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A characterization of convex cones of matrices with constant regular inertia

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Abstract

Let $\mathscr{A}$ be a convex cone of $n \times n$ matrices. In this paper, we present a necessary and sufficient condition for $\mathscr{A}$ to contain matrices with a constant regular inertia, based on a version of the Lyapunov equation. The condition involves only the normalized extreme points of $\mathscr{A}$. This extends a previous paper by the authors, where a robust stability criterion for $\mathscr{A}$ was obtained. © 2000 Published by Elsevier Science Inc. All rights reserved.

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1. Introduction

The verification of Hurwitz stability of a convex set $\mathscr{A}$ of complex $n \times n$ matrices, robust stability in engineering terminology, is known to be of interest, see e.g. [1,5–8,14,16–18]. Specifically, let $\{A_1, \ldots, A_p\}$ be the extreme points of a convex set $\mathscr{A}$. Note that p need not be finite or even countable. It is well known that the stability of the extreme points is not sufficient in general to guarantee robust stability. Some of the results in this area can be found in [5].

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A similar situation occurs when one wishes to check the constancy of inertia, or the absence of singular matrices, throughout this convex set. These verification problems are all known to be NP-hard with respect to the matrix size n, see e.g. [14–17] for more details. The dependence on the number p is of a more subtle nature, see e.g. [4, Theorem 4]. See also [12], where a full rank convex hull of p rectangular matrices, was characterized.

The case $p = 2$, i.e. a pair of matrices A, B, is classical: non-singularity amounts to $A^{-1}B$ having no (real) non-positive eigenvalues; while robust stability amounts to $L_A(H)^{-1}L_B(H)$ having the same property [5]. Here, $L_A(H)$ is the Lyapunov operator on the space of Hermitian matrices, defined by $L_A(H) := HA + A^*H$. The case of constant inertia can be studied using similar ideas, although the complete result there is somewhat more involved [6].

The case $p > 2$ is much more complicated, and for obvious reasons cannot be treated in a similar fashion. The natural approach used in the engineering robust stability literature is to cover the given set $\mathscr{A}$ by a finite system of neighborhoods $\mathscr{A}_j$, such that each $\mathscr{A}_j$ has a common Lyapunov factor H_j, which in the case of robust stability should be, under our conventions, negative definite, see [18].

We refer to these methods as the *finite coverage of* $\mathscr{A}$ approach, since the whole convex set $\mathscr{A}$ is involved in the process, rather than its extreme points.

An essentially different approach was proposed by the authors in [1]. Extending a result of Johnson [10], the authors have obtained a robust stability verification procedure which uses only $\{A_1, \ldots, A_p\}$, the extreme matrices in $\mathscr{A}$, and which we shall therefore call the *extreme points method*.

In this paper, we extend the extreme points method for the case of constant *regular* inertia, i.e. no imaginary eigenvalues, and describe some of its computational advantages. The extension, given in Section 2, is quite straightforward. Some computational aspects are briefly presented in Section 3, and in Section 4, we discuss various cases in which the verification of fixed inertia can be reduced to the verification of non-singularity. In particular, we show that this is the case for every convex *invertible* cone, i.e. convex cone $\mathscr{C}$ such that whenever $\det(A) \neq 0$ for some $A \in \mathscr{C}$, it implies that $A^{-1} \in \mathscr{C}$ as well, see [2,3,13]. We make several comments on the practical sides of non-singularity verification in such cones.

2. The extreme points approach

Throughout this paper, a matrix implies a complex $n \times n$ matrix and a vector, a complex n-vector. We shall denote by $\mathscr{H}$ the set of Hermitian matrices and by $\mathscr{P}$ its subset of positive definite ones. A^* will denote the complex conjugate transpose of a matrix A. The inertia of A is a triple (ν, δ, π), in which ν, δ, π record the number of eigenvalues with negative, zero, and positive real part, respectively. In particular, the inertia is called *regular* if there is no imaginary eigenvalue, namely if $\delta = 0$, and anti-stable if $\pi = n$.

Our starting point is the following criterion for regular inertia of a single matrix $A \in \mathbb{C}^{n\times n}$.

Theorem 1. *A has regular inertia if and only if for every non-zero vector x there exists a matrix $H = H(x) \in \mathscr{H}$ such that*

$$\mathrm{Re}(x^* HAx) > 0. \tag{1}$$

Proof. Assume that A has regular inertia. According to the Inertia Theorem (see e.g. [9, Theorem 2.4.10]),

$$(HA + A^*H) \in \mathscr{P}, \tag{2}$$

is satisfied by some (non-singular) $H \in \mathscr{H}$. Multiplying on the left and right by x^* and x, respectively, (1) is obtained. Conversely, assume that A does not have regular inertia, namely there exists a non-zero vector x and a real scalar r so that $Ax = \mathrm{i}rx$. Then, for all $H \in \mathscr{H}$ one has that $2\,\mathrm{Re}(x^*HAx) = x^*(HA + A^*H)x = x^*Hx(\mathrm{i}r - \mathrm{i}r) = 0$, violating (1). □

According to the Inertia Theorem, if A and H satisfy (2) then they have the same inertia. If for example A is stable (i.e. has inertia $(n, 0, 0)$) one can use this fact to guarantee a solution H of (2) which is negative definite. Here, (2) reduces to the classical Lyapunov equation. A similar simplification occurs if A is anti-stable (i.e. has positive inertia), in which case $H \in \mathscr{P}$.

Theorem 1 for the stable case, with H in (1) negative definite, appeared originally in [10, Theorem 1], see also [9, Theorem 2.4.11]. Theorem 1 here offers a twofold extension, first the inertia of the matrix A needs to be regular only, and second H may be singular. The latter fact guarantees that for given A and x, the set of all $H \in \mathscr{H}$ satisfying (1), is convex.

In order to discuss the robust version of Theorem 1, valid for general sets of matrices, we find it convenient to introduce the following notation: for each set of matrices $\mathscr{A}$ and each non-zero vector x,

(i) $\mathscr{H}(\mathscr{A})$ will denote the set of all $H \in \mathscr{H}$ for which (2) holds for all $A \in \mathscr{A}$.

(ii) $\mathscr{H}(\mathscr{A}, x)$ will denote the set of all $H \in \mathscr{H}$ for which (1) holds for all $A \in \mathscr{A}$.

By definition,

$$\mathscr{H}(\mathscr{A}) = \bigcap_{x\neq 0} \mathscr{H}(\mathscr{A}, x).$$

Hence, clearly the condition $\mathscr{H}(\mathscr{A}) \neq \emptyset$ implies that $\mathscr{H}(\mathscr{A}, x) \neq \emptyset$ for all $x \neq 0$, but the converse need not be true, see Example 7. From a combination of the Inertia Theorem and Theorem 1, it follows that whenever $\mathscr{A}$ is a singleton, the converse holds as well. (Another case will be given towards the end of the next section.)

In the anti-stable case, due to the simplification $H \in \mathscr{P}$ discussed earlier, instead of $\mathscr{H}(\mathscr{A})$ and $\mathscr{H}(\mathscr{A}, x)$ one may use the smaller sets $\mathscr{P}(\mathscr{A}) = \mathscr{P} \cap \mathscr{H}(\mathscr{A})$ and $\mathscr{P}(\mathscr{A}, x) = \mathscr{P} \cap \mathscr{H}(\mathscr{A}, x)$.

The following robust version of Theorem 1 appeared as in [1, Theorem 2], extending a former result in [10] from $p = 2$ to p arbitrary.

Theorem 2. *Let $\mathscr{E} \subset \mathbb{C}^{n\times n}$ be a compact set of matrices and let $\mathscr{A}$ be the convex hull of $\mathscr{E}$. Then, the following are equivalent:*
(i) *All matrices in $\mathscr{A}$ are anti-stable.*
(ii) *$\mathscr{P}(\mathscr{A}, x) \neq \emptyset$ for every non-zero vector x.*
(iii) *$\mathscr{P}(\mathscr{E}, x) \neq \emptyset$ for every non-zero vector x.*

On the face of it, the problem of characterizing constant regular inertia is much harder than that of preserving stability. This intuition may be supported by comparing between Theorems 4.1 and 4.2 in [6]. Nevertheless, it turns out that Theorem 2 can be easily extended from positive inertias to general regular inertias.

Theorem 3. *Let $\mathscr{E} \subset \mathbb{C}^{n\times n}$ be a compact set of matrices and let $\mathscr{A}$ be the convex hull of $\mathscr{E}$. Then, the following are equivalent:*
(i) *All matrices in $\mathscr{A}$ have the same regular inertia.*
(ii) *$\mathscr{H}(\mathscr{A}, x) \neq \emptyset$ for every non-zero vector x.*
(iii) *$\mathscr{H}(\mathscr{E}, x) \neq \emptyset$ for every non-zero vector x.*

The proof of Theorem 3 is identical to the proof of Theorem 2, which can be found in [1], and will be omitted here. The only change required in this proof is the use of the full power of Theorem 1, for a general inertia, whereas only negative inertia is required in Theorem 2.

In practice, how can Theorem 3 be used algorithmically? Here (1) is adopted as a starting point and one strives to construct a *test set* $H_1, \ldots, H_m \in \mathscr{H}$, and a finite covering $\mathscr{X}_1, \ldots, \mathscr{X}_m$ of the unit sphere in $\mathbb{C}^n$ such that $H_j \in \mathscr{H}(\mathscr{E}, \mathscr{X}_j)$ for all $j = 1, \ldots, m$. In the sequel, we shall refer to the minimal such m as the "complexity of the extreme points method".

A word of caution concerning the use of Theorem 3: even when all matrices in the set $\mathscr{A}$ are real, the vector x in $\mathscr{H}(\mathscr{A}, x)$, should vary over the whole unit sphere in $\mathbb{C}^n$. This and other aspects of the extreme points approach, are illustrated below for a case where $n = 2$ and $p = 3$.

Example 4. Consider the set $\mathscr{E} = \{E_1, E_2, E_3\}$, where

$$E_1 = \begin{pmatrix} 1 & 0 \\ -1 & -2 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 4 & 17 \\ 0 & -2 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 2 & -7 \\ 0 & -1 \end{pmatrix}.$$

Direct calculation shows that $a_1E_1 + a_2E_2 + a_3E_3$ has inertia $(1, 0, 1)$ for all $a_k \geqslant 0$, $\sum_{k=1}^{3} a_k = 1$. In other words, the convex set $\mathscr{A}$ generated by these three matrices has constant regular inertia. We shall show that the complexity m of the extreme points method in this case is $m = 2$ using the test set

$$H_1 = \operatorname{diag}\left\{1, \frac{-145}{16}\right\}, \quad H_2 = \operatorname{diag}\left\{1, \frac{-127}{16}\right\}.$$

First, we check the vector $x = \binom{0}{z}$, $0 \neq z \in \mathbb{C}$. All expressions of the form $\operatorname{Re}(x^* H_j E_k x)$, where $k = 1, \ldots, 2, 3$ and $j = 1, 2$ are indeed positive.

Any other complex vector $x\binom{z_1}{z_2}$, $z_1 \neq 0$, may be scaled to the form $x(z) := \binom{1}{z}$. Since scaling preserves the sign of $\operatorname{Re}(x^* H A x)$, it does not affect our procedure. The function $p_{jk}(z) := x(z)^* H_j E_k x(z)$ is a real quadratic expression in z. The set of all $z \in \mathbb{C}$ for which $p_{jk}(z) > 0$ will be denoted by C_{jk}. Define the sets

$$\varDelta := \bigcup_{j=1,2} \left\{ \bigcap_{k=1,\ldots,3} C_{jk} \right\}.$$

Condition (iii) in Theorem 3 is equivalent to $\varDelta$ coinciding with the whole complex plane.

Due to the quadratic nature of the functions $p_{jk}(z)$, each C_{jk} turns out to be the complement of a closed disc of some radius. This includes some limit cases when the radius is zero or infinite; namely, C_{jk} may be the empty set, or the whole complex plane, or the complement of a single point. In our example, we duly compute these six sets:

$$p_{11}(z) = \frac{145}{4}\left(\left|z + \frac{1}{4}\right|^2 - \frac{17}{4^2 145}\right),$$

$$p_{12}(z) = \frac{145}{4}\left(\left|z + \frac{68}{145}\right|^2 + \frac{4^2}{145^2}\right),$$

$$p_{13}(z) = \frac{145}{8}\left(\left|z - \frac{56}{145}\right|^2 + \frac{4^2 94}{145^2}\right),$$

$$p_{21}(z) = \frac{127}{4}\left(\left|z + \frac{1}{4}\right|^2 + \frac{1}{4^2 127}\right),$$

$$p_{22}(z) = \frac{127}{4}\left(\left|z + \frac{68}{127}\right|^2 - \frac{4^2 35}{127^2}\right),$$

$$p_{23}(z) = \frac{127}{8}\left(\left|z - \frac{56}{127}\right|^2 + \frac{4^2 58}{127^2}\right).$$

In other words, if $B(a, r)$ is the closed complex disc of center a and radius r, and $B(a, r)^{\mathrm{c}}$ is its complement in $\mathbb{C}$, we have

$$C_{11} = B\left(\frac{-1}{4}, \frac{\sqrt{17}}{4\sqrt{145}}\right)^{\mathrm{c}}, \quad C_{22} = B\left(\frac{-68}{127}, \frac{4\sqrt{35}}{127}\right)^{\mathrm{c}}.$$

Since each of the other four sets is equal to the whole plane, $\Delta = C_{11} \cup C_{22}$. Note now that, if we denote $Q_1 := \{z : \mathrm{Re}(z) < (-1/4)(1 + \sqrt{17/145}) \approx -0.3356\}$ and $Q_2 := \{z : \mathrm{Re}(z) > (4/127)(\sqrt{35} - 17) \approx -0.3491\}$, then on the one hand $Q_1 \subset C_{11}$ and $Q_2 \subset C_{22}$; on the other hand, $\mathbb{C} = Q_1 \cup Q_2$, thus the construction is complete.

This calculation shows that the complexity for the extreme points method in Example 4 is $m \leqslant 2$. From Example 7 it will turn out that indeed $m = 2$.

3. Computational aspects

Recall that in the extreme points method, described in the previous section, one is searching for a test set $H_1, \ldots, H_m \in \mathscr{H}$ and a covering $\mathscr{X}_1, \ldots, \mathscr{X}_m$ of the unit sphere in $\mathbb{C}^n$ such that $H_j \in \mathscr{H}(\mathscr{E}, \mathscr{X}_j)$ for all $j = 1, \ldots, m$.

In contrast, adopting (2) as an alternative starting point for proving constant inertia, one would wish to construct a test set $H_1', \ldots, H_l' \in \mathscr{H}$ together with a finite covering $\mathscr{A}_1, \ldots, \mathscr{A}_l$ of $\mathscr{A}$ such that $H_j' \in \mathscr{H}(\mathscr{A}_j)$ for all $j = 1, \ldots, l$. We shall refer to the minimal such l as the "complexity of the finite coverage of $\mathscr{A}$ method". Unfortunately, there is no transparent connection between l on the one hand and n and p on the other. The simplest, if not most efficient, way to obtain a valid test set is by repeated bisection of $\mathscr{A}$, as suggested by [18] in the case of a stable interval matrix.

The complexity of both methods is exactly 1 when $\mathscr{H}(\mathscr{A}) \neq \emptyset$. In general, the two approaches may differ quite substantially, and a comprehensive comparison between their complexities is not available. The constant inertia verification problem is known to be NP-hard (see e.g. [14,15,17]), and so in both approaches the complexity has exponential growth in terms of n, the size of the matrices involved. In practice, however, one can often get good clues as to how to construct a sensible test set of relatively small complexity.

It appears that the extreme points method has several advantages for n large. First, the verification is restricted to the p extreme points of $\mathscr{A}$, which are typically a given finite collection, rather than to the full set $\mathscr{A}$. Secondly, there is no need to check the positive definiteness of a large number of matrices, which is extremely time consuming.

The detailed discussion in [1, Section III] (albeit restricted to the stable case) points at an additional important advantage of the extreme points method: further reduction of complexity, due to passing from matrix multiplication to vector inner products. This is presented next.

For given A_k $(k = 1, \ldots, p)$ and x with $\|x\| = 1$, define the vectors $y_k := A_k x$. If we fix $H \in \mathscr{H}$ we can also define $h := Hx$ and observe that x^*h is real. Moreover, if H belongs to $\mathscr{H}(\mathscr{E}, x)$, then $\mathrm{Re}(h^* y_k) > 0$. Conversely, given a vector h such that x^*h is real and $\mathrm{Re}(h^* y_k) > 0$, it is easy to construct $H \in \mathscr{H}$ for which $h = Hx$. Namely, take $H \in \mathscr{H}$ so that $H = rhh^* + \tilde{H}$, where $r \in \mathbb{R}$ is appropriately chosen and $\tilde{H}x = 0$. Consequently, $H \in \mathscr{H}(\mathscr{E}, x)$. (In the special case of stability the condition was $x^*h < 0$.)

Moreover, if an h with these properties has been found, one can find a whole neighborhood $\mathscr{X}$ of x for which $\min_{k=1,\ldots,p} \mathrm{Re}(x^*Hy_k) > 0$. The complexity of our procedure depends on the ability to find a covering of the unit sphere in $\mathbb{C}^n$ by a small number of such neighborhoods.

Once $\mathscr{X} = \mathscr{X}_1 \cup \cdots \cup \mathscr{X}_j$ has been calculated this way, a new unit vector $x_{j+1} \notin \mathscr{X}$ has to be selected, and the process repeated, creating the new neighborhood $\mathscr{X}_{j+1}$, etc. A flow chart of the entire procedure would be (for $k = 1, \ldots, p$):

(1) Start: set $j = 0$, $\mathscr{X} = \emptyset$ fix $\epsilon > 0$ and normalize $A_k \longrightarrow \frac{1}{\|A_k\|} A_k$.
(2) $j \longrightarrow j + 1$.
(3) Choose $x = x_j \notin \mathscr{X}$, where $\|x\| = 1$.
(4) Denote $y_k := A_k x$ and $\eta(h) := \min_{k=1,\ldots,p} \mathrm{Re}(h^* y_k)$, where $h \in \mathbb{C}^n$, $\|h\| \leqslant 1$.
(5) Find $\eta_j := \max_{h^*x \in \mathbb{R}} \eta(h)$.
(6) If $\eta_j < \epsilon$ stop (up to the prescribed precision), the set $\mathscr{A}$ does not have a fixed regular inertia.
Otherwise, let h_j be a maximizer of $\eta_j = \eta(h_j)$.
(7) Construct a matrix $H_j \in \mathscr{H}$ so that $h_j = H_j x$. There is a neighborhood $\mathscr{X}_j$ of x_j on the unit sphere in $\mathbb{C}^n$, so that $\min_{k=1,\ldots,p} \mathrm{Re}(x^* H_j A_k x) \geqslant \epsilon$ for all $x \in \mathscr{X}_j$.
(8) $\mathscr{X} \longrightarrow \mathscr{X} \cup \mathscr{X}_j$.
(9) If $\mathscr{X}$ does not cover the entire unit sphere, repeat the process from (2).
Otherwise, set $j = m$ and stop. All matrices in the set $\mathscr{A}$ share the same regular inertia.
This can be proved through the Hermitian matrices $H_1, \ldots, H_m$.

Now, there are two remarks in order. First note that in H there are n^2 real parameters, while in h there are only $2n$ real parameters. Moreover, the search is over all feasible h vectors, which form a *convex* subset of the closed unit ball in $\mathbb{C}^n$. Thus, at the critical optimization stage, (4) and (5), the problem is simplified. Second, we do not know yet how to efficiently choose in step (3) $x_j \notin \mathscr{X}$ for the next iteration. In fact, similar to the finite coverage of $\mathscr{A}$ method, unfortunately we cannot produce an a priori upper bound (based on the parameters n and p) on m, the number of sets $\mathscr{X}_j$ required to cover the unit sphere in $\mathbb{C}^n$.

In addition to the above, the structure of the set $\mathscr{A}$ can often be exploited in order to further alleviate the computational burden of the test. For example, from [2, Theorem 4.2] it follows that, if $\mathscr{A}$ is a convex hull of a pair of Hermitian matrices, then $\mathscr{H}(\mathscr{A}) \neq \emptyset$ if and only if $\mathscr{H}(\mathscr{A}, x) \neq \emptyset$ for all $x \neq 0$. In other words, a minimal test set, if it exists, has size $m = 1$. This is based on a result of Johnson and Rodman [11].

We conclude this section by pointing out that, since positive scaling preserves inertia, without loss of generality, we may restrict our discussion to convex *cones*. Consequently, this involves technical modifications such as roughly making $\mathscr{E}$ a compact set of normalized extreme points of a convex cone.

4. Can we check non-singularity instead?

If a given convex set of matrices, $\mathscr{A}$, has the constant regular inertia property, all the matrices in $\mathscr{A}$ are necessarily non-singular. The converse statement is false in general, although it does hold for some sets of interest, such as convex sets of Hermitian matrices, or more generally, matrices with real eigenvalues (see [7,8,14,16], for relevant material).

Another class of sets for which the converse is automatically guaranteed is the class of *invertible* convex cones, i.e. convex cones $\mathscr{C}$ such that $A \in \mathscr{C}$ and $\det(A) \neq 0$ implies $A^{-1} \in \mathscr{C}$. Cones of this type were studied in [2], where they were called convex invertible cones, or *cic* for short. See also [3,13].

Proposition 5 [2, Proposition 2.6]. *For a cic $\mathscr{C}$ of complex matrices, the following properties are equivalent*:
(i) *$\mathscr{C}$ contains no singular matrices.*
(ii) *No matrix in $\mathscr{C}$ has imaginary eigenvalues.*
(iii) *All the matrices in $\mathscr{C}$ have the same regular inertia.*

The set of *cic*s is rich, and contains many examples of interest, for example, the sets of upper triangular, Hermitian, diagonal, or positive definite matrices. The Lyapunov equation itself defines a matrix *cic* $\mathscr{L}_H$. Namely, for a fixed $H \in \mathscr{H}$ we define

$$\mathscr{L}_H := \{A \in \mathbb{C}^{n\times n} : HA + A^*H \in \mathscr{P}\}.$$

Note that $\mathscr{L}_{\pm I}$ are the sets of dissipative and accretive matrices.

Using this terminology, the finite coverage method for establishing robust inertia amounts to covering a given set $\mathscr{A}$ by a finite union of convex sets $\mathscr{A}_j$, where $\mathscr{A}_j := \mathscr{L}_{H_j} \cap \mathscr{A}$ for $j = 1, \ldots, l$. In order to decrease the number l, the sets $\mathscr{A}_j$ should be made as large as possible. It turns out that the largest such sets are of the form $\mathscr{L}_H$.

Proposition 6 [2, Observation 3.2, Proposition 3.7; 13, Theorem 7.1(i)–(iii)]. *For an arbitrary $H \in \mathscr{H}$ the following is true*:
(i) *$\mathscr{L}_H$ is a cic.*
(ii) *inertia(A) = inertia(H) for all $A \in \mathscr{L}_H$.*
(iii) *$\mathscr{L}_H$ is a maximal non-singular convex cone open in $\mathbb{C}^{n\times n}$.*

We remark that in [13, Proposition 9] a result similar to Proposition 6 is given in the context of the matrix *Riccati* equation: For a pair of $n \times n$ matrices H_a, H_b so that $(H_a + H_b)$ is non-singular, a set $\mathscr{R}_{H_a,H_b}$ of $2n \times 2n$ Hamiltonian matrices is defined. These Hamiltonians are associated with Riccati equations sharing H_a and $-H_b$ as a common pair of Hermitian solution. $\mathscr{R}_{H_a,H_b}$ turns out to be a maximal *cic* with *inertia* = $(n, 0, n)$.

According to Proposition 5, verifying constant inertia for a given *cic* of matrices $\mathscr{C}$ is reduced to checking that all the matrices in $\mathscr{C}$ are not singular. Just like the verification of constant inertia, the verification of non-singularity in convex sets of matrices is NP-hard [15]; however, there are interesting simplifications. For example, Polyak and Rohn [15] show that if an interval matrix contains a singular element then one of its edges contains a singular element. Hollot and Bartlett [8] show that an interval matrix with real eigenvalues only is robustly stable if and only if all its corner matrices are stable; and Rohn [16] shows that only 2^{2n} corner matrices need be checked in this case, rather than the entire set of 2^{n^2} corner matrices.

For a set $\mathscr{M}$ of $n \times n$ matrices, denote by $\mathscr{C}(\mathscr{M})$ (resp. $\mathrm{conv}(\mathscr{M})$) the smallest *cic* (resp. convex cone) which contains $\mathscr{M}$. Using Theorem 3 on the set $\mathscr{M}$, one can guarantee the constant inertia of $\mathrm{conv}(\mathscr{M})$, but not necessarily of $\mathscr{C}(\mathscr{M})$, which is typically larger.

Example 7. Consider the set $\mathscr{E}$ from Example 4 and its subset $\mathscr{M} = \{E_1, E_2\}$. As we have shown in that example, $\mathscr{H}(\mathscr{E}, x) \neq \emptyset$ for every non-zero vector x, implying constant inertia $(1, 0, 1)$ throughout $\mathrm{conv}(\mathscr{M})$. However, the matrix $(E_1 + 4E_2^{-1})$ which belongs to $\mathscr{C}(\mathscr{M})$, is stable. Hence, the inertia in $\mathscr{C}(\mathscr{M})$ cannot be constant.

Recall that by Proposition 5 a *cic* has constant regular inertia if it is a subset of $\mathscr{L}_H$ for some non-singular Hermitian matrix H. Thus, $\mathscr{C}(\mathscr{M})$ in not a subset of any $\mathscr{L}_H$. This amounts to $\mathscr{H}(\mathscr{C}(\mathscr{M}))$ being empty. By [2, Proposition 3.9(i)] we conclude that $\mathscr{H}(\mathscr{M}) = \mathscr{H}(\mathscr{C}(\mathscr{M})) = \emptyset$. This implies that the number m calculated in Example 4 is larger than 1, hence $m = 2$.

The *normalized* extreme points of $\mathscr{C}(\mathscr{M})$ are typically a continuum, rather than a discrete set, like for instance the extreme points of an interval matrix or the normalized extreme points of a polygonal convex cone. As a result, the verification of non-singularity on general cics is problematic. It would therefore be nice to identify cics which are *polygonal*, i.e. have a finite set of (normalized) extreme points.

Recall also that if $\mathscr{M}$ is an arbitrary set of matrices, typically $\mathbb{R}_+ \cdot \mathrm{conv}(\mathscr{M}, \mathscr{M}^{-1})$ is a *proper* subset of $\mathscr{C}(\mathscr{M})$: For example, if $A = \mathrm{diag}\{3 + \mathrm{i}4, 4 + \mathrm{i}3\}$ then, on the one hand $(\alpha A + \beta A^{-1}) \notin \mathbb{R}^{2\times 2}$ for all $\alpha, \beta \in \mathbb{R}$, but on the other hand $I \in \mathscr{C}(A)$. Similarly, if $B = \mathrm{diag}\{2, 3, 4\}$, then for all $\alpha, \beta \in \mathbb{R}$ the matrix $(\alpha B + \beta B^{-1})$ has at least one non-unit diagonal element but $I \in \mathscr{C}(B)$. In this respect, the case of real 2×2 matrices is of special interest.

Observation 8. *Let $\mathscr{M}$ be a set of real 2×2 matrices. Then the following are true*:

(i) *If* $\det(A)$ *has the same* (*non-zero*) *sign for every matrix* $A \in \mathrm{conv}(\mathscr{M})$, *then* $A^{-1} \in \mathrm{conv}(\mathscr{M}^{-1})$.

(ii) *If* $\mathrm{conv}(\mathscr{M}, \mathscr{M}^{-1})$ *has constant regular inertia*, *then* $\mathscr{C}(\mathscr{M}) = \mathbb{R}_+ \cdot \mathrm{conv}(\mathscr{M}, \mathscr{M}^{-1})$ *and this cic is non-singular.*

Proof. First note that if $A = \sum_k \alpha_k E_k$ for some $E_k \in \mathbb{C}^{2\times 2}$ and $\alpha_k \in \mathbb{C}$, then clearly, $\mathrm{adj}(A) = \sum_k \alpha_k \,\mathrm{adj}(E_k)$ and hence the relation,

$$A^{-1} = \sum_k \alpha_k \frac{\det(E_k)}{\det(A)} E_k^{-1}, \tag{3}$$

holds whenever all the inverses involved do exist.

In case $\alpha_k \geqslant 0$ and $\mathscr{M} := \{E_1, E_2, \ldots\} \subset \mathbb{R}^{2\times 2}$ is so that all matrices in $\mathrm{conv}(\mathscr{M})$ have determinant with same sign, then for all k, the quantity $\alpha_k(\det(E_k)/\det(A))$ in (3) is positive, so (i) is established.

A closer scrutiny of (3) reveals that if $C := bB + aA^{-1}$, where A, B are 2×2 matrices and $a, b \in \mathbb{C}$, all arbitrary, then, whenever all the inverses involved do exist, $C^{-1} = \hat{b}B^{-1} + \hat{a}A$, with $\hat{b} := b(\det(B)/\det(C))$ and $\hat{a} := a/\det(C)\det(A)$. Now, since by assumption $\mathrm{conv}(\mathscr{M}, \mathscr{M}^{-1})$ has constant regular inertia, the coefficients $\hat{a}, \hat{b}$ are real and positive, whenever a and b are. Namely, the convex cone $\mathbb{R}_+ \cdot \mathrm{conv}(\mathscr{M}, \mathscr{M}^{-1})$ is closed under inversion, thus (ii) is established and the proof is complete. □

References

[1] N. Cohen, I. Lewkowicz, A necessary and sufficient criterion for the stability of a convex set of matrices, IEEE Trans. Autom. Control AC-38 (4) (1993) 611–615.

[2] N. Cohen, I. Lewkowicz, Convex invertible cones and the Lyapunov equation, Linear Algebra Appl. 250 (1997) 105–131.

[3] N. Cohen, I. Lewkowicz, Convex invertible cones of state space systems, Math. Control Signals Syst. 10 (1997) 265–285.

[4] L. Elsner, T. Szulc, Convex combinations of matrices – nonsingularity and Schur stability characterization, Linear and Multilinear Algebra 44 (1998) 301–312.

[5] D. Hershkowitz, On cones and stability, Linear Algebra Appl. 275–276 (1998) 249–259.

[6] D. Hershkowitz, H. Schneider, On the inertia of intervals of matrices, SIAM J. Matrix Anal. Appl. 11 (4) (1990) 565–574.

[7] D. Hertz, The extreme eigenvalue and stability of real symmetric interval matrices, IEEE Trans. Autom. Control 35 (4) (1992) 532–535.

[8] C.V. Hollot, A.C. Bartlett, On the eigenvalues of interval matrices, in: Prceedings of the 26th Conference on Decision and Control, Los-Angeles, California, December 1987, pp. 794–799.

[9] R.A. Horn, C.R. Johnson, Topics in Matrix Analysis, Cambridge University Press, Cambridge, 1991.

[10] C.R. Johnson, A local Lyapunov theorem and the stability of sums, Linear Algebra Appl. 13 (1976) 37–43.

[11] C.R. Johnson, L. Rodman, Convex sets of Hermitian matrices with constant inertia, SIAM J. Algebraic Discrete Meth. 6 (3) (1985).

[12] B. Kolodziejczak, T. Szulc, Convex combination of matrices – full rank characterization, Linear Algebra Appl. 287 (1999) 215–222.

[13] I. Lewkowicz, Convex invertible cones of matrices – a unified framework for the equations of Sylvester, Lyapunov and Riccati, Linear Algebra Appl. 286 (1999) 107–133.

[14] A. Nemirovskii, Several NP-hard problems arising in robust stability analysis, Math. Control Signals Syst. 6 (1993) 99–105.

[15] S. Poljak, J. Rohn, Checking robust non-singularity is NP-hard, Math. Control Signals Syst. 6 (1993) 1–9.
[16] J. Rohn, Stability of interval matrices: the real eigenvalue case, IEEE Trans. Autom. Control AC-37 (10) (1992) 1604–1066.
[17] J.N. Tsitsiklis, V.D. Blondel, The Lyapunov exponent and joint spectral radius of pairs of matrices are hard – when not impossible to compute and to approximate, Math. Control Signals Syst. 10 (1997) 31–40.
[18] K. Wang, A.N. Michel, D. Liu, Necessary and sufficient condition for the Hurwitz and Schur stability of interval matrices, IEEE Trans. Autom. Control 39 (6) (1994) 1251–1255.