On the Rankin–Selberg Method for Functions Not of Rapid Decay on Congruence Subgroups

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In this work we apply the Rankin–Selberg method to automorphic forms of congruence subgroups which are not of rapid decay. As an example, we obtain a simple proof of a generalized Maass–Selberg relation.

1. INTRODUCTION

The Rankin–Selberg method is one of the most important ideas in the theory of automorphic forms. The Mellin transform of the constant term in the Fourier expansion of an automorphic function can be written as the convolution of the automorphic function with an Eisenstein series. Hence it inherits the analytic properties of the Eisenstein series such as the functional equation.

This technique was initially used by Rankin and Selberg independently to give new estimates for the Fourier coefficients of automorphic forms on GL(2). Other applications of the Rankin–Selberg method include proofs of nonvanishing theorems, Doi–Naganuma liftings, and the Shimura correspondence.

Jacquet, Piatetskii–Shapiro, and Shalika, and others, have generalized this method to a large number of cases where the forms involved are cuspidal. In Zagier’s paper [5] a method was introduced whereby the Rankin–Selberg method could be applied to an automorphic form of the full modular group which is not of rapid decay. The method involved the application of the “folding–unfolding” trick to a truncated domain. This was the first of the very few examples in the literature giving a detailed formulation of this kind. Other cases can be seen in Patterson [4] and Lieman [3]. But Zagier’s idea is often used in some other setting, for instance [1] and [3]. It is particularly useful when one considers the
convolutions of various theta series and metaplectic Eisenstein series, which are not of rapid decay. This is because the Fourier coefficients of such functions encode number theoretic information. The Rankin–Selberg convolutions collect this information in a Dirichlet series, and the analytic properties of the convolution shed light on the nature of the coefficients.

In this work we give a theorem which deals with automorphic forms of congruence subgroups which are not of rapid decay. We apply the "folding-unfolding" technique to all the cusps of the congruence subgroup, unlike Zagier's case which is on the full modular group. As an example, we obtain from the theorem a simple proof of a generalized Maass-Selberg relation. Another potential application for this theorem would be to the convolution of two metaplectic Eisenstein series. A consequence would then be the derivation of sharp mean value estimates for squares of quadratic Dirichlet \( L \)-functions.

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2. THE MAIN THEOREM

Let \( \Gamma \) be a congruence subgroup of \( SL(2, \mathbb{Z}) \), which is reduced at infinity, i.e.,

\[
\Gamma_{\infty} = \{ g \in \Gamma | g(\infty) = \infty \} = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \mid b \in \mathbb{Z} \right\}.
\]

Let \( \Gamma \) operate on \( \mathbb{H} \), the upper half plane, in the usual way. Let \( \infty = \kappa_1, \kappa_2, \ldots, \kappa_h \) be the finite set of non-equivalent cusps of a reduced congruence subgroup \( \Gamma \). Let \( \Gamma' = \{ \sigma \in \Gamma | \sigma \kappa = \kappa \} \). Then there are \( \kappa_i \in GL(2, \mathbb{Q}) \) for each \( \kappa_i \), such that \( \kappa_i \infty = \kappa_i \) and \( \kappa_i^{-1} \Gamma \kappa_i = \Gamma_{\infty} \). In particular, \( \kappa_1 = I_{\mathbb{Z}+2} \).

We define the Eisenstein series \( E_i(z, s) \) at the cusp \( \kappa_i \), as

\[
E_i(z, s) = \sum_{\delta \in \kappa_i \Gamma' \kappa_i} \gamma^s(\kappa_i^{-1} \delta z),
\]

where \( \gamma(x + iy) = y \). Set

\[
\tilde{E}(z, s) = \begin{bmatrix}
E_1(z, s) \\
E_2(z, s) \\
\vdots \\
E_h(z, s)
\end{bmatrix}.
\]
Then there is a matrix of functions usually denoted by $\Phi(s)$, such that $E(z, s)$ has the functional equation

$$E(z, s) = \Phi(s) \tilde{E}(z, 1 - s),$$

where

$$\Phi(s) = (\phi_{ij})_{h \times h}$$

and

$$\phi_{ij}(s) = \pi^{1/2} \frac{\Gamma(s - 1/2)}{\Gamma(s)} \phi_{ij,0}(s)$$

For the details of $E(z, s)$ and $\Phi(s)$, see [2]. Note that $E_i(z, s)$ is the Eisenstein series at cusp $\infty$. We will write this as $E_{\infty}(z, s)$. For this reason we will often use the subindex $\infty$ instead of 1 for corresponding quantities.

Now let us state the main theorem of this paper.

**Theorem.** Let $F(z)$ be a continuous function invariant under the action of a congruence subgroup $\Gamma$, which is reduced at infinity. Let $F(x, z)$ have a Fourier expansion $F(x, z) = \sum_{m \in \mathbb{Z}} a_m(y) e(mx)$, where $a_m(z)$ is the constant term of $F(x, z)$ at $c_i$. Further suppose that

$$F(x, z) = \psi_1(y) + O(y^{-N}) \quad \forall N \quad y = \Im z \to \infty,$$

where $\psi_1(y)$ is a function of the form

$$\psi_1(y) = \sum_{j=1}^i \frac{c_{ij}}{n_{ij}^\delta} y^{\kappa_i} \log^n y,$$

where $c_{ij}, \kappa_i \in C, n_{ij} \in N \geq 0$. Define the Rankin–Selberg transform of $F$ at cusp $\kappa_i$ by

$$R_i(F, s) = \int_0^\infty (a_m(y) - \psi_1(y)) y^{s-2} \, dy$$

Thus $R_i(F, s)$ converges absolutely for $\Re s$ sufficiently large.
Then $R_i(F, s)$ can be meromorphically continued to all $s$, the only possible poles being at $s = 0, 1, \pi_i, 1 - \pi_i$ and $\rho/2$ ($\rho$ = non-trivial zero of the Riemann zeta-function).

More precisely we have the functional equation

$$ R_i(F, s) = \Phi(s) \left( \begin{array}{c} R_0(F, 1-s) \\ R_1(F, 1-s) \\ \vdots \\ R_{n-1}(F, 1-s) \end{array} \right) = \Phi(s) R_i(F, 1-s) $$

and

$$ \zeta^*(2s) R_i(F, s) = \zeta^*(2s) \tilde{h}_i(s) + \zeta^*(2s) \Phi(s) \tilde{h}_i(1-s) + \frac{\text{entire func. of } s}{s(s-1)}, $$

where $\Phi(s)$ is as defined above, $\zeta^*(2s) = \pi^{-s} \Gamma(s) \zeta(2s)$ and $\tilde{h}_i(s)$ is a $h \times 1$ column vector defined as

$$ \tilde{h}_i(s) = \left( -\sum_{j=1}^{l} \frac{c_{ij}}{(1 - \pi_i - s)^{n_1+1}} \right)_{h \times 1}. $$

Proof. Let $F$ be an arbitrary continuous function on $\Gamma \backslash \mathcal{H}$. Let $\mathcal{D} = \{z \in \mathcal{H} \mid |z| > 1, |x| \leq \frac{1}{2}\}$, the standard fundamental domain for the action of $SL(2, Z)$ on $\mathcal{H}$. Let $\mathcal{D}_T$ be a fundamental domain of $\Gamma$ with $|x| \leq \frac{1}{2}$.

Let $S_{\omega}(T) = \{z \in \mathcal{H} \mid \text{Im } z > T, |x| \leq \frac{1}{2}\}$. Let $S_{\omega}(T) = \{z \in \mathcal{H} \mid \pi_i^{-1}z \in S_{\omega}(T)\} = \omega S_{\omega}(T)$. Consider the truncated domain $\mathcal{D}_T = \mathcal{D}_T \cup \bigcup S_{\omega}(T)$, where $T$ is sufficiently large. Then $\mathcal{D}_T$ is the fundamental domain for the action of $\Gamma$ on $\mathcal{H}_T$. $\mathcal{H}_T$ is defined as

$$ \mathcal{H}_T = \bigcup_{\gamma \in \Gamma} \gamma \mathcal{D}_T $$

$$ = \{z \in \mathcal{H} \mid \max_{\delta \in \Gamma, \ i \geq 1} \text{Im } \delta z \leq T\} $$

$$ = \{z \in \mathcal{H} \mid \text{Im } z - T \leq \bigcup_{c \geq 1} \bigcup_{(a, c) \equiv 1} S_{\omega/c}, $$

where $\omega^{-1}z \in T$, $\delta \in \Gamma$ and $\delta(a/c) = \kappa_i$, for some $i \geq 1$. In fact, $S_{\omega/c}$ is an open disk in the upper half plane tangent to $x$-axis, and its closure touches the $x$-axis at the point $(a/c, 0)$.

Thus

$$ \Gamma_0 \backslash \mathcal{H}_T = \{x + iy \mid 0 \leq x \leq 1, 0 \leq y \leq T\} - \bigcup_{c \geq 1} \bigcup_{(a, c) \equiv 1} S_{\omega/c}. $$
We denote $\chi_T$ the characteristic function of $\mathcal{H}_T$. Applying Rankin-Selberg identity, that is,

$$\int_{\mathcal{H}_T} F(z) E(z,s) \, du = \int_{\mathcal{H}_T} F(z) \, y^s \, du,$$

Re $s > 1$

to the $\Gamma$-invariant function $F \cdot \chi_T$, we have

$$\int_{\mathcal{H}_T} F(z) E(z,s) \, du = \int_{\mathcal{H}_T} F(z) \, y^s \, du$$

$$= \int_{\mathcal{H}_T} \int_0^1 F(z) \, y^s \, du - \sum_{e=1}^{\infty} \sum_{a \pmod{e}} \int_{S_{\gamma e}} F(z) \, y^s \, du. \quad (1)$$

To compute the summation on the right side of (1), we divide it into two cases.

Case 1. The cusp $a/c$ is equivalent to $\infty$ i.e. $a/c \sim \infty$.

Let $\gamma_0 = \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right) \in \Gamma$ with the first column $\left( \begin{array}{c} 0 \\ 1 \end{array} \right)$, then $\gamma_0^{-1} S_{a/c} = \{ z \in \mathcal{H} \mid \text{Im } z > T \}$. We have

$$\int_{S_{\gamma_0}} F(z) \, y^s \, du = \int_T^{-1/2} F(z) \, \text{Im}(\gamma_0 z)^s \, du$$

$$= \int_T^{-1/2} F(z) \sum_{n=-\infty}^{\infty} \text{Im}(\gamma_0(z+n)) \, du$$

$$= \int_{S_{\gamma_0}(T)} F(z) \sum_{\gamma = \left( \begin{array}{cc} a & \ast \\ c & \ast \end{array} \right) \in \Gamma} \text{Im}(\gamma \gamma_0)^s \, du,$$

where the sum is over all $\gamma \in \Gamma$ with the first column $\left( \begin{array}{c} \ast \\ \ast \end{array} \right)$. (All such $\gamma$ have the form $\gamma_0 \left( \begin{array}{c} 1 \\ 0 \end{array} \right)$ for some $n \in \mathbb{Z}$.)

We have

$$\{ \Gamma_\infty \backslash \Gamma \mid \pm 1 \} = \left\{ \bigcup_{e > 0} \bigcup_{a \equiv c \equiv \infty} \bigcup_{e \equiv \infty} \left( \begin{array}{cc} a & \ast \\ c & \ast \end{array} \right) \in \Gamma \right\} \cup \left\{ \left( \begin{array}{cc} 1 \\ 1 \end{array} \right) \right\}.$$ 

Thus

$$\sum_{e = 1}^{\infty} \sum_{a \equiv c \equiv \infty} \int_{S_{\gamma_0}} F(z) \, y^s \, du = \int_{S_{\gamma_0}(T)} F(z)(E_{\infty}(z,s) - y^s) \, du. \quad (2)$$
Case 2. The cusp $a/c$ is equivalent to $\kappa_i, a/c \sim \kappa_i$, i.e. $\delta(a/c) = \kappa_i$ where $\delta \in \Gamma$, and $i > 1$.

$$\sum_{c \equiv 1 \pmod{c_i}} \sum_{(\alpha, e) = 1} \int_{\gamma_0 \cdot \tau} F(z) \ y^s \ du = \int_{\gamma_0 \cdot \tau} F(z) \ E_{a/z}(z, s) \ du$$

$$= \int_{\gamma_0 \cdot \tau} F(z) \ E_{a/z}(z, s) \ du$$

$$= \int_{\gamma_0 \cdot \tau} F(z) \ E_{a/z}(z, s) \ du \cdot (3)$$

Substituting (2) and (3) in the LHS of (1) we get

$$\int_0^\tau a_0(y) \ y^{s-2} \ dy = \sum_{i=1}^h \int_{\gamma_i \cdot \tau} F(x, z)\left[ E_{a/z}(x, z, s) - \delta_{x, y} \right] \ dy.$$

Now use the following details of the Eisenstein series.

Let $c_{x,z} = c_{x,z}(y, s)$ be the constant term in the expansion of $E_{a/z}(x, z, s)$ at $\kappa_i$; that is, the function is given by the equation

$$c_{x,z} = \int_0^1 E_{a/z}(x, z, s) \ dx.$$ 

It is a well known fact that for $s \neq 0, \frac{1}{2}, 1, c_{x,z} = \delta_{x, y} + \phi_{x, y} y^{1-s}$. Thus for $\Re s$ sufficiently large

$$\int_{\gamma_i \cdot \tau} F(z) \ E_{a/z}(z, s) \ du = \int_0^\tau a_0(y) \ y^{s-2} \ dy - \int_{\gamma_i \cdot \tau} F(x, z) \left[ E_{a/z}(x, z, s) - c_{x,z} \right] \ dy$$

$$- \sum_{i=1}^h \int_{\gamma_i \cdot \tau} F(x, z) \phi_{x, y} y^{1-s} \ du$$

$$= \int_0^\tau a_0(y) \ y^{s-2} \ dy - \sum_{i=1}^h \int_{\gamma_i \cdot \tau} F(x, z)$$

$$\times \left[ E_{a/z}(x, z, s) - c_{x,z} \right] \ dy$$

$$- \sum_{i=1}^h \phi_{x, y} \int_0^\tau a_0(y) \ y^{-1-s} \ dy.$$
The difference \[ E_n(x, z) - e_{n, s} \] is an entire function of \( s \) and is of rapid decay with respect to \( y \).

\[
\int_0^T a_0''(y) \, y^{s-2} \, dy - \sum_{i=1}^k \phi_{n, i} \int_T^\infty a_i''(y) \, y^{-1-s} \, dy \\
= \left[ \int_{i=1}^n F(z) \, E_n(z, s) \, du + \sum_{i=1}^n \left[ \int_{S_n(T)} F(x, z) \, \left[ E_n(x, z, s) - e_{n, s} \right] \, du \right] \right].
\] (4)

We evaluate the left hand side. Consider

\[
\int_0^T a_0''(y) \, y^{s-2} \, dy = \int_0^T (a_0''(y) - \psi_s(y)) \, y^{s-2} \, dy + \int_0^T \psi_s(y) \, y^{s-2} \, dy.
\]

Define

\[
h_T^*(s) = \int_0^T \psi_s(y) \, y^{s-2} \, dy = \sum_{n \in \mathbb{N}} \frac{C_n}{n!} \frac{\partial^{n-1}}{\partial s^{n-1}} \left( \frac{T^{s+n-1}}{s+n-1} \right)
\]

and

\[
\vec{h}_T(s) = (h_T^*(s))_{n \times 1}.
\]

Thus

\[
\int_0^T a_0''(y) \, y^{s-2} \, dy = R_0(F, s) - \int_T^\infty (a_0''(y) - \psi_s(y)) \, y^{s-2} \, dy + h_T^*(s).
\] (5)

Consider

\[
\int_T^\infty a_0''(y) \, y^{-s-2} \, dy = \int_T^\infty (a_0''(y) - \psi_s(y)) \, y^{-s-2} \, dy + \int_T^\infty \psi_s(y) \, y^{-s-2} \, dy.
\]

Since \( \psi_s(y) = \frac{1}{y^{1-s}} \),

\[
\int_T^\infty \psi_s(y) \, y^{-s-1} \, dy = -\int_0^T \psi_s(y) \, y^{-s-1} \, dy = -h_T^*(1-s).
\]

Thus

\[
\int_T^\infty a_0''(y) \, y^{-s-1} \, dy = \int_T^\infty (a_0''(y) - \psi_s(y)) \, y^{-s-1} \, dy - h_T^*(1-s).
\] (6)
Substituting (5) and (6) in (4), we get

\[ R_\alpha(F, s) - \int_0^\infty \left( a'_\alpha(y) - \psi_\alpha(y) \right) y^{s-2} \, dy \]

\[ - h_T^r(s) - \sum_{i=1}^h \phi_{i,e} \left[ \int_0^\infty (a'_\alpha(y) - \psi_\alpha(y)) y^{s-1} \, dy - h'_T(1-s) \right] \]

\[ = \int_{\partial T} F(z) E_\infty(z, s) \, d\mu + \int_{\partial S(T)} \sum_{i=1}^h F(\alpha_i z) (E_\infty(\alpha_i z, s) - e_{i,e}) \, d\mu. \]

Since

\[ \int_{\partial S(T)} \sum_{i=1}^h F(\alpha_i z) e_{i,e} \, d\mu = \int_0^\infty a'_\alpha(y) y^{s-2} \, dy + \sum_{i} \phi_{i,e} \int_0^\infty a'_\alpha(y) y^{s-1} \, dy, \]

the previous equality gives the equation

\[ R_\alpha(F, s) + h_T^r(s) + \sum_{i=1}^h \phi_{i,e} h'_T(1-s) \]

\[ = \int_{\partial T} F(z) E_\infty(z, s) \, d\mu + \int_{\partial S(T)} \sum_{i=1}^h F(\alpha_i z) (E_\infty(\alpha_i z, s) - e_{i,e}) \, d\mu \]

\[ + \int_0^\infty (a'_\alpha(y) - \psi_\alpha(y)) y^{s-2} \, dy \]

\[ + \sum_{i} \phi_{i,e} \left[ \int_0^\infty (a'_\alpha(y) - \psi_\alpha(y)) y^{s-1} \, dy \right] \]

\[ = \int_{\partial T} F(z) E_\infty(z, s) \, d\mu \]

\[ + \int_{\partial S(T)} \sum_{i=1}^h (F(\alpha_i z) E_\infty(\alpha_i z, s) - \psi_\alpha(y) e_{i,e}) \, d\mu. \]

(7)

Let \( F(z) \) have an expansion at \( \kappa \) as \( f(z) = F(\sigma z) = \sum_{m \in Z} a_m^\kappa(y) e(mx) \) whose constant term is \( a_0^\kappa(y). \) Then this expansion is the same as the expansion of \( f(z) \) at \( \infty. \) Thus we have

\[ R_\alpha(f, s) = R_\alpha(F, s) \]

\[ = \left. \int_0^\infty (a'_\alpha(y) - \psi_\alpha(y)) y^{s-2} \, dy \right|_{y=0} \]
The function \( f(z) = F(xz) \) is invariant under \( \sigma^{-1} \Gamma_n = \tilde{\Gamma} \). Let \( \varphi_{\Gamma} = \varphi_{\sigma^{-1} \Gamma_n} = \sigma^{-1} \varphi_{\Gamma} \). The cusps of \( \tilde{\Gamma} \) are \( \{ \tilde{\kappa}_i \} = \{ \sigma^{-1} \kappa_i \} \). Thus \( \tilde{\kappa}_i \propto \kappa_i \), where \( \tilde{\kappa}_i = \sigma^{-1} \kappa_i \).

Consider
\[
E_{\sigma}(\tilde{\kappa}_i, z, s) = \sum_{\gamma \in \Gamma_n \setminus \Gamma} \text{Im}(\gamma \tilde{\kappa}_i, z)^s
= \sum_{\gamma \in \Gamma_n \setminus \Gamma} \text{Im}(\sigma^{-1} \gamma \sigma^{-1} \kappa_i, z)^s
= \sum_{\gamma \in \Gamma_n \setminus \Gamma} \text{Im}(\sigma^{-1} \gamma \kappa_i, z)^s
= E_{\sigma}(\kappa_i, z, s).
\]

Also
\[
f(\tilde{\kappa}_i, z) = F(\sigma \tilde{\kappa}_i, z)
= F(\sigma^{-1} \kappa_i, z)
= F(\kappa_i, z).
\]

Hence
\[
\tilde{\psi}_i(y) = \text{polynomial part of the constant term of } f(\tilde{\kappa}_i, z)
= \text{polynomial part of the constant term of } F(\kappa_i, z)
= \psi_i(y).
\]

And
\[
\tilde{c}_{i, \infty} = \text{constant term of } E_{\sigma}(\tilde{\kappa}_i, z)
= \text{constant term of } E_{\sigma}(\kappa_i, z)
= c_{i, \infty}.
\]

Thus (7) gives
\[
R_{\sigma}(F, s) + h^c_{\sigma}(s) + \sum_{i=1}^h \phi_{\mu_i}(s) h^*_{\sigma_i}(1 - s)
= \int_{S}\int_{S} F(z) E_{\sigma}(z, s) \, d\mu
+ \int_{S} \sum_{i=1}^h \left( F(\kappa_i, z) E_{\sigma}(\kappa_i, z) - \psi_i(y) c_{i, \infty} \right) \, d\mu. 
\]
Now to convert to a matrix from, let
\[ \tilde{R}(F, s) = (R_i(F, s))_{h \times 1} \]
\[ = \left( \int_0^\infty (\alpha_i'(y) - \psi_i(y)) y^{s-2} dy \right)_{h \times 1} \]
\[ \tilde{h}_T(s) = (h'_T(s))_{h \times 1} \]
\[ = \left( \int_0^\infty \psi_i(y) y^{s-2} dy \right)_{h \times 1} \]
\[ \tilde{E}(z, s) = (E_i(z, s))_{h \times 1}, \]
\[ \tilde{e}_i(y, s) = \begin{pmatrix} e_{iy} \\ e_{i\phi} \end{pmatrix} = (y^s + \Phi(s) y^{1-s}) \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}. \]

Thus (8) in vector form is
\[ \tilde{R}(F, s) + \tilde{h}_T(s) + \Phi(s) \tilde{h}_T(1-s) \]
\[ = \int_{\gamma} F(z) \tilde{E}(z, s) \, d\mu \]
\[ + \int_{S(z)} \left( \sum_{i=1}^n (F(x, z) \tilde{E}(x, z, s) - \tilde{\psi}(y) \tilde{e}(y, s)) \right) \, d\mu. \]  

(9)

Recalling that we have defined
\[ \tilde{h}_i(s) = \left( - \sum_{j=1}^n \frac{c_{ij}}{(1 - \alpha_j - s)^{n+1}} \right)_{h \times 1} \]  

(10)
in the theorem above, observe that
\[ \tilde{h}_T(s) - \tilde{h}_i(s) = \left( \sum_{j=1}^n \frac{c_{ij}}{n_j} \frac{\partial^{n_j}}{s^{n_j} - 1} \left( \frac{T^{-s} - 1}{s + \alpha_j - 1} \right) \right)_{h \times 1} \]  

(11)
is an entire function of \( s \).

Combining (9), (11), we get the statements about the poles in the theorem. To prove the functional equation, we check that
\[ \Phi(s) (\tilde{h}_T(1-s) + \Phi(1-s) \tilde{h}_T(s)) = \Phi(s) \tilde{h}_T(1-s) + \tilde{h}_T(s), \]
and

\[ \Phi(s) F(x, z) \bar{E}(x, z, 1-s) - \psi_{ij}(y) \bar{e}(y, 1-s) \]
\[ = F(x, z) \bar{E}(x, z, s) - \psi_{ij}(y) \Phi(s) \bar{e}(y, 1-s) \]
\[ = F(x, z) \bar{E}(x, z, s) - \psi_{ij}(y) \bar{e}(y, s). \]

Also \( \Phi(s) F(z) \bar{E}(z, 1-s) = F(z) \bar{E}(z, s). \)

Thus all the terms except \( \Phi(F, s) \) in (9) have transformation as

\[ \Phi(s) \text{[function of } 1-s]\text{]} \text{= function of } s. \]

Thus \( \Phi(F, s) \) has the same transformation formula. So we have

\[ \Phi(s) \Phi(F, 1-s) = \Phi(F, s). \]

3. AN EXAMPLE

Let \( F(z) = E_{\omega, c}(z, s), s_1 \in c \) be the Eisenstein series at \( \infty \) as in the theorem. For \( s_1 \neq 0, \frac{1}{2} \) and 1, we have \( \psi_{ij}(y) = e_{\omega, c} = \delta_{ij} y^{s_1} + \phi_{\omega, c}(s_1) y^{1-s_1}. \)

Thus \( \Phi(F, s_1) = 0, \) and the equation (7), for \( \text{Re } s \text{ sufficiently large, becomes} \)

\[ \int_{S_\infty} E_{\omega, c}(z, s_1) E_{\omega, c}(z, s) d\mu \]
\[ + \int_{S_\infty} \sum_{i=1}^{h} (E_{\omega, c}(z, s_1) E_{\omega, c}(z, s) - e_{\omega, c}(s_1) e_{\omega, c}(s)) \]
\[ = \frac{T^{s_1-s_1} - 1}{s + s_1 - 1} + \phi_{\omega, c}(s_1) \frac{T^{s-s_1}}{s - s_1} + \phi_{\omega, c}(s) \frac{T^{s_1-s}}{s_1 - s} \]
\[ + \sum_{i=1}^{h} \phi_{\omega, c}(s_1) \phi_{\omega, c}(s) \frac{T^{1-s_1-s}}{1 - s - s_1}. \] (12)

Further, both sides of (12) are meromorphic functions of \( s. \) Thus (12) holds for all \( s \) except at \( s = s_1, 1 - s_1, 0, \frac{1}{2} \) and 1. To work out the cases for the values of \( s = s_1, 1 - s_1, 0, \frac{1}{2} \) or 1 and for \( s_1 = 0, \frac{1}{2} \) or 1 will be an interesting exercise.

Formula (12) is a generalized “Maass-Selberg relation” (see eg. [2], Theorem 2.3.1).
REFERENCES