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Pointwise Convergence of Double Trigonometric Series*

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The pointwise convergence problem of the rectangular partial sums of a certain type of double trigonometric series is considered. This type of series obeys certain conditions on the finite-order differences of its coefficients. We prove that if the Césaro sums of the double series converge unrestrictedly, then so do its partial sums. It is pointed out that the converse of the last statement may not hold for the same kind of double trigonometric series. As a corollary, it is shown that the double Fourier series of the mentioned type converges unrestrictedly almost everywhere. Generalizations of the above results to the restricted case are also established. These results generalize the theorems of Chen. © 1993 Academic Press, Inc.

1. INTRODUCTION

Let T^2 be the torus, defined by $T^2 = \{(x, y) \in R^2 : -\pi \leqslant x, y < \pi\}$. Consider the double trigonometric series

$$\sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} c_{jk} e^{i(jx+ky)}, \tag{1.1}$$

where $\{c_{jk} : -\infty < j, k < \infty\}$ is a double sequence of complex numbers. The partial sums $s_{mn}(x, y)$ and the Césaro sums $\sigma_{mn}(x, y)$ of (1.1) are defined as

$$s_{mn}(x, y) = \sum_{|j| \leqslant m} \sum_{|k| \leqslant n} c_{jk} e^{i(jx+ky)} \qquad (m, n \geqslant 0), \tag{1.2}$$

$$\sigma_{mn}(x, y) = \frac{1}{(m+1)(n+1)} \sum_{j=0}^{m} \sum_{k=0}^{n} s_{jk}(x, y) \qquad (m, n \geqslant 0). \tag{1.3}$$

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If there exists an $f \in L_1(T^2)$ such that

$$c_{jk} = \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(x, y)\, e^{-i(jx+ky)}\, dx\, dy \qquad (-\infty < j, k < \infty), \qquad (1.4)$$

then (1.1) is said to be the double Fourier series of f, and the c_{jk} $(-\infty < j, k < \infty)$ are called the Fourier coefficients of f. In this case, we will frequently write $s_{mn}(f; x, y)$ and $\sigma_{mn}(f; x, y)$ instead of $s_{mn}(x, y)$ and $\sigma_{mn}(x, y)$, respectively.

Let $E \subset T^2$ and $\{g_{mn}: m, n \geqslant 0\}$ be a double sequence of functions defined on E. We say that g_{mn} converges uniformly on E in the unrestricted sense if there is a complex-valued function g defined on E such that g_{mn} converges to g uniformly on E as $\min(m, n)$ tends to infinity. In contrast, we say that g_{mn} converges uniformly on E in the restricted sense if there is a complex-valued function g defined on E such that

$$\lim_{\substack{a \leqslant m/n \leqslant b \\ m, n \to \infty}} g_{mn}(x, y) = g(x, y) \qquad \text{uniformly on } E \qquad (1.5)$$

for all $0 < a < b < \infty$. Here (1.5) means that for every $\varepsilon > 0$, there exists a positive integer N such that $|g_{mn}(x, y) - g(x, y)| < \varepsilon$ for all $m, n \geqslant 0$ satisfying $\min(m, n) > N$ and $a \leqslant m/n \leqslant b$, and for all $(x, y) \in E$. We also say that $g_{mn}(x_0, y_0)$ converges unrestrictedly (or restrictedly) to $g(x_0, y_0)$ if g_{mn} converges uniformly on E to g in the unrestricted (or restricted) sense, where $E = \{(x_0, y_0)\}$. Conventionally we say that (1.1) has the mentioned property if (1.2) does.

During the past thirty years, the almost everywhere convergence problem of the single Fourier series has been quickly developed. A typical result in this direction is that the Fourier series of $f \in L_2(T)$ converges almost everywhere. A natural question arises: Does this hold for the double Fourier series of $f \in L_2(T^2)$? Unfortunately, Fefferman [3] gave a negative answer to the question. He constructed a continuous function f on T^2 such that $s_{mn}(f; x, y)$ diverges everywhere as $\min(m, n) \to \infty$. Hence, it is meaningful to find a new class of integrable functions f on T^2 for which $s_{mn}(f; x, y)$ converge unrestrictedly almost everywhere. In [6], Móricz and Waterman proved that if the c_{jk} $(-\infty < j, k < \infty)$ are the Fourier coefficients of some $f \in L_1(T^2)$ and there exists a $1 < \lambda_0 \leqslant 2$ such that

$$\lim_{|k| \to \infty} \sum_{j=-\infty}^{\infty} |\Delta_{10}\, c_{jk}| = 0, \qquad \text{(MW1)}$$

$$\lim_{|j| \to \infty} \sum_{k=-\infty}^{\infty} |\Delta_{01}\, c_{jk}| = 0, \qquad \text{(MW2)}$$

$$\overline{\lim_{m\to\infty}} \frac{1}{[\lambda m]-m+1} \sum_{|j|=m}^{[\lambda m]} |j| \sum_{k=-\infty}^{\infty} |\Delta_{11} c_{jk}| \leqslant K, \tag{MW3}$$

$$\overline{\lim_{n\to\infty}} \frac{1}{[\lambda n]-n+1} \sum_{|k|=n}^{[\lambda n]} |k| \sum_{j=-\infty}^{\infty} |\Delta_{11} c_{jk}| \leqslant K \tag{MW4}$$

for all $1<\lambda\leqslant\lambda_0$, where K is a finite constant not depending on λ, then

$$\text{(1.1) converges unrestrictedly to } f \text{ a.e. if } |f| \log^+ |f| \in L_1(T^2); \tag{U}$$

$$\text{(1.1) converges restrictedly to } f \text{ a.e. if } f\in L_1(T^2). \tag{R}$$

(Notice that $\Delta_{10} c_{jk} = c_{jk} - c_{j+1,k}$, $\Delta_{01} c_{jk} = c_{jk} - c_{j,k+1}$, $\Delta_{11} c_{jk} = c_{jk} - c_{j,k+1} - c_{j+1,k} + c_{j+1,k+1}$.)

In this paper we establish the conclusions (U) and (R) under similar but different conditions from (MW1)–(MW4). Our conditions are

$$\lim_{\lambda\downarrow 1} \overline{\lim_{m\to\infty}} \sum_{k=-\infty}^{\infty} \max_{m\leqslant |j|\leqslant \lambda m} |c_{jk}| = 0, \tag{A1}$$

$$\lim_{\lambda\downarrow 1} \overline{\lim_{n\to\infty}} \sum_{j=-\infty}^{\infty} \max_{n\leqslant |k|\leqslant \lambda n} |c_{jk}| = 0, \tag{A2}$$

$$\lim_{\lambda\downarrow 1} \overline{\lim_{m\to\infty}} \sum_{k=-\infty}^{\infty} \sum_{|j|=m}^{[\lambda m]} |\Delta_1^p c_{jk}| = 0, \tag{A3}$$

$$\lim_{\lambda\downarrow 1} \overline{\lim_{n\to\infty}} \sum_{j=-\infty}^{\infty} \sum_{|k|=n}^{[\lambda n]} |\Delta_2^q c_{jk}| = 0, \tag{A4}$$

where p and q are nonnegative integers, and $\Delta_1^p c_{jk}$ and $\Delta_2^q c_{jk}$ are the pth difference of c_{jk} for the j-index and the qth difference of c_{jk} for the k-index, respectively. We note that our result generalizes [2, Corollary 2].

It is well known that, for the one-dimensional case, the pointwise convergence of a trigonometric series automatically implies that of its Césaro sums. In this paper, we show by an example that the corresponding statement may not be true for double trigonometric series even under the assumptions of (A1)–(A4). Simultaneously, we prove that the converse holds for those double trigonometric series with the conditions (A1)–(A4). This generalizes [2, Theorem 1]. As a corollary, the result mentioned in the last paragraph follows from this.

We also show that (R) is valid even under weaker conditions, (B1)–(B4), which involve the concept of restricted limit superior as defined in Section 3.

The organization of the paper is the following: In Section 2, we prove that under the conditions (A1)–(A4), the unrestricted convergence of σ_{mn}

implies that of s_{mn}. The method of summation by parts as was introduced in [2] is used in our proofs. In Section 3, we introduce the concept of restricted limit superior and then extend the result in Section 2 to the restricted case. We conclude in Section 4 with an application of the results in the above two sections to the pointwise convergence problem of double Fourier series.

2. Unrestricted Summability

It is a well-known fact that the pointwise convergence of a single trigonometric series automatically implies that of its Césaro sums. However, the corresponding statement may not be true for double trigonometric series, in general. For instance, let $(x_0, y_0) \in T^2$. Consider the double sequence $\{c_{jk}: -\infty < j, k < \infty\}$ defined by

$$c_{jk} = (-1)^k e^{-i(jx_0 + ky_0)}/j \qquad (j \geq 1, k = 0, 1),$$
$$c_{jk} = 0 \qquad (\text{otherwise}).$$

By a routine calculation, we find that such a double sequence satisfies (A1)–(A4), and $s_{mn}(x_0, y_0)$ converges unrestrictedly to 0. However, from

$$\sigma_{mn}(x_0, y_0) = \frac{1}{(m+1)(n+1)} \sum_{j=1}^{m} \left(1 + \frac{1}{2} + \cdots + \frac{1}{j}\right) \qquad (m, n \geq 1),$$

we see that $\sigma_{mn}(x_0, y_0)$ diverges as $\min(m, n) \to \infty$. This example illustrates that even under the conditions (A1)–(A4), the unrestricted convergence of s_{mn} does not guarantee that of σ_{mn}. It is a surprising fact that the converse of the mentioned statement is true under the assumptions of (A1)–(A4). In this section, we establish this fact.

Let $\{c_{jk}: -\infty < j, k < \infty\}$ be a double sequence of complex numbers, and $(x, y) \in T^2$. For $m \geq 0$, and $k = 0, \pm 1, \pm 2, \ldots$, define

$$s_{m(k)}(x, y) = \sum_{|j| \leq m} c_{jk} e^{i(jx + ky)}, \tag{2.1}$$

$$\sigma_{m(k)}(x, y) = \frac{1}{m+1} \sum_{j=0}^{m} s_{j(k)}(x, y). \tag{2.2}$$

By an elementary calculation, we obtain

$$\sigma_{m(k)}(x, y) = \sum_{|j| \leq m} \left(1 - \frac{|j|}{m+1}\right) c_{jk} e^{i(jx + ky)} \qquad (m \geq 0, k = 0, \pm 1, \pm 2, \ldots), \tag{2.3}$$

$$s_{mn}(x, y) = \sum_{|k| \leqslant n} s_{m(k)}(x, y) \qquad (m, n \geqslant 0), \tag{2.4}$$

$$\sigma_{mn}(x, y) = \sum_{|k| \leqslant n} \left(1 - \frac{|k|}{n+1}\right) \sigma_{m(k)}(x, y) \qquad (m, n \geqslant 0). \tag{2.5}$$

For $\alpha \geqslant 0$, define $\Delta_1^\alpha c_{jk}$ by the following recursive relations: for $j > 0$ and $k = 0, \pm 1, \pm 2, ...,$

$$\Delta_1^0 c_{jk} = c_{jk}, \qquad \Delta_1^0 c_{-j,k} = c_{-j,k},$$

$$\Delta_1^1 c_{jk} = c_{jk} - c_{j+1,k}, \qquad \Delta_1^1 c_{-j,k} = c_{-j,k} - c_{-j-1,k},$$

$$\Delta_1^\alpha c_{jk} = \Delta_1^1(\Delta_1^{\alpha-1} c_{jk}), \qquad \Delta_1^\alpha c_{-j,k} = \Delta_1^1(\Delta_1^{\alpha-1} c_{-j,k}) \qquad (\alpha \geqslant 2).$$

It is easy to see that for $j > 0$, $k = 0, \pm 1, \pm 2, ...,$ and $\alpha \geqslant 0$,

$$\Delta_1^\alpha c_{jk} = c_{jk} - \binom{\alpha}{1} c_{j+1,k} + \binom{\alpha}{2} c_{j+2,k} - \cdots + (-1)^\alpha \binom{\alpha}{\alpha} c_{j+\alpha,k},$$

$$\Delta_1^\alpha c_{-j,k} = c_{-j,k} - \binom{\alpha}{1} c_{-j-1,k} + \binom{\alpha}{2} c_{-j-2,k} - \cdots + (-1)^\alpha \binom{\alpha}{\alpha} c_{-j-\alpha,k}.$$

Similarly, $\Delta_2^\alpha c_{jk}$ is defined in the same way as $\Delta_1^\alpha c_{jk}$ by interchanging the role of j and k. The purpose of this section is to establish the following result, which extends [2, Theorem 1] from single trigonometric series to double trigonometric series.

THEOREM 1. *Let* $\{c_{jk} : -\infty < j, k < \infty\}$ *be a double sequence of complex numbers such that* (A1)–(A4) *hold for some* p *and* q. *Suppose* $E \subset T^2$ *satisfies*

$$x_0 \equiv \inf\{|x| : (x, y) \in E\} \neq 0,$$

$$y_0 \equiv \inf\{|y| : (x, y) \in E\} \neq 0.$$

If σ_{mn} *converges uniformly on* E *in the unrestricted sense, then so does* s_{mn}. *Moreover, the condition* $x_0 \neq 0$ *can be eliminated for the case* $p = 0$. *Similarly, the condition* $y_0 \neq 0$ *is not required for the case* $q = 0$.

Remark. Theorem 1 applies to many particular cases. Before proving the theorem, let us investigate these applications. The first case we want to investigate is that in which

(i) $c_{jk} = a_j b_k$ $(-\infty < j, k < \infty)$, $a_j = 0$ except perhaps for a finite number of j, $\{b_k : -\infty < k < \infty\}$ is a null sequence, and $\lim_{\lambda \downarrow 1} \overline{\lim}_{n \to \infty} \sum_{|k|=n}^{[\lambda n]} |\Delta^q b_k| = 0$ for some nonnegative integer q, where $\Delta^q b_k$ denotes the qth difference of b_k. Obviously, such a double sequence satisfies (A1)–(A4). Therefore, Theorem 1 can be applied to any double trigonometric series of type (i), in particular to any double trigonometric series of the type

> $c_{jk} = a_j b_k$ $(-\infty < j, k < \infty)$, $a_j = 0$ except perhaps for a finite number of j, and $\{b_k : -\infty < k < \infty\}$ is a null sequence of types (i) to (vii) described in [2, pp. 292–294].

If we choose $a_0 = 1$ and $a_j = 0$ for all $j \neq 0$, then Theorem 1 reduces to the situation of [2, Theorem 1]. Hence, the theorem generalizes [2, Theorem 1].

The second case we want to investigate is the following type:

(ii) $\{c_{jk} : -\infty < j, k < \infty\}$ is a lacunary double sequence satisfying (A1) and (A2).

The notion of a lacunary sequence was introduced by Hadamard in the study of the "over-convergence" problem of a power series (cf. [1, 2] for the definition of a lacunary sequence). We extend this concept to a double sequence as follows: $\{c_{jk} : -\infty < j, k < \infty\}$ is said to be a lacunary double sequence if there is a $\gamma > 1$ such that for each $-\infty < j < \infty$, $\{c_{jk} : -\infty < k < \infty\}$ is a lacunary sequence with a degree of lacunarity $\geqslant \gamma$, and for each $-\infty < k < \infty$, $\{c_{jk} : -\infty < j < \infty\}$ is a lacunary sequence with a degree of lacunarity $\geqslant \gamma$. From the definition, we find that for all $m, n \geqslant 1$ and for all λ with $1 < \lambda < \gamma$,

$$\sum_{k=-\infty}^{\infty} \sum_{|j|=m}^{[\lambda m]} |c_{jk}| \leqslant 2 \left(\sum_{k=-\infty}^{\infty} \max_{m \leqslant |j| \leqslant \lambda m} |c_{jk}| \right)$$

and

$$\sum_{j=-\infty}^{\infty} \sum_{|k|=n}^{[\lambda n]} |c_{jk}| \leqslant 2 \left(\sum_{j=-\infty}^{\infty} \max_{n \leqslant |k| \leqslant \lambda n} |c_{jk}| \right).$$

From here, we see that (A3) and (A4) follow from (A1) and (A2), respectively. Hence, Theorem 1 can apply to such a case, in particular to the case

$$\lim_{\lambda \downarrow 1} \overline{\lim_{n \to \infty}} \sum_{|k|=n}^{[\lambda n]} |c_{kk}| = 0 \qquad \text{and} \qquad c_{jk} = 0 \qquad \text{for all} \quad j \neq k.$$

Proof of Theorem 1. For simplicity we frequently omit (x, y) in the sums as $s_{mn}(x, y)$, $\sigma_{mn}(x, y)$, etc. Also, as in [2], we write λ_m for $[\lambda m]$, and λ_n for $[\lambda n]$. We have

$$s_{mn} - \sigma_{mn} = \Sigma_1(m, n) + \Sigma_2(m, n), \tag{2.6}$$

where

$$\Sigma_1(m, n) = \sum_{|k| \leq n} (s_{m(k)} - \sigma_{m(k)}),$$

and

$$\Sigma_2(m, n) = \frac{1}{n+1} \sum_{|k| \leq n} |k| \, \sigma_{m(k)}.$$

We show that $s_{mn} - \sigma_{mn}$ converges uniformly on E to zero in the unrestricted sense. To do this, we first estimate $\Sigma_2(m, n)$. Fix $\lambda > 1$. Using (2.5) for $\sigma_{m\lambda_n}$ and σ_{mn}, we get

$$\sigma_{m\lambda_n} - \sigma_{mn} = \frac{\lambda_n - n}{(\lambda_n + 1)(n+1)} \sum_{|k| \leq n} |k| \, \sigma_{m(k)} + \sum_{|k| = n+1}^{\lambda_n} \left(1 - \frac{|k|}{\lambda_n + 1}\right) \sigma_{m(k)}.$$

Multiplying both sides by $(\lambda_n + 1)/(\lambda_n - n)$ and then transposing, we obtain

$$\Sigma_2(m, n) = \frac{\lambda_n + 1}{\lambda_n - n} (\sigma_{m\lambda_n} - \sigma_{mn}) - \Sigma_{21}(m, n, \lambda), \tag{2.7}$$

where

$$\Sigma_{21}(m, n, \lambda) = \frac{\lambda_n + 1}{\lambda_n - n} \sum_{|k| = n+1}^{\lambda_n} \left(1 - \frac{|k|}{\lambda_n + 1}\right) \sigma_{m(k)}.$$

It follows from (2.3) that

$$\begin{aligned} &|\Sigma_{21}(m, n, \lambda)| \\ &= \left| \sum_{|j| \leq m} \left(1 - \frac{|j|}{m+1}\right) \sum_{|k| = n+1}^{\lambda_n} \frac{\lambda_n + 1 - |k|}{\lambda_n - n} c_{jk} e^{i(jx + ky)} \right| \\ &\leq \sum_{|j| \leq m} \left| \sum_{|k| = n+1}^{\lambda_n} \frac{\lambda_n + 1 - |k|}{\lambda_n - n} c_{jk} e^{iky} \right|. \end{aligned} \tag{2.8}$$

As in [2], we define $E_n^*(y)$ and $E_{-n}^*(y)$ for $n > 0$ as

$$E_n^*(y) = \frac{e^{i(n+1)y}}{2ie^{iy/2} \sin(y/2)} = \sum_{k=0}^{n} e^{iky} + \frac{1}{2ie^{iy/2} \sin(y/2)}, \tag{2.9}$$

$$E_{-n}^*(y) = E_n^*(-y). \tag{2.10}$$

Fix j. Using summation by parts, we get

$$\sum_{k=n+1}^{\lambda_n} \frac{\lambda_n+1-|k|}{\lambda_n-n} c_{jk} e^{iky}$$

$$= \sum_{k=n+1}^{\lambda_n} \frac{\lambda_n+1-|k|}{\lambda_n-n} c_{jk}(E_k^*(y)-E_{k-1}^*(y))$$

$$= \sum_{k=n+1}^{\lambda_n} \frac{\lambda_n+1-|k|}{\lambda_n-n} (\Delta_2^1 c_{jk})\, E_k^*(y)+\phi(j, n, \lambda; y), \tag{2.11}$$

where

$$\phi(j, n, \lambda; y)=\frac{1}{\lambda_n-n} \sum_{k=n+1}^{\lambda_n} c_{j,k+1} E_k^*(y)-c_{j,n+1} E_n^*(y).$$

For $(x, y) \in E$, we have

$$|\phi(j, n, \lambda; y)| \leqslant \frac{1}{\sin(y_0/2)} \max_{n+1 \leqslant k \leqslant \lambda_n+1} |c_{jk}| \leqslant \Psi_0(j, n, \lambda), \tag{2.12}$$

where

$$\Psi_0(j, n, \lambda)=\frac{1}{\sin(y_0/2)} \max_{n+1 \leqslant |k| \leqslant \lambda_n+1} |c_{jk}|.$$

By (2.9), (2.11), and (2.12) we get

$$\left|\sum_{k=n+1}^{\lambda_n} \frac{\lambda_n+1-|k|}{\lambda_n-n} c_{jk} e^{iky}\right|$$

$$\leqslant \frac{1}{2\sin(y_0/2)}\left|\sum_{k=n+1}^{\lambda_n} \frac{\lambda_n+1-|k|}{\lambda_n-n} (\Delta_2^1 c_{jk})\, e^{iky}\right|+\Psi_0(j, n, \lambda). \tag{2.13}$$

For $\alpha=0, 1, 2, \ldots, q-1$, define

$$\Psi_\alpha(j, n, \lambda)=\frac{1}{\sin(y_0/2)} \max_{n+1 \leqslant |k| \leqslant \lambda_{n+1}} |\Delta_2^\alpha c_{jk}|.$$

Set

$$\Psi(j, n, \lambda)=\sum_{\alpha=0}^{q-1} \frac{1}{(2\sin(y_0/2))^\alpha} \Psi_\alpha(j, n, \lambda).$$

Notice that if $q=0$, then $\Psi(j, n, \lambda)$ is defined to be 0 and the condition $y_0 \neq 0$ is not required. Replace c_{jk} in (2.13) by $\Delta_2^1 c_{jk}$, $\Delta_2^2 c_{jk}$, etc. Then

$$\begin{aligned}
&\left| \sum_{k=n+1}^{\lambda_n} \frac{\lambda_n+1-|k|}{\lambda_n-n} c_{jk} e^{iky} \right| \\
&\quad \leqslant \frac{1}{2\sin(y_0/2)} \left| \sum_{k=n+1}^{\lambda_n} \frac{\lambda_n+1-|k|}{\lambda_n-n} (\Delta_2^1 c_{jk}) e^{iky} \right| + \Psi_0(j, n, \lambda) \\
&\quad \leqslant \frac{1}{(2\sin(y_0/2))^2} \left| \sum_{k=n+1}^{\lambda_n} \frac{\lambda_n+1-|k|}{\lambda_n-n} (\Delta_2^2 c_{jk}) e^{iky} \right| \\
&\qquad + \frac{1}{2\sin(y_0/2)} \Psi_1(j, n, \lambda) + \Psi_0(j, n, \lambda) \\
&\quad \leqslant \cdots \\
&\quad \leqslant \frac{1}{(2\sin(y_0/2))^q} \left| \sum_{k=n+1}^{\lambda_n} \frac{\lambda_n+1-|k|}{\lambda_n-n} (\Delta_2^q c_{jk}) e^{iky} \right| + \Psi(j, n, \lambda) \\
&\quad \leqslant \frac{1}{(2\sin(y_0/2))^q} \sum_{k=n}^{\lambda_n} |\Delta_2^q c_{jk}| + \Psi(j, n, \lambda).
\end{aligned} \tag{2.14}$$

Go through the procedure from (2.11) to (2.14), with slight modification, for

$$\sum_{k=n+1}^{\lambda_n} \frac{\lambda_n+1-|k|}{\lambda_n-n} c_{j,-k} e^{-iky},$$

and then add the result to (2.14). We finally obtain

$$\begin{aligned}
&\left| \sum_{|k|=n+1}^{\lambda_n} \frac{\lambda_n+1-|k|}{\lambda_n-n} c_{jk} e^{iky} \right| \\
&\qquad \leqslant \frac{1}{(2\sin(y_0/2))^q} \sum_{|k|=n}^{\lambda_n} |\Delta_2^q c_{jk}| + 2\Psi(j, n, \lambda).
\end{aligned} \tag{2.15}$$

Therefore, by (A2), (A4), (2.8), and (2.15) we find that to each $\varepsilon > 0$, there corresponds a $\delta > 0$ such that the following holds:

For every λ with $0 < \lambda - 1 < \delta$, there exists a positive integer N_λ for which $|\Sigma_{21}(m, n, \lambda)| < \varepsilon$ for all $n \geqslant N_\lambda$, all $m \geqslant 0$, and all $(x, y) \in E$. (2.16)

From now on, the convergence involved is considered in the unrestricted sense. For each fixed $\lambda > 1$,

$$\frac{\lambda_n + 1}{\lambda_n - n}(\sigma_{m\lambda_n} - \sigma_{mn}) \to 0 \qquad \text{uniformly on } E. \tag{2.17}$$

This follows from the assumption on σ_{mn}. From (2.7), (2.16), and (2.17) we find that

$$\Sigma_2(m, n) \to 0 \qquad \text{uniformly on } E. \tag{2.18}$$

Next, we estimate $\Sigma_1(m, n)$. Fix $\lambda > 1$. With the help of (2.3), we find that

$$s_{m(k)} - \sigma_{m(k)} = \frac{\lambda_m + 1}{\lambda_m - m}(\sigma_{\lambda_m(k)} - \sigma_{m(k)}) - \sum_{|j| = m+1}^{\lambda_m} \frac{\lambda_m + 1 - |j|}{\lambda_m - m} c_{jk} e^{i(jx + ky)}.$$

This implies that

$$\Sigma_1(m, n) = \Sigma_{11}(m, n, \lambda) - \Sigma_{12}(m, n, \lambda), \tag{2.19}$$

where

$$\Sigma_{11}(m, n, \lambda) = \frac{\lambda_m + 1}{\lambda_m - m} \sum_{|k| \leq n} (\sigma_{\lambda_m(k)} - \sigma_{m(k)}),$$

and

$$\Sigma_{12}(m, n, \lambda) = \sum_{|k| \leq n} \sum_{|j| = m+1}^{\lambda_m} \frac{\lambda_m + 1 - |j|}{\lambda_m - m} c_{jk} e^{i(jx + ky)}.$$

Using (2.5) for $\sigma_{\lambda_m n}$ and σ_{mn}, we obtain

$$\sigma_{\lambda_m n} - \sigma_{mn} = \sum_{|k| \leq n} (\sigma_{\lambda_m(k)} - \sigma_{m(k)}) - \frac{1}{n+1} \sum_{|k| \leq n} |k| \, (\sigma_{\lambda_m(k)} - \sigma_{m(k)}).$$

Multiplying both sides by $(\lambda_m + 1)/(\lambda_m - m)$ and then transposing, we get

$$\Sigma_{11}(m, n, \lambda) = \frac{\lambda_m + 1}{\lambda_m - m}(\sigma_{\lambda_m n} - \sigma_{mn}) + \frac{\lambda_m + 1}{\lambda_m - m}(\Sigma_2(\lambda_m, n) - \Sigma_2(m, n)). \tag{2.20}$$

Obviously, we have

$$\frac{\lambda_m+1}{\lambda_m-m}(\sigma_{\lambda_m n}-\sigma_{mn})\to 0 \qquad \text{uniformly on } E. \tag{2.21}$$

By (2.18) we find that

$$\frac{\lambda_m+1}{\lambda_m-m}(\Sigma_2(\lambda_m, n)-\Sigma_2(m, n))\to 0 \qquad \text{uniformly on } E. \tag{2.22}$$

Combining (2.21) with (2.22), we find that for each fixed $\lambda>1$,

$$\Sigma_{11}(m, n, \lambda)\to 0 \qquad \text{uniformly on } E. \tag{2.23}$$

Go through the procedure from (2.11) to (2.15) with slight modification, and then we get

$$\begin{aligned}|\Sigma_{12}(m, n, \lambda)| &\leqslant \sum_{|k|\leqslant n}\left|\sum_{|j|=m+1}^{\lambda_m}\frac{\lambda_m+1-|j|}{\lambda_m-m}c_{jk}e^{ijx}\right| \\ &\leqslant \frac{1}{(2\sin(x_0/2))^p}\sum_{|k|\leqslant n}\sum_{|j|=m}^{\lambda_m}|\Delta_1^p c_{jk}| \\ &\quad +2\sum_{|k|\leqslant n}\Psi^*(m, k, \lambda),\end{aligned} \tag{2.24}$$

where

$$\Psi^*(m, k, \lambda)=\sum_{\alpha=0}^{p-1}\frac{1}{(2\sin(x_0/2))^\alpha}\Psi_\alpha^*(m, k, \lambda),$$

and

$$\Psi_\alpha^*(m, k, \lambda)=\frac{1}{\sin(x_0/2)}\max_{m+1\leqslant|j|\leqslant\lambda_{m+1}}|\Delta_1^\alpha c_{jk}|.$$

Notice that if $p=0$, then $\Psi^*(m, k, \lambda)$ is defined to be 0 and the condition $x_0\neq 0$ is not required. By (A1) and (A3), we find that to each $\varepsilon>0$ there corresponds a $\delta>0$ such that the following holds:

> For every λ with $0<\lambda-1<\delta$, there exists a positive integer M_λ for which $|\Sigma_{12}(m, n, \lambda)|<\varepsilon$ for all $m\geqslant M_\lambda$, all $n\geqslant 0$, and all $(x, y)\in E$. (2.25)

From (2.19), (2.23), and (2.25), we infer that $\Sigma_1(m, n)$ converges to 0 uniformly on E. With the help of (2.6), the desired result follows.

3. RESTRICTED SUMMABILITY

Let $\{d_{jk}: -\infty < j, k < \infty\}$ be a double sequence of extended real numbers. For $0 < a < b < \infty$, we define the concept of restricted limit superior as

$$\overline{\lim_{\substack{a \leqslant m/n \leqslant b \\ m,n \to \infty}}} d_{mn} = \inf_{\substack{a \leqslant m/n \leqslant b \\ m,n > 0}} \Big(\sup_{\substack{a \leqslant j/k \leqslant b \\ j \geqslant m, k \geqslant n}} d_{jk} \Big).$$

It is easy to see that for any $\rho > 0$,

$$\overline{\lim_{\substack{a \leqslant m/n \leqslant b \\ m,n \to \infty}}} d_{mn} = \lim_{n \to \infty} \Big(\sup_{\substack{a \leqslant j/k \leqslant b \\ j \geqslant n\rho, k \geqslant n}} d_{jk} \Big).$$

This tells us that the restricted limit superior of the double sequence $\{d_{jk}: -\infty < j, k < \infty\}$ is completely determined by those d_{jk} with j and k large enough. Using this, we can extend Theorem 1 in the following way:

THEOREM 2. *Let $\{c_{jk}: -\infty < j, k < \infty\}$ be a double sequence of complex numbers such that for all $0 < a < b < \infty$,*

$$\lim_{\lambda \downarrow 1} \overline{\lim_{\substack{a \leqslant m/n \leqslant b \\ m,n \to \infty}}} \sum_{|k| \leqslant n} \Big(\max_{m \leqslant |j| \leqslant \lambda m} |c_{jk}| \Big) = 0, \tag{B1}$$

$$\lim_{\lambda \downarrow 1} \overline{\lim_{\substack{a \leqslant m/n \leqslant b \\ m,n \to \infty}}} \sum_{|j| \leqslant m} \Big(\max_{n \leqslant |k| \leqslant \lambda n} |c_{jk}| \Big) = 0, \tag{B2}$$

and there exist two nonnegative integers p and q, depending on a and b, such that

$$\lim_{\lambda \downarrow 1} \overline{\lim_{\substack{a \leqslant m/n \leqslant b \\ m,n \to \infty}}} \sum_{|k| \leqslant n} \sum_{|j| = m}^{[\lambda m]} |\Delta_1^p c_{jk}| = 0, \tag{B3}$$

$$\lim_{\lambda \downarrow 1} \overline{\lim_{\substack{a \leqslant m/n \leqslant b \\ m,n \to \infty}}} \sum_{|j| \leqslant m} \sum_{|k| = n}^{[\lambda n]} |\Delta_2^q c_{jk}| = 0. \tag{B4}$$

Let E be as in Theorem 1. If σ_{mn} converges uniformly on E in the restricted sense, then so does s_{mn}. Moreover, the condition $x_0 \neq 0$ can be eliminated for the case $p = 0$. Similarly, the condition $y_0 \neq 0$ is not required for the case $q = 0$.

Remark. It is evident that (B1)–(B4) are weaker than (A1)–(A4). Hence, Theorem 2 generalizes Theorem 1 for the case of restricted convergence. As explained in the remark of Theorem 1, we see that

Theorem 2 can apply to any of the cases (i) to (ii) stated there. Besides these applications, Theorem 2 can also apply to the case

(iii) $c_{jk} = a_j b_k$ $(-\infty < j, k < \infty)$, $|j|^\alpha a_j = O(1)$ $(|j| \to \infty)$ for some $\alpha > 1$, and $k b_k = O(1)$ $(|k| \to \infty)$.

The reason for this is as follows: for $m, n \geqslant 3$, $\lambda > 1$, and $(\lambda - 1) m \geqslant 1$, we have

$$\sum_{|k| \leqslant n} \sum_{|j| = m}^{[\lambda m]} |c_{jk}| = \Big(\sum_{|k| \leqslant n} |b_k| \Big) \Big(\sum_{|j| = m}^{[\lambda m]} |a_j| \Big) \leqslant M(\log n)(\lambda - 1) m^{1-\alpha},$$

and

$$\sum_{|j| \leqslant m} \sum_{|k| = n}^{[\lambda n]} |c_{jk}| = \Big(\sum_{|j| \leqslant m} |a_j| \Big) \Big(\sum_{|k| = n}^{[\lambda n]} |b_k| \Big) \leqslant M(1^{-\alpha} + 2^{-\alpha} + 3^{-\alpha} + \cdots) \left(\frac{1}{n} + \log \lambda \right),$$

where M is an absolute constant. From these estimates, we find that (B1)–(B4) are satisfied. A special example of case (iii) is as follows: $c_{jk} = 1/(j^2 k)$ for all $j, k \geqslant 1$, and $c_{jk} = 0$ for other cases. We know that Theorem 2 is applicable to this example. However, for $p \geqslant 0$, $m \geqslant 1$, and $\lambda > 1$, we have

$$\sum_{k = -\infty}^{\infty} \sum_{|j| = m}^{[\lambda m]} |\Delta_1^p c_{jk}| = \sum_{k = 1}^{\infty} \sum_{j = m}^{[\lambda m]} |\Delta_1^p (1/j^2 k)| = +\infty.$$

This leads to the conclusion that (A3) fails. Thus, Theorem 1 can not work on this example.

Proof of Theorem 2. Let $0 < \alpha < \beta < \infty$ be fixed. We claim that

$$\lim_{\substack{\alpha \leqslant m/n \leqslant \beta \\ m, n \to \infty}} (\sigma_{mn} - s_{mn}) = 0 \qquad \text{uniformly on } E.$$

Following (2.8)–(2.15), we find that to each $\varepsilon > 0$, there corresponds a $\delta > 0$ such that the following holds:

> For every λ with $0 < \lambda - 1 < \delta$, there exists a positive integer N_λ for which $|\Sigma_{21}(m, n, \lambda)| < \varepsilon$ for all m and n satisfying $n \geqslant N_\lambda$ and $\alpha \leqslant m/n \leqslant 2\beta$, and for all $(x, y) \in E$.

This is obtained by applying (B2) with $a = \alpha/2$, $b = 2\beta$, and (B4) with $a = \alpha$,

$b = 2\beta$ to the inequalities (2.8) and (2.15). From now on, σ_{mn} is assumed to converge uniformly on E in the restricted sense. Then for each fixed $\lambda > 1$, (2.17) is true in the restricted sense. With the help of (2.7), we find that

$$\lim_{\substack{\alpha \leqslant m/n \leqslant 2\beta \\ m,n \to \infty}} \Sigma_2(m, n) = 0 \qquad \text{uniformly on } E. \tag{*}$$

Next, we consider the term $\Sigma_1(m, n)$. Fix $1 < \lambda < 2$. It follows from the assumption on σ_{mn} that

$$\frac{\lambda_m + 1}{\lambda_m - m} (\sigma_{\lambda_m n} - \sigma_{mn}) \to 0 \qquad \text{uniformly on } E$$

in the restricted sense. From (*), we know that

$$\lim_{\substack{\alpha \leqslant m/n \leqslant \beta \\ m,n \to \infty}} \frac{\lambda_m + 1}{\lambda_m - m} (\Sigma_2(\lambda_m, n) - \Sigma_2(m, n)) = 0 \qquad \text{uniformly on } E.$$

By (2.20), we infer that for each fixed $1 < \lambda < 2$,

$$\lim_{\substack{\alpha \leqslant m/n \leqslant \beta \\ m,n \to \infty}} \Sigma_{11}(m, n, \lambda) = 0 \qquad \text{uniformly on } E.$$

Apply (B1) with $a = \alpha$, $b = 2\beta$, and (B3) with $a = \alpha$, $b = \beta$ to (2.24), and we find that to each $\varepsilon > 0$, there corresponds a $\delta > 0$ such that the following holds:

> For every λ with $0 < \lambda - 1 < \delta$, there exists a positive integer M_λ for which $|\Sigma_{12}(m, n, \lambda)| < \varepsilon$ for all m and n satisfying $m \geqslant M_\lambda$ and $\alpha \leqslant m/n \leqslant \beta$, and for all $(x, y) \in E$.

With the help of (2.19), we infer that

$$\lim_{\substack{\alpha \leqslant m/n \leqslant \beta \\ m,n \to \infty}} \Sigma_1(m, n) = 0 \qquad \text{uniformly on } E.$$

From (2.6), the desired result follows.

4. Applications to Double Fourier Series

To apply our preceding theorems to the pointwise convergence problem of double Fourier series, we need the following lemma, which was established in [4, 5]. (See also [7, Vol. 2, pp. 308–309].)

LEMMA 3. *Let $f \in L_1(T^2)$ and (1.1) be its double Fourier series. Then $\sigma_{mn}(f; x, y)$ converges restrictedly to $f(x, y)$ a.e. In addition, if $|f| \log^+ |f| \in L_1(T^2)$, then $\sigma_{mn}(f; x, y)$ converges unrestrictedly to $f(x, y)$ a.e.*

With the help of this lemma, Theorems 1 and 2 have the following consequences:

THEOREM 4. (U) *Let $|f| \log^+ |f| \in L_1(T^2)$ and (1.1) be the double Fourier series of f. If* (A1)–(A4) *hold for some p and q, then $s_{mn}(f; x, y)$ converges unrestrictedly to $f(x, y)$ a.e.*

(R) *Let $f \in L_1(T^2)$ and (1.1) be its double Fourier series. If* (B1)–(B4) *hold, then $s_{mn}(f; x, y)$ converges restrictedly to $f(x, y)$ a.e.*

Remark. As explained in the Remark after Theorem 1, we see that Theorem 4(U) can be applied to either of the cases (i), (ii) stated there. Hence, this theorem generalizes [2, Corollary 2]. In the following, we will present an example to which Theorem 4(U) can apply, but Móricz and Waterman's result is not applicable. This example is constructed in such a way that (1.1) is the double Fourier series of some $f \in L_2(T^2)$, $c_{jk} = 0$ for $j \leqslant 0$ or $k \neq 0$, and

$$(\lambda(w) - 1)^{1/2} \leqslant \varlimsup_{m \to \infty} \sum_{j=m}^{[\lambda(w) m]} |\Delta_1^1 c_{j0}| \leqslant \frac{3}{2} (\lambda(w) - 1)^{1/2} \tag{4.1}$$

for all $w = 1, 2, 3, \ldots$, where $\lambda(w) = 1 + 2^{-w}$. The construction is given as follows: Pick up a sequence $\{w_s : s \geqslant 1\}$ of positive integers such that to each positive integer w there corresponds infinitely many s with $w_s = w$. Choose an increasing sequence $\{m_s : s \geqslant 1\}$ of positive integers such that for all $s \geqslant 1$,

$$4m_s + 1 < m_{s+1} \qquad \text{and} \qquad (\lambda(w_s) - 1)^{1/2} m_s \geqslant s^2.$$

Define the double sequence $\{c_{jk} : -\infty < j, k < \infty\}$ as follows: if j is an odd integer with $m_s \leqslant j \leqslant \lambda(w_s) m_s$ for some s, then

$$c_{j0} = (\lambda(w_s) - 1)^{-1/2} m_s^{-1} = 2^{(w_s)/2} m_s^{-1}; \tag{4.2}$$

and $c_{jk} = 0$ for other cases. Then

$$\sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} |c_{jk}|^2 \leqslant \sum_{s=1}^{\infty} (\lambda(w_s) - 1)^{-1} m_s^{-2} (\lambda(w_s) m_s - m_s + 1)$$

$$\leqslant 2 \sum_{s=1}^{\infty} s^{-2} < +\infty.$$

This implies that the c_{jk} $(-\infty < j, k < \infty)$ are the Fourier coefficients of some function f in $L_2(T^2)$. (Notice that this implies $|f| \log^+ |f| \in L_1(T^2)$.) Next, we claim that (4.1) holds for all $w = 1, 2, \ldots$. Let w be any given positive integer. Consider those m with $m_s \leqslant m \leqslant \lambda(w_s)\, m_s$ for some s. Then $\lambda(w)m \leqslant 4m_s$. If $w_s \geqslant w$, it follows from (4.2) that

$$\begin{aligned}\sum_{j=m}^{[\lambda(w)m]} |\Delta_1^1 c_{j0}| \leqslant \sum_{j=m_s}^{4m_s} |\Delta_1^1 c_{j0}| &= \sum_{j=m_s}^{[\lambda(w_s)\, m_s]} |\Delta_1^1 c_{j0}| \\ &\leqslant (\lambda(w_s) - 1)^{-1/2}\, m_s^{-1} (\lambda(w_s)\, m_s - m_s + 1) \\ &\leqslant (\lambda(w) - 1)^{1/2} + s^{-2}. \end{aligned} \tag{4.3}$$

If $w_s < w$, it follows from (4.2) that

$$\begin{aligned}\sum_{j=m}^{[\lambda(w)m]} |\Delta_1^1 c_{j0}| &\leqslant (\lambda(w_s) - 1)^{-1/2}\, m_s^{-1} (\lambda(w)m - m + 1) \\ &\leqslant \frac{3}{2} (\lambda(w) - 1)^{1/2} + s^{-2}. \end{aligned} \tag{4.4}$$

The inequalities (4.3) and (4.4) give the conclusion that if $m_s \leqslant m \leqslant \lambda(w_s)\, m_s$, then

$$\sum_{j=m}^{[\lambda(w)m]} |\Delta_1^1 c_{j0}| \leqslant \frac{3}{2} (\lambda(w) - 1)^{1/2} + s^{-2}. \tag{4.5}$$

Consider those m with $\lambda(w_s)\, m_s < m < m_{s+1}$. Then (4.2) gives

$$\sum_{j=m}^{[\lambda(w)m]} |\Delta_1^1 c_{j0}| \leqslant (\lambda(w_{s+1}) - 1)^{-1/2}\, m_{s+1}^{-1} + \sum_{j=m_{s+1}}^{[\lambda(w)\, m_{s+1}]} |\Delta_1^1 c_{j0}|.$$

By (4.5), we obtain

$$\sum_{j=m}^{[\lambda(w)m]} |\Delta_1^1 c_{j0}| \leqslant \frac{3}{2} (\lambda(w) - 1)^{1/2} + 2(s+1)^{-2}.$$

From the above discussion, we find that if $m_s \leqslant m < m_{s+1}$, then

$$\sum_{j=m}^{[\lambda(w)m]} |\Delta_1^1 c_{j0}| \leqslant \frac{3}{2} (\lambda(w) - 1)^{1/2} + 2s^{-2}.$$

From here, we infer that

$$\overline{\lim_{m \to \infty}} \sum_{j=m}^{[\lambda(w)m]} |\Delta_1^1 c_{j0}| \leqslant \frac{3}{2} (\lambda(w) - 1)^{1/2}. \tag{4.6}$$

On the other hand, for $w_s = w$, we have

$$\sum_{j=m_s}^{[\lambda(w)m_s]} |\Delta_1^1 c_{j0}| \geqslant (\lambda(w_s)-1)^{-1/2} m_s^{-1}([\lambda(w_s)m_s]-m_s)$$
$$\geqslant (\lambda(w)-1)^{1/2} - s^{-2}.$$

Since there exists infinitely many s with $w_s = w$,

$$\varlimsup_{m\to\infty} \sum_{j=m}^{[\lambda(w)m]} |\Delta_1^1 c_{j0}| \geqslant (\lambda(w)-1)^{1/2}. \tag{4.7}$$

Combining (4.6) with (4.7), we find that (4.1) holds. For $w = 1, 2,\ldots$, we have

$$\frac{1}{[\lambda(w)m]-m+1} \sum_{|j|=m}^{[\lambda(w)m]} |j| \sum_{k=-\infty}^{\infty} |\Delta_{11} c_{jk}|$$
$$\geqslant \frac{2m}{(\lambda(w)-1)m+1} \sum_{j=m}^{[\lambda(w)m]} |\Delta_1^1 c_{j0}|.$$

Applying (4.1) to this inequality, we find that

$$\varlimsup_{m\to\infty} \frac{1}{[\lambda(w)m]-m+1} \sum_{|j|=m}^{[\lambda(w)m]} |j| \sum_{k=-\infty}^{\infty} |\Delta_{11} c_{jk}| \geqslant (\lambda(w)-1)^{-1/2}.$$

Since $(\lambda(w)-1)^{-1/2}$ diverges to infinity as $w\to\infty$, we find that (MW3) is not satisfied. Consequently, Móricz and Waterman's result can not apply to this case. In contrast, (A1), (A2), and (A4) with $q=0$ hold. This follows from the definition of the double sequence. From (4.1), we conclude that (A3) holds for $p=1$. Hence, Theorem 4(U) can apply to this example.

As explained in the Remark after Theorem 2, we see that Theorem 4(R) can be applied to any of the cases (i) to (iii), which are stated in the Remarks after Theorems 1 and 2. A special example of case (iii) is as follows: $c_{jk} = 1/(j^2 k)$ ($j>0$, k odd), and $c_{jk}=0$ for other cases. We have

$$\sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} |c_{jk}|^2 < \infty,$$

which implies that (1.1) is the double Fourier series of some $f \in L_2(T^2)$. Hence, Theorem 4(R) can apply to this example. However, for $p \geqslant 0$, $m \geqslant 1$, and $\lambda > 1$, we have

$$\sum_{k=-\infty}^{\infty} \sum_{|j|=m}^{[\lambda m]} |\Delta_1^p c_{jk}| = \sum_{k=0}^{\infty} \sum_{j=m}^{[\lambda m]} |\Delta_1^p((j^{-2})(2k+1)^{-1})| = +\infty.$$

This leads to the conclusion that (A3) fails. For $j \geqslant 1$, we have

$$\sum_{k=-\infty}^{\infty} |\Delta_{11} c_{jk}| \geqslant \sum_{k=1}^{\infty} (j^{-2} - (j+1)^{-2})(k+1)^{-1} = +\infty,$$

which gives the conclusion that (MW3) fails. From the above discussion, we find that neither Theorem 4(U) nor Móricz and Waterman's result is applicable to this example.

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