Topology, Vol. 9, pp. 149—151. Pergamon Press, 1970. Printed in Great Britain

COUNTING TOPOLOGICAL MANIFOLDS†

J. CHEEGER and J. M. KISTER

(*Received* 25 *June* 1969; *revised* 6 *October* 1969)

WE CONSIDER the class $\mathscr{C}$ of all compact topological manifolds, boundaries permitted. It is known that there are only a countable number of homotopy types in $\mathscr{C}$, [1] and [2]. The subclass $\mathscr{C}_{PL}$ of piecewise linear manifolds has only a countable number of topologically distinct elements, since each could be regarded as a finite simplicial complex and a simple argument shows there are only a countable number of those, up to isomorphism. Recently, however, Kirby and Siebenmann [3] have discovered some examples of topological manifolds admitting no *PL* structure, so that route for counting homeomorphism types in $\mathscr{C}$ is rather unpromising (whether manifolds can be triangulated without a *PL* structure is still open).

We take a direct approach and with the aid of a very useful result of Edwards and Kirby [4] examine overlapping coordinate neighborhoods to show that $\mathscr{C}$ has only countably many elements up to homeomorphism. Our argument is similar to one in a differential setting in [3].

Siebenmann has informed us that he and Kirby obtained the same result earlier for closed manifolds having dimension at least 6, and for bounded manifolds of dimension at least 7, as a result of their topological handlebody theory [6]. This approach will not give the general result, however, since they also show that there is a closed manifold of dimension 4 or 5 that has no handle decomposition. Our proof is much more elementary and it is purely geometric. An interesting by-product is a topological submersion theorem and a proof of a key lemma in [5]. See the remark after the proof.

THEOREM. *There are precisely a countable number of compact topological manifolds (boundary permitted), up to homeomorphism.*

Proof. We first consider those with empty boundaries. Suppose not, then there are an uncountable number of some dimension, say n. Let $\{M_\alpha{}^n\}_{\alpha\in A}$ be such a collection. For each $M_\alpha{}^n$ find a collection of imbeddings $h_{\alpha j}\colon B(2) \to M_\alpha{}^n$, $j = 1, 2,\ldots, k_\alpha$, such that $\{h_{\alpha j}(B(1))\}_{j=1}^{k\alpha}$ covers $M_\alpha{}^n$, where $B(r)$ is the closed ball of radius r and center 0 in R^n. By possibly choosing an uncountable subcollection, we can assume without loss of generality that $k_\alpha = k$ for all α. We can also assume, by reparametrizing, that $h_{\alpha j}\,|\,B(1)$ can be extended to an imbedding of $B(k+1)$ into M_α, $j = 1, \ldots, k$. We shall also regard each $M_\alpha{}^n$ as a subset of R^l, e.g. let $l = 2n+1$. If d is the metric in R^l, define $\varepsilon_{\alpha jm} = d(h_{\alpha j}(B(m)), M_\alpha - h_{\alpha j}(B(m+1)))$

†This research was partially supported by NSF Grant GP-8484.

and let $\varepsilon_\alpha = \min_{j,m} \{\varepsilon_{\alpha jm}\}$. By choosing a subcollection again we can assume there exists an $\varepsilon > 0$ such that $\varepsilon_\alpha > \varepsilon$ for all α in A.† Each M_α^n determines an imbedding $g_\alpha : B(k+1) \to R^{kl} = R^l \times R^l \times \cdots \times R^l$ by $g_\alpha(x) = (h_{\alpha 1}(x), \ldots, h_{\alpha k}(x))$. The set of all such imbeddings under the uniform metric: $d_u(g_\alpha, g_\beta) = \max_{x B(k+1)} d(g_\alpha(x), g_\beta(x))$, is separable metric, hence some g_{α_0} is a limit point of a sequence of distinct imbeddings $g_{\alpha_1}, g_{\alpha_2}, \ldots$. We will show that M_{α_0} is homeomorphic to M_{α_i}, for i sufficiently large. Furthermore, this homeomorphism can be taken to be arbitrarily close to the identity as measured by the metric d, which we use in all that follows.

Let $V_j(m) = h_{\alpha_0 j}(B(m))$, $j = 1, 2, \ldots, k$, $m = 1, 2, \ldots, k+1$. To simplify notation we denote M_{α_i} by M', for fixed but arbitrarily large i, and we denote $h_{\alpha_i j}(B(m)) \subset M'$ by $V_j'(m)$, $j = 1, 2, \ldots, k$, $m = 1, 2, \ldots, k+1$. Now let

$$U_j(m) = \bigcup_{p=1}^{j} V_p(m) \text{ and } U_j'(m) = \bigcup_{p=1}^{j} V_p'(m).$$

Note that $U_k(1) = M$ and $U_k'(1) = M'$. Define $f_j : V_j(k+1) \to V_j'(k+1)$ as $h_{\alpha_i j} \circ h_{\alpha_0 j}^{-1}$ $j = 1, \ldots, k$ and note that each f_j can be taken as close to the identity as we like for M' sufficiently far out in the sequence $M_{\alpha_1}, M_{\alpha_2}, \ldots$. We now proceed to construct a small homeomorphism from M to M' inductively on the sets $U_j(m)$ as follows.

Suppose we can construct an imbedding $g_j : U_j(m) \to M'$ as close as we like to the identity by choosing M' sufficiently far out in the sequence. We will show that we can construct an imbedding $g_{j+1} : U_{j+1}(m-1) \to M'$ as close to the identity as we please. Hence, starting off by letting $g_1 = f_1$ and $m = k$, in $k - 1$ steps we will have an imbedding $g_k : U_k(1) \to M'$, the desired homeomorphism.

First we see that $g_j(U_j(m) \cap V_{j+1}(m)) \subset V_{j+1}'(m+1)$ if M' is chosen sufficiently far out and g_j is close to the identity (relative to our previous ε). Then $f_{j+1}^{-1} g_j$ is defined on $U_j(m) \cap V_{j+1}(m)$ and close to the identity. Letting N be an open set in M with $U_j(m-1) \cap V_{j+1}(m-1) \subset N \subset U_j(m) \cap V_{j+1}(m)$ we can extend $f_{j+1}^{-1} g_j | N : N \to V_{j+1}(m)$ to an onto homeomorphism $h : V_{j+1}(m) \to V_{j+1}(m)$ close to the identity, using the theorem of [4].

Now define $g_{j+1} : U_{j+1}(m-1) \to M'$ by

$$g_{j+1}(x) = \begin{cases} g_j(x) \text{ for } x \text{ in } U_j(m-1) \\ f_{j+1} h(x) \text{ for } x \text{ in } V_{j+1}(m-1). \end{cases}$$

By the definition of h, g_{j+1} is well-defined. Since g_{j+1} can be extended to N, as well, using either half of the definition, it is easily seen that g_{j+1} is a local homeomorphism, since it is an imbedding on the two open sets $U_j(m-1) \cup (N \cap U_{j+1}(m-1))$ and $V_{j+1}(m-1) \cup (N \cap U_{j+1}(m-1))$. It would fail to be an imbedding only if $g_{j+1}(x) = g_{j+1}(y)$ for some x and y in $U_j(m-1) - N$ and $V_{j+1}(m-1) - N$ respectively, two compact disjoint sets with

†We are indebted to Mr. A. Shilepsky for pointing out that this condition is necessary to obtain an inclusion we need later.

a positive distance between them. Since these two sets are independent of the choice of M, and the g_j we start off with, we choose M' sufficiently far out in the sequence and g_j, and hence g_{j+1}, close enough to the identity so that g_{j+1} is 1–1. This completes the induction and the proof of the non-bounded case.

To obtain the theorem in case the boundary is non-empty we vary the argument by assuming (without loss of generality, using the preceding case) that each M_α^n is an n-manifold with boundary homeomorphic to a fixed $(n-1)$-manifold B^{n-1}. Then each has a collar given by an imbedding $h_{\alpha 1} : B^{n-1} \times [0, k+1] \to M_\alpha^n$ with $h_{\alpha 1}(b, 0) = b$. We will also assume $h_{\alpha j} : B(k+1) \to M_\alpha^n\, j = 2, \ldots, k$ are imbeddings such that $\{h_{\alpha 1}(B^{n-1} \times [0, 1]), h_{\alpha 2}(B(1)), \ldots, h_{\alpha k}(B(1))\}$ covers M_α^n. As before, each M_α^n is a subset of R^l, and we can define a map $g_\alpha : B^{n-1} \times [0, k+1] \times B(k+1) \to R^{kl}$ by $g_\alpha(x, t, y) = (h_{\alpha 1}(x, t), h_{\alpha 2}(y), \ldots, h_{\alpha k}(y))$. Find g_{α_0} a limit point of $g_{\alpha_1}, g_{\alpha_2}, \ldots$. Let $V_1(m) = h_{\alpha_0 1}(B^{n-1} \times [0, m])$ and define $V_j(m)$, $j = 2, \ldots, k$, and $U_j(m), j = 1, \ldots, k, m = 1, 2, \ldots, k+1$, as before. The proof now proceeds in exactly the same way to produce a small homeomorphism from M_{α_0} to M_{α_i} for i sufficiently large.

Remark. Let $f : W \to Y$ be a proper monotone map satisfying the following condition. For every x in W there are closed neighborhoods U of $f(x)$ in Y and V of x in W and a homeomorphism $h : B(2) \times U \to V$ such that $f \circ h$ is the projection map onto U. Letting $M_y = f^{-1}(y)$, it is easy to see M_y is a compact topological manifold, and if we fix y_0 we can find a collection of imbeddings $\{h_{yj} : B(2) \to M_y |_{y \text{ in } u}^{j=1,2,\ldots K}\}$, where U is a closed neighborhood of y_0 in Y, and $\{h_{yj}(B(1))\}_{j=1}^k$ covers M_y. Furthermore, for fixed j, the h_{yj} vary continuously in y as imbeddings of $B(2)$ in W (here we assume a metric on W and use the uniform topology).

Our method of constructing a homeomorphism $g_y : M_{y_0} \to M_y$, for y near y_0, say y in a small neighborhood U' of y_0, is canonical and continuous in y, since the result we used from [3] was also. Then $g : M_{y_0} \times U' \to W$, defined by $g(x, y) = g_y(x)$ is a local trivialization of f.

This shows that if W is a compact connected topological $(n+1)$-manifold, $Y = [0, 1]$, and f is a topological Morse function without critical points, then f is a trivial bundle map with fiber a compact n-manifold, a useful result in the topological Morse Theory in [5].

REFERENCES

1. M. Mather: Counting homotopy types of manifolds, *Topology* **4** (1965), 93–94.
2. J. Kister: Homotopy type of ANR's, *Proc. Am. math. Soc.* **19** (1968), 195.
3. R. Kirby and L. Siebenmann: to appear.
4. R. Edwards and R. Kirby: Deformations of spaces of imbeddings (abstract), *Notices math. Soc.* **16** (1969), 434.
5. J. Cheeger: Finiteness Theorems for Riemannian manifolds, *Am. J. Math.* to appear.
6. L. Siebenmann: *Notices Am. math. Soc.* to appear.

The University of Michigan