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The n-Width of the Unit Ball of H^q

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Let $\mathbf{E}$ be a compact subset of the open unit disc Δ and let H^q be the Hardy space of analytic functions f on Δ for which $|f|^q$ has a harmonic majorant. We determine the value of the Kolmogorov, Gel'fand, and linear n-widths in $L^p(\mathbf{E}, \mu)$ of the restriction to $\mathbf{E}$ of the unit ball of H^q when $p \leqslant q$ or when $1 \leqslant q < p < \infty$ and $\mathbf{E}$ is "small."

INTRODUCTION

Let Δ be the open unit disc in the complex plane, $\mathbf{E}$ a compact subset of Δ, and μ a positive measure on $\mathbf{E}$. In this paper we establish the precise value of the n-width of the unit ball of the Hardy space H^q in the space $L^p(\mathbf{E}, \mu)$ in the case when $1 \leqslant p \leqslant q \leqslant \infty$ and in certain cases when $1 \leqslant q < p \leqslant \infty$. These results extend results of Fisher and Micchelli for the cases $q = \infty$, $1 \leqslant p \leqslant \infty$, and $p = q = 2$ (see [FM1; FM2], respectively). When $p \leqslant q$, $\mathbf{E}$ is the circle $\{z: |z| = r\}$, and μ is restricted to a special class of measures, the value of the width was obtained by O. G. Parfenov [Pa].

In Section 1 we establish our notation, give all the requisite definitions, and state and prove the main theorem. We conclude in Section 2 with several results concerning the more difficult case when $1 \leqslant q < p \leqslant \infty$.

SECTION 1

Let X be a Banach space and $\mathbf{A}$ a (convex, compact, centrally symmetric) subset of X.

The *Kolmogorov n-width* of $\mathbf{A}$ in X is defined by

$$d_n(\mathbf{A}, X) := \inf_{X_n} \sup_{f \in \mathbf{A}} \inf_{g \in X_n} \|f - g\|,$$

where X_n runs over all n dimensional subspaces of X.

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The *Gel'fand n-width* of $\mathbf{A}$ in X is defined by

$$d^n(\mathbf{A}, X) := \inf_{L^n} \sup_{x \in L^n \cap \mathbf{A}} \|x\|,$$

where L^n runs over all subspaces of codimension n.

The *linear n-width* of $\mathbf{A}$ in X is defined by

$$\delta_n(\mathbf{A}, X) := \inf_{T_n} \sup_{f \in \mathbf{A}} \|f - T_n f\|,$$

where T_n varies over all linear operators of rank n which map X into itself.

Much information on n-widths is in the book by A. Pinkus [Pi].

We shall take $\mathbf{A}$ to be the restriction to the compact set $\mathbf{E}$ of the closed unit ball A_q of the Hardy space H^q. We say that *sampling is optimal* for A_q if there are points $z_1, ..., z_n$ in Δ, L^p functions $c_1, ..., c_n$ on $\mathbf{E}$, and a linear operator T_n of the form

$$(T_n f)(z) = \sum_{k=1}^{n} c_k(z) f(z_k), \qquad f \in H^q$$

such that

$$\delta_n(\mathbf{A}_q, L^p) = \sup_{f \in A_q} \|f - T_n f\|_{L^p}.$$

(Repetitions among the points $z_1, ..., z_n$ are allowed with the usual understanding that if z_i is repeated k times, the values of f at z_i are the consecutive derivatives of f at z_i of order zero through $k-1$.)

The values of the n-widths are expressed in terms of Blaschke products. A *Blaschke product of degree n* is an analytic function B on Δ of the form

$$B(z) = \lambda \prod_{j=1}^{n} (z - a_j)/(1 - \bar{a}_j z), \qquad a_1, ..., a_n \in \Delta, \quad |\lambda| = 1.$$

We denote the collection of all Blaschke products of degree n or less by $\mathfrak{B}_n$.

The proof of our main theorem depends in an essential way on the following extremal problem: for $1 \leqslant p,\ q < \infty$, and a measure μ on $\mathbf{E}$ define

$$\delta(p, q; \mu) := \sup\{\|g\|_{L^p(\mathbf{E}, \mu)} / \|g\|_{H^q} : g \in H^q\}. \tag{1}$$

It is evident that solutions to (1) exist and that any solution is an outer function (division by a nonconstant inner factor would not affect the H^q norm while strictly increasing the $L^p(\mathbf{E}, \mu)$ norm). We shall call a solution g of (1) *normalized* if g has H^q norm one and is positive at the origin.

PROPOSITION 1. *Let g be a normalized solution of* (1). *Then*

$$\delta^p\,|g(e^{i\theta})|^q = \int_E |g(w)|^p\, P(e^{i\theta}; w)\, d\mu(w) \tag{2}$$

for all θ, where $P(e^{i\theta}; w)$ is the Poisson kernel for w at $e^{i\theta}$ and δ is short for $\delta(p, q; \mu)$.

Proof. Let v be a real harmonic function on Δ which is continuous on the closed unit disc and ε a small positive or negative number. Then

$$\delta\left\{\int_T |g|^q\, e^{\varepsilon q v}\, d\theta\right\}^{1/q} \geqslant \left\{\int_E |g|^p\, e^{\varepsilon p v}\, d\mu\right\}^{1/p}$$

where T is the unit circle $\{e^{i\theta}: 0 \leqslant \theta \leqslant 2\pi\}$. After expanding the exponential terms and using the binomial theorem and the fact that g is a normalized solution to (1), we obtain

$$\begin{aligned}\delta^p &\int_T |g(e^{i\theta})|^q\, v(e^{i\theta})\, d\theta \\ &= \int_E |g(w)|^p\, v(w)\, d\mu(w) \\ &= \int_E |g(w)|^p \int_T v(e^{i\theta}) P(e^{i\theta}; w)\, d\theta\, d\mu(w) \\ &= \int_T v(e^{i\theta}) \int_E |g(w)|^p\, P(e^{i\theta}; w)\, d\mu(w)\, d\theta.\end{aligned}$$

Since v is an arbitrary continuous function on T, this gives (2). ∎

We shall be able to give the n-width in the case when $p \leqslant q$ or when $p > q$ and **E** is sufficiently "small" in the following sense.

DEFINITION. The *hyperbolic radius* of a compact set **E** in the unit disc Δ is the infimum of all those numbers r such that there is a conformal mapping Φ of Δ onto Δ such that $\Phi(\mathbf{E})$ lies inside a circle of radius r centered at the origin.

PROPOSITION 2. *Suppose that $1 \leqslant p \leqslant q < \infty$; then there is but one normalized solution of* (1). *Moreover, the same conclusion holds if $1 \leqslant q < p < \infty$ provided that the hyperbolic radius r_0 of* **E** *satisfies*

$$\arctan(2r_0/(1 - r_0^2)) < q\pi/2p.$$

Proof. Let g_1 and g_2 be two normalized solutions of (1). Then

$$|g_1(e^{i\theta})/g_2(e^{i\theta})|^q$$
$$= \int_E |g_1(w)/g_2(w)|^p \, |g_2(w)|^p \, P(e^{i\theta}; w) \, d\mu(w) \Big/ \int_E |g_2(w)|^p \, P(e^{i\theta}; w) \, d\mu(w).$$

The measure $d\beta(w) = |g_2(w)|^p \, P(e^{i\theta}; w) \, d\mu(w) / \int_E |g_2(w)|^p \, P(e^{i\theta}; w) \, d\mu(w)$ is a probability measure so the above equality gives (for each θ)

$$|g_1(e^{i\theta})/g_2(e^{i\theta})|^q \leqslant \sup_{w \in E} |g_1(w)/g_2(w)|^p. \tag{3}$$

Since g_1 and g_2 are any two normalized solutions, (3) holds with the roles of g_1 and g_2 interchanged. Moreover, $g_1/g_2 = \exp(u + iv)$, so that (3), and its counterpart with g_1 and g_2 interchanged, can be rephrased as

$$\sup_T u(e^{i\theta}) \leqslant \{p/q\} \sup_{w \in E} u(w)$$

and

$$-\inf_T u(e^{i\theta}) \leqslant -\{p/q\} \inf_{w \in E} u(w).$$

When we add these two inequalities we obtain

$$\sup_T u(e^{i\theta}) - \inf_T u(e^{i\theta}) \leqslant \{p/q\} \{ \sup_{w \in E} u(w) - \inf_{w \in E} u(w) \}. \tag{4}$$

If $q \geqslant p$, this clearly implies (by the maximum principle) that u is a constant; that is, g_1 is a constant multiple of g_2. This constant must be 1 since g_1 and g_2 are both normalized.

If $q < p$, then we have to work a little harder. Assume that u is not identicaly constant. Adding a constant to u and then multiplying by a positive scalar clearly does not change (4). Hence, we may suppose that $-1 \leqslant u \leqslant 1$ on T and that the left-hand side of (4) is equal to 2. The following lemma is now needed.

LEMMA. *Suppose that u is a real-valued harmonic function on Δ satisfying $-1 \leqslant u \leqslant 1$. If the hyperbolic radius of $\mathbf{E}$ is r, then*

$$\sup_{w, \zeta \in \mathbf{E}} \{u(w) - u(\zeta)\} \leqslant (4/\pi) \arctan(2r/(1 - r^2)).$$

Proof. Clearly the problem is conformally invariant, so there is no loss in assuming that $\mathbf{E}$ lies within the disc of radius r centered at the origin. We shall use the maximum principle and the Poisson integral formula for u:

$$\begin{aligned}\sup_{w,\zeta\in E}\{u(\zeta)-u(w)\} &\leqslant \sup\{u(\zeta)-u(w) : |\zeta|=|w|=r\}\\ &\leqslant \sup\left\{(1/2\pi)\int |P(e^{i\theta};\zeta)-P(e^{i\theta};w)|\,d\theta : |\zeta|=|w|=r\right\}\\ &= (1/2\pi)\int |P(e^{i\theta};r)-P(e^{i\theta};-r)|\,d\theta\\ &= (4/\pi)\arctan(2r/(1-r^2)).\end{aligned}$$

This concludes the proof of the lemma.

We apply the conclusion of the lemma to (4). Thus, if $\arctan(2r/(1-r^2))<\pi q/2p$, then once again we obtain a contradiction. This establishes that u is identically constant and hence that $g_1=g_2$. The proof of uniqueness is complete. ∎

Our main result is this.

THEOREM 1. *Suppose that $1\leqslant p\leqslant q<\infty$ or that the hyperbolic radius r_0 of E satisfies*

$$\arctan(2r_0/(1-r_0^2))<\pi q/2p.$$

Then

$$d_n(A_q, L^p)=d^n(A_q, L^p)=\delta_n(A_q, L^p)=\inf_{B\in\mathfrak{B}_n}\ \sup_{g\in A_q}\|gB\|_{L^p}. \tag{5}$$

Moreover, sampling is optimal for A_q.

Proof. There is an odd continuous mapping σ of the sphere S^{2n+1} into $\mathscr{B}_n$. This mapping was first used in [FM] and is simple to define: let $z_0, ..., z_n$ be $n+1$ distinct points of Δ; for each $n+1$-tuple $\mathbf{w}=(w_0, ..., w_n)$ of complex numbers whose moduli sum to 1, the Pick–Nevalinna theorem guarantees that there is a unique positive scalar ρ and a unique Blaschke product B of degree at most n with $\rho B(z_j)=w_j$, $j=0, ..., n$. (A proof of the Pick–Nevalinna theorem can be found, for instance, in [F].) The map σ is then defined by $\sigma(\mathbf{w})=B$.

We now use the map σ and Proposition 2 to establish the lower bound. For each Blaschke product B of degree n or less, let g_B be the unique normalized solution of (1) with respect to the measure $|B|^p\,d\mu$. Let τ be the mapping from the sphere S^{2n+1} into A_q defined by

$$\tau(\mathbf{x})=\sigma(\mathbf{x})\,g_{\sigma(\mathbf{x})},\qquad \mathbf{x}\in S^{2n+1}.$$

Then τ is an odd mapping from the sphere S^{2n+1} into A_q; further, τ is continuous into the weak topology on H^q. In particular, the mapping τ is continuous from S^{2n+1} into $L^p(E,\mu)$.

We now apply standard arguments involving Borsuk's theorem to prove that

$$d^n(A_q, L^p), d_n(A_q, L^p) \geqslant \inf_{B \in \mathfrak{B}_n} \sup_{g \in A_q} \|gB\|_{L^p}.$$

To obtain the lower bound for the Gel'fand n-width, let $l_1, ..., l_n$ be n continuous linear functionals on L^p. The mapping $\mathbf{x} \mapsto \{l_j(\tau(\mathbf{x}))\}$ is continuous and odd from S^{2n+1} into C^n. From Borsuk's theorem we conclude that this map has a zero; that is, that there is a $B \in \mathfrak{B}_n$ such that $l_j(Bg_B) = 0$, $j = 1, ..., n$. Hence,

$$\sup\{\|f\| : l_j(f) = 0 \text{ and } f \in A_q\} \geqslant \|Bg_B\| \geqslant \inf_{B \in \mathfrak{B}_n} \sup_{g \in A_q} \|gB\|_{L^p}.$$

When we minimize over all choices of $l_1, ..., l_n$ we obtain the desired lower bound for the Gel'fand width. The lower bound for the Kolmogorov width is established in this way. Let X_n be any n dimensional subspace of $L^p(\mathbf{E}, \mu)$ and let $y_1, ..., y_n$ be a basis for X_n. We shall assume that $p > 1$; the case $p = 1$ follows by a limit argument. Each function $f \in A_q$ has a unique best approximation from X_n and this best approximation varies continuously with f. In particular, this is true of the functions $\tau(\mathbf{x})$ as $\mathbf{x}$ varies over S^{2n+1}. Let the best approximation to $\tau(\mathbf{x})$ be $\sum c_j(\mathbf{x}) y_j$. The n-tuple $\{c_j(\mathbf{x})\}$ is a continuous, odd function of $\mathbf{x}$ and hence by Brosuk's theorem, there is a choice of $\mathbf{x}$ which makes all the c_j simultaneously equal to zero. That is, there is a Blaschke product B_0 such that the best approximation to $B_0 g_{B_0}$ from X_n is zero. This then gives

$$\sup_{f \in A_q} \inf_{h \in X_n} \|f - h\| \geqslant \inf_{h \in X_n} \|B_0 g_{B_0} - h\| = \|B_0 g_{B_0}\| \geqslant \inf_{B \in \mathscr{B}_n} \|Bg_B\|.$$

This is the lower bound for the Kolmogorov n-width.

We shall next establish (for all p and q) the upper bound

$$\delta_n(A_q, L^p) \leqslant \inf_{B \in \mathfrak{B}_n} \sup_{g \in A_q} \|gB\|_{L^p}. \tag{6}$$

This will complete the proof of Theorem 1 since δ_n exceeds both d^n and d_n (see [Pi]). To see (6) we shall use Theorem 3 of [MR]. Let B be any Blaschke product of degree n with zeros at $z_1, ..., z_n$. Using the notation of [MR], let $X = H^q$, $K = A_q$, $Z = L^p(E, \mu)$, $Uf =$ the restriction of f to the compact set $\mathbf{E}$, $Y = C^n$, and $I(f) = (f(z_1), ..., f(z_n))$. Let G be defined by

$$G(a_1, ..., a_n)(z) = \sum_{k=1}^{n} a_k B_k(z), \qquad (a_1, ..., a_n) \in C^n,$$

where B_k is a constant multiple of the Blaschke product with zeros at z_j, $j \neq k$, the constant being chosen so that $B_k(z_k) = 1$. According to Theorem 3 of [MR],

$$\sup\{\|f\|_{L^p} : f \in A_q \text{ and } f(z_k) = 0, k = 1, ..., n\}$$
$$= \inf_A \sup\{\|f - A(I(f))\| : f \in A_q\},$$

where A ranges over all transformations from C^n into $L^p(E, \mu)$. Moreover, G is an optimal algorithm; that is,

$$\sup\{\|f\|_{L^p} : f \in A_q \text{ and } f(z_k) = 0, k = 1, ..., n\}$$
$$= \sup\{\|f - G(I(f))\| : f \in A_q\}. \tag{7}$$

The left-hand side of (7) is exactly

$$\sup\{\|Bg\| : g \in A_q\}$$

while the right-hand side of (7) is surely at least as large as the linear n-width of A_q in $L^p(E, \mu)$. We may now take the infimum over all Blaschke products of degree n to obtain the desired inequality. ∎

EXAMPLE 1. We use Theorem 1 to determine the n-width of A_q in L^p when E is the circle $|z| = r$, $d\mu = d\theta$, and $q \geqslant p$ or $\arctan(2r/(1-r^2)) < \pi q/2p$. In (5) take $B(z) = z^n$; we know that the normalized extremal g from (1) must be unique and it follows from the choices of E, μ, and B that g must also be rotation invariant. Therefore, it must be that $g(z)$ is identically equal to 1. Hence,

$$d_n = d^n = \delta_n \leqslant r^n.$$

On the other hand,

$$d_n = d^n = \delta_n = \inf_{B \in \mathfrak{B}_n} \sup_{g \in A_q} \|Bg\| \geqslant \inf_{B \in \mathfrak{B}_n} \|B\| = r^n$$

since it is not hard to establish that among all Blaschke products of degree n or less, $B(z) = z^n$ has the minimal L^p norm over $\{|z| = r\}$ with respect to $d\theta$. This result for d^n and δ_n when $p \leqslant q$ was obtained by O. Parfenov [Pa].

Remark. Suppose that μ is a measure on Δ whose support is not compact but nonetheless the restriction operator which maps H^q into $L^p(\mu)$ is compact. Examples of such measures are not difficult to construct. In this case, we can again ask for the values of the n-widths of the unit ball of H^q in L^p. The analysis given above (when $p < q$) carries over immediately to this more general case and, of course, the answer is exactly the same. The case $p = q$ then follows by a limit argument.

SECTION 2. THE CASE $1 \leqslant q < p \leqslant \infty$

This section has several results, most of which are examples which show that the situation when $q<p$ and $\mathbf{E}$ is not hyperbolically small is quite different from the other case.

EXAMPLE 2. Uniqueness of solutions of (1) may fail when $q<p$. To see this, take $\mathbf{E}$ to be the circle $|z|=r$ and take $d\mu$ to be $d\theta$. If the normalized solution to (1) were unique, it would have to be $g(z) \equiv 1$ since it would be rotation invariant. Thus the value of δ would be 1. On the other hand, if we take any $a \neq 0$ in the unit disc and set

$$f(z) = [(1-|a|^2)/(1-az)^2]^{1/q}$$

then f lies in the unit sphere of H^q. Hence, because $p>q$ and because f is not constant, the L^p norm of f on the unit circle with respect to $d\theta$ is strictly larger than 1. Thus, the L^p norm of f on the circle of radius r with respect to $d\theta$ is larger than 1, when r is near enough to 1. This contradiction establishes that uniqueness cannot hold.

On the other hand, Osipenko and Stessin in [OS1] prove that when $q=2$, $p=\infty$, $\mathbf{E}$ is the circle of radius r, and μ is Lebesgue measure, then the Gel'fand and linear widths coincide and are equal to

$$r^n/(1-r^2)^{1/2}.$$

It is not hard to show in this case that this is in turn equal to

$$\inf_{B \in \mathfrak{B}_n} \sup_{g \in A_2} \|Bg\|_\infty.$$

However, this happy coincidence of the answer for the case $q \geqslant p$ with the case $q<p$ seems to be more of an accident than a rule. We begin with the following result which is valid for all compact sets $\mathbf{E}$.

THEOREM 2. *Let* $\mathbf{E}$ *be a compact set and* μ *a positive measure on* $\mathbf{E}$. *Then*

$$d^n(A_2, L^\infty) = \delta_n(A_2, L^\infty) = \inf_{g_1, \ldots, g_n} \sup_{z \in \mathbf{E}} \left\{ 1/(1-|z|^2) - \sum_{j=1}^n |g_j(z)|^2 \right\} \qquad (8)$$

where $g_1, \ldots, g_n$ *vary over all sets of* n *orthonormal functions in* H^2.

Proof. For any particular set of n orthonormal functions, we note that

$$\left\{ 1/(1-|z|^2) - \sum_{j=1}^n |g_j(z)|^2 \right\} = K_S(z, z),$$

where $K_S(z, w)$ is the reproducing kernel for $w \in \Delta$ with respect to S, the orthogonal complement of the linear span of $g_1, ..., g_n$. (For each fixed $w \in \Delta$, $K_S(\cdot, w)$ is a member of S; $K_S(z, w)$ is an analytic function of z and also of $\bar{w}$.) To establish the lower bound for d^n, let S be a subspace of H^2 of codimension n and let $g_1, ..., g_n$ be an orthonormal basis for the orthogonal complement of S in H^2. Then for $f \in A_2$

$$\sup_{f \in S} \sup_{z \in E} |f(z)| \geqslant \sup_{w \in E} \sup_{z \in E} \{|K_S(z, w)|/K_S(w, w)\}^{1/2}$$
$$\geqslant \sup_{w \in E} \{K_S(w, w)\}^{1/2}.$$

After taking the infimum over all such subspaces S, equivalently, over all orthonormal sets $g_1, ..., g_n$, this gives the lower bound. Since $|f(z)| \leqslant \{K_S(z, z)\}^{1/2}$ for all $f \in A_2 \cap S$ and all $z \in \Delta$, we also obtain the right-hand side of (8) as an upper bound of d^n.

The upper bound for δ_n is obtained by noting that any orthonormal set $g_1, ..., g_n$ gives a rank n operator from H^2 to L^∞ by the simple formula

$$(T_n f)(z) = \sum_{j=1}^{n} g_j(z) \int_0^{2\pi} f \bar{g}_j \, d\theta$$

and so

$$\delta_n \leqslant \|I - T_n\| \leqslant \sup \left\{ \|f\| : f \in A_2, \int_0^{2\pi} f \bar{g}_j \, d\theta = 0, j = 1, ..., n \right\}$$
$$\leqslant \sup_{z \in E} \{K_S(z, z)\}^{1/2}. \quad \blacksquare$$

With Theorem 2 proved, we consider the following example.

EXAMPLE 3. We compute the Gel'fand 1-width of the unit ball of H^2 in $L^\infty(E, \mu)$ where E is the interval $[-r, r]$, $0 < r \leqslant 1/2$, and $d\mu$ is dx. A computation establishes that

$$\inf_{B \in \mathfrak{B}_1} \sup_{z \in E} |B(z)|/(1 - |z|^2)^{1/2} = r/(1 - r^2)^{1/2}.$$

On the other hand, the function $g(z) = (1 - r^4)^{1/2}/(1 - r^2 z^2)$ has H^2 norm one and some simple calculus (here is where you use $r \leqslant 1/2$) shows that

$$\sup_{z \in E} \{(1 - |z|^2)^{-1} - |g(z)|^2\} < r/(1 - r^2)^{1/2}.$$

This shows that formula (5) of Theorem 1 does not always hold in the case when $q < p$.

EXAMPLE 4. Even when E is the circle $|z|=r$ and $d\mu=d\theta$, formula (5) of Theorem 1 may not hold. Fix q, $1\leqslant q<2$, and take $p=\infty$. Let

$$\varphi(z)=((1-rz)^{-1}-rz)^{2/q}/((1-r^2)^{-1}-r^2)^{1/q}.$$

Then it is not overly hard to establish that φ lies in the unit sphere of H^q and that φ satisfies the integral identity for each $g\in H^q$

$$\int_0^{2\pi} \bar{\varphi}(e^{i\theta})\, |\varphi(e^{i\theta})|^{q-2}\, g(e^{i\theta})\, d\theta = c_1\, g(r)+c_2\, g'(0),$$

where c_1 and c_2 are two constants. It follows from [OS2] that φ is a solution of the extremal problem

$$\gamma := \sup\{|f(r)| : f\in A_q \text{ and } f'(0)=0\}$$

and hence $\gamma=((1-r^2)^{-1}-r^2)^{1/q}$. We take the subspace M of H^q of codimension one determined by

$$M=\{f\in H^q : f'(0)=0\}.$$

Then surely

$$\delta_1(A_q, L^\infty)\leqslant \sup\{\|f\|_\infty : f\in M\cap A_q\}=((1-r^2)^{-1}-r^2)^{1/q}.$$

For r near enough to 1, this last quantity is strictly smaller than $r/(1-r^2)^{1/q}$ which is the value of

$$\inf_{B\in\mathfrak{B}_1}\ \sup_{g\in A_q} \|Bg\|_\infty.$$

Hence, formula (5) of Theorem 1 cannot hold here.

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