j''$.

Remark. Suppose the conditions of the lemma hold and that (1) is reducible with respect to a decomposition $W_1 \oplus \cdots \oplus W_l$ of E^n. Then we can show in a similar way that there is a minimal decomposition $V_1' \oplus \cdots \oplus V_k'$ and a partition $\{1,..., k\} = J_1 \cup J_2 \cup \cdots \cup J_l$ such that

$$W_i = \bigoplus_{j \in J_i} V_j'.$$

COROLLARY 2. *Let $A(t)$ and $V_1 \oplus \cdots \oplus V_k$ be as in Corollary 1 and let $m_1,..., m_l$ be positive integers such that $\sum_{i=1}^{l} m_i = n$. Then there exists $\delta > 0$ such that system (44) is $(m_1,..., m_l)$-reducible for any continuous matrix function $B(t)$ with $\sup_{t \geq 0} |B(t)| < \delta$ if and only if there is a partition $J_1 \cup J_2 \cup \cdots \cup J_l$ of $\{1,..., k\}$ such that for $i = 1,..., l$,*

$$\sum_{j \in J_i} \dim V_j = m_i. \tag{48}$$

Proof. If δ is sufficiently small it follows from Corollary 2 in [8] that (44) is $(n_1, n_2,..., n_k)$-exponentially separated, where $n_i = \dim V_i$. It follows by Lemma 1 in [8] that it is also $(n_1, n_2,..., n_k)$-reducible and hence $(m_1,..., m_l)$-reducible if the conditions (48) hold. This proves the sufficiency.

To prove the necessity perturb the system as in Corollary 1. If δ is small enough, the perturbed system (44) is $(m_1,..., m_l)$-reducible. Let $W_1 \oplus \cdots \oplus W_l$ be the corresponding decomposition of E^n so that $\dim W_i = m_i$. It follows from Lemma 8 and the remark after it that there is a minimal decomposition $V_1' \oplus \cdots \oplus V_k'$ and a partition $J_1 \cup \cdots \cup J_l$ of $\{1,..., k\}$ such that for $i = 1,..., l$,

$$W_i = \bigoplus_{j \in J_i} V_j'.$$

Hence $m_i = \dim W_i = \sum_{j \in J_i} \dim V_j' = \sum_{j \in J_i} \dim V_j$.

Remark. The case $l = n$, $m_i = 1$ for all i was already treated in Millionščikov [6] and Palmer [8].

9. Systems on the Whole Line

Let $A(t)$ be an $n \times n$ matrix function, bounded and continuous on $(-\infty, \infty)$. If the system (1) is exponentially separated (this is defined for systems on $(-\infty, \infty)$ as for systems on $[0, \infty)$, cf. [8]), then it follows from the roughness of exponential separation and the reducibility of exponentially separated systems that small perturbations of (1) are reducible.

On the other hand if small perturbations of (1) are reducible it follows, by restricting to the half-lines, that (1) is exponentially separated on both $[0, \infty)$ and $(-\infty, 0]$. However in general, this is not a sufficient condition. For let $A(t)$ be a 3×3 matrix function such that system (1) has minimal decompositions $V_1^+ \oplus V_2^+$ and $V_1^- \oplus V_2^-$ on $[0, \infty)$ and $(-\infty, 0]$, respectively, where $\dim V_1^+ = \dim V_2^- = 2$ but $V_1^+ \neq V_2^-$. By perturbing the system as in Corollary 1 (we do this separately on $[0, \infty)$ and $(-\infty, 0]$; clearly we can choose both perturbations to be zero at $t = 0$ so that the resulting perturbation is continuous on $(-\infty, \infty)$), we can ensure that V_1^+ and V_2^- are irreducible. Suppose now that the perturbed system is reducible on $(-\infty, \infty)$ with corresponding splitting $W_1 \oplus W_2$ of E^3. Assuming without loss of generality that $\dim W_1 = 2$, it follows from Lemma 8 that $W_1 = V_1^+$ and from its analogue for systems on $(-\infty, 0]$ that $W_1 = V_2^-$. This is impossible since $V_1^+ \neq V_2^-$.

Nevertheless even if a system is not exponentially separated on $(-\infty, \infty)$ it can still happen that small perturbations of it are reducible. For let (1) be a system which is *integrally separated* (cf. [2, 8]) on $[0, \infty)$ but just exponentially separated on $(-\infty, 0]$. Any perturbed system will have the same property. By Lemma 3 in [7] we can choose a basis $l_1, l_2, \ldots, l_n$ for E^n such that the one-dimensional subspaces V_i generated by these are exponentially separated on $[0, \infty)$ and so that there is a proper subset J of $\{1, \ldots, n\}$ such that the vectors l_i, $i \in J$, generate the last subspace W in the minimal decomposition of the perturbed system on $(-\infty, 0]$. Take V as the subspace generated by the vectors l_i, $i \notin J$. It follows by Corollary 1 in [8] that the decomposition $V \oplus W$ is exponentially separated on $(-\infty, 0]$ and hence certainly reducible. On the other hand it is also reducible on $[0, \infty)$ because the system is diagonalizable on $[0, \infty)$ with respect to the decomposition $V_1 \oplus \cdots \oplus V_n$.

Summing up, it seems that a system on $(-\infty, \infty)$ has the property that small perturbations of it are reducible if and only if it is exponentially separated on both half-lines and there is some relationship between the two minimal decompositions. However, the author has not been able to discover what this relationship is, in general.

This seems to be a good place to fill in a gap in the proof of Corollary 5 in [8]. First, we lose no generality in assuming that $V_1 \oplus \cdots \oplus V_k$ is a minimal decomposition which is then unique by Theorem 3(iii) in [8]. So the projections P_i are uniquely determined. Next, if $A(t + t_l) \to \tilde{A}(t)$ uniformly with respect to t as $l \to \infty$ and if $X(t_l) P_i X^{-1}(t_l) \to \tilde{P}_i$ for $i = 1, \ldots, k$, it follows using Theorem 1 in [8] and an argument similar to that used in the proof of Lemma 1 in [3, p. 70] that the system

$$\dot{x} = \tilde{A}(t)x$$

will be $(n_1, \ldots, n_k)$-exponentially separated with respect to the decomposition $W_1 \oplus \cdots \oplus W_k$, where W_i is the range of $\tilde{P}_i$. If this decomposition were not minimal we could use the fact that $\tilde{A}(t - t_l) \to A(t)$ uniformly to deduce that (1) were exponentially separated with respect to a decomposition with more than k subspaces, contradicting the minimality of $V_1 \oplus \cdots \oplus V_k$. Now the assertion made in the proof of Corollary 5 that the functions $X(t+H) P_i X^{-1}(t)$ are almost periodic will follow using arguments similar to those used in the proofs of Proposition 4 in [3, p. 72] and Theorem 2 in reference [5] in [8].

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References

1. I. U. Bronshtein and V. F. Chernii, Linear extensions satisfying Perron's condition, I, *Differential Equations* **14** (1978), 1234–1243.
2. B. F. Bylov, R. E. Vinograd, D. M. Grobman, and V. V. Nemyckii, "Theory of Lyapunov Exponents and Its Application to Problems of Stability," Nauka, Moscow, 1966. [Russian]
3. W. A. Coppel, "Dichotomies in Stability Theory," Lecture Notes in Mathematics, No. 629, Springer-Verlag, Berlin, 1978.
4. Ju. L. Daleckiĭ and M. G. Kreĭn, "Stability of Solutions of Differential Equations in Banach Space," Amer. Math. Soc. Translations, Providence, R. I., 1974.
5. J. L. Massera and J. J. Schäffer, "Linear Differential Equations and Function Spaces," Academic Press, New York, 1966.
6. V. M. Millionščikov, Coarse properties of linear systems of differential equations, *Differential Equations* **5** (1969), 1314–1321.
7. K. J. Palmer, Exponential dichotomy, integral separation, and diagonalizability of linear systems of ordinary differential equations, *J. Differential Equations* **43** (1982), 184–203.
8. K. J. Palmer, Exponential separation, exponential dichotomy, and spectral theory for linear systems of ordinary differential equations, *J. Differential Equations* **46** (1982), 324–345.
9. R. J. Sacker and G. R. Sell, A spectral theory for linear differential systems, *J. Differential Equations* **27** (1978), 320–358.
10. R. J. Sacker and G. R. Sell, The spectrum of an invariant submanifold, *J. Differential Equations* **38** (1980), 135–160.