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On two inequalities for the Hadamard product and the Fan product of matrices ☆

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ABSTRACT

If A and B are $n \times n$ nonsingular M-matrices, a lower bound on the smallest eigenvalue $\tau(A\bigstar B)$ for the Fan product of A and B is given. In addition, using the estimate on the perron root of nonnegative matrices, we also obtain an upper bound on the spectral radius $\rho(A \circ B)$ for nonnegative matrices A and B. These bounds improve some existing results.

1. Introduction

For a positive integer n, N denotes the set $\{1, 2, \ldots, n\}$. The set of all $n \times n$ complex matrices is denoted by $C^{n\times n}$ and $R^{n\times n}$ denotes the set of all $n \times n$ real matrices throughout.

Let $A = (a_{i,j})$ and $B = (b_{i,j})$ be two real $n \times n$ matrices. Then, $A \geqslant B(> B)$ if $a_{i,j} \geqslant b_{i,j}(> b_{i,j})$ for all $1 \leqslant i \leqslant n, 1 \leqslant j \leqslant n$. If O is the null matrix and $A \geqslant O(> O)$, we say that A is a nonnegative (positive)

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matrix. The spectral radius of A is denoted by $\rho(A)$. If A is a nonnegative matrix, the Perron–Frobenius theorem guarantees that $\rho(A) \in \sigma(A)$, where $\sigma(A)$ denotes the spectrum of A.

For $n \geqslant 2$, an $n \times n$ $A \in C^{n\times n}$ is reducible if there exists an $n \times n$ permutation matrix P such that

$$P^TAP = \begin{bmatrix} A_{1,1} & A_{1,2} \\ 0 & A_{2,2} \end{bmatrix},$$

where $A_{1,1}$ is an $r \times r$ submatrix and $A_{2,2}$ is an $(n-r) \times (n-r)$ submatrix, where $1 \leqslant r < n$. If no such permutation matrix exists, then A is irreducible. If A is a 1×1 complex matrix, then A is irreducible if its single entry is nonzero, and reducible otherwise.

Let A be an irreducible nonnegative matrix. It is well known that there exists a positive vector u such that $Au = \rho(A)u$, u being called right Perron eigenvector of A.

The *Hadamard product* of $A \in C^{n\times n}$ and $B \in C^{n\times n}$ is defined by $A \circ B \equiv (a_{i,j}b_{i,j}) \in C^{n\times n}$.

In [3, p. 358], there is a simple estimate for $\rho(A \circ B)$: if $A, B \in R^{n\times n}$, $A \geqslant O$, and $B \geqslant O$, then $\rho(A \circ B) \leqslant \rho(A)\rho(B)$. From Exercise [3, p. 358], we know this inequality can be very weak by taking $B = J$, the matrix of all ones. For example, If $A = I$, $B = J$, then we have

$$\rho(A \circ B) = \rho(A) = 1 \ll \rho(A)\rho(B) = n$$

when n is very large. But also clearly show that equality can occur (let $A = I$ and $B = I$).

Recently, Fang [4] gave an upper bound for $\rho(A \circ B)$, that is,

$$\rho(A \circ B) \leqslant \max_{1\leqslant i\leqslant n} \{2a_{i,i}b_{i,i} + \rho(A)\rho(B) - a_{i,i}\rho(B) - b_{i,i}\rho(A)\} \tag{1}$$

which is shaper than the bound $\rho(A)\rho(B)$ in [3, p. 358].

For two nonnegative matrices A, B, we will give a new upper bound for $\rho(A \circ B)$ in Section 2. The bound is shaper than the bound $\rho(A)\rho(B)$ in [3, p. 358] and the bound $\max_{1\leqslant i\leqslant n}\{2a_{i,i}b_{i,i} + \rho(A)\rho(B) - a_{i,i}\rho(B) - b_{i,i}\rho(A)\}$ in [4].

The set $Z_n \subset R^{n\times n}$ is defined by

$$Z_n = \{A = (a_{i,j}) \in R^{n\times n} : a_{i,j} \leqslant 0 \quad \text{if } i \neq j,\ i,j = 1,\dots,n\}$$

the simple sign patten of the matrices in Z_n has many striking consequences. Let $A = (a_{i,j}) \in Z_n$ and suppose $A = \alpha I - P$ with $\alpha \in R$ and $P \geqslant O$. Then $\alpha - \rho(P)$ is an eigenvalue of A, every eigenvalue of A lies in the disc $\{z \in C : |z - \alpha| \leqslant \rho(P)\}$, and hence every eigenvalue λ of A satisfies $Re\lambda \geqslant \alpha - \rho(P)$. In particular, A is an M-matrix if and only if $\alpha > \rho(P)$. If A is an M-matrix, one may always write $A = \gamma I - P$ with $\gamma = \max\{a_{i,i} : i = 1,\dots,n\}$, $P = \gamma I - A \geqslant O$; necessarily, $\gamma > \rho(P)$.

If $A = (a_{i,j}) \in Z_n$, and if we denote $\min\{Re(\lambda) : \lambda \in \sigma(A)\}$ by $\tau(A)$. Basic for our purpose are the following simple facts (see Problem 16, 19 and 28 in Section 2.5 of [3]):

(i) $\tau(A) \in \sigma(A)$; $\tau(A)$ is called the minimum eigenvalue of A.
(ii) If $A, B \in Z_n$, and $A \geqslant B$, then $\tau(A) \geqslant \tau(B)$.
(iii) If $A \in Z_n$, then $\rho(A^{-1})$ is the Perron eigenvalue of the nonnegative matrix A^{-1}, and $\tau(A) = \frac{1}{\rho(A^{-1})}$ is a positive real eigenvalue of A.

Let A be an irreducible nonsingular M-matrix. It is well known that there exists a positive vector u such that $Au = \tau(A)u$, u being called right Perron eigenvector of A.

Let $A \in C^{n\times n}$, $B \in C^{n\times n}$. The *Fan product* of A and B is denoted by $A \bigstar B \equiv C = (c_{i,j}) \in C^{n\times n}$ and is defined by

$$c_{i,j} = \begin{cases} -a_{i,j}b_{i,j}, & \text{if } i \neq j, \\ a_{i,i}b_{i,i}, & \text{if } i = j. \end{cases}$$

If $A, B \in Z_n$ are M-matrices, then so is $A \bigstar B$. In [3, p. 359], a lower bound for $\tau(A \bigstar B)$ was given: Let $A, B \in Z_n$ be M-matrices. Then $A^{-1} \circ B^{-1} \geqslant (A \bigstar B)^{-1}$, and hence $\tau(A \bigstar B) \geqslant \tau(A)\tau(B)$. Fang [4] gave a sharper lower bound for $\tau(A \bigstar B)$, that is,

$$\tau(A \bigstar B) \geqslant \min_{1\leqslant i\leqslant n} \{a_{i,i}\tau(B) + b_{i,i}\tau(A) - \tau(A)\tau(B)\}. \tag{2}$$

For two nonsingular M-matrices A and B, we will give a new lower bound for $\tau(A\bigstar B)$ in Section 3.

2. Inequalities for the Hadamard product of nonnegative matrices

In this section, we will give an upper bound for $\rho(A\circ B)$. In order to prove our results, we first give some Lemmas.

Lemma 1 [1]. *Let $A\in R^{n\times n}$ be given. Then either A is irreducible or there exists a permutation P such that*

$$P^TAP=\begin{bmatrix} R_{1,1} & R_{1,2} & \cdots & R_{1,m}\\ O & R_{2,2} & \cdots & R_{2,m}\\ \vdots & \vdots & \ddots & \vdots\\ O & O & \cdots & R_{m,m}\end{bmatrix},\tag{3}$$

where each square submatrix $R_{j,j}$, $1\leqslant j\leqslant m$, is either irreducible or a 1×1 null matrix.

Remark 1. Eq. (3) is said to be the normal form of a reducible matrix A. Clearly, the eigenvalues of A are the eigenvalues of the square submatrices $R_{j,j}$, $1\leqslant j\leqslant m$ (cf. [5]).

Lemma 2 [1]. *Let $A\in R^{n\times n}$ be a nonnegative matrix. If A_k is a principal submatrix of A, then $\rho(A_k)\leqslant\rho(A)$. If, in addition, A is irreducible and $A_k\neq A$, then $\rho(A_k)<\rho(A)$.*

Lemma 3 [2]. *Let $A=(a_{i,j})\in R^{n\times n}$ be a nonnegative matrix. Then*

$$\rho(A)\leqslant\max_{i\neq j}\frac{1}{2}\left\{a_{i,i}+a_{j,j}+\left[(a_{i,i}-a_{j,j})^2+4\sum_{k\neq i}a_{i,k}\sum_{k\neq j}a_{j,k}\right]^{\frac{1}{2}}\right\}.$$

Lemma 4 [3]. *Let $A,B\in C^{n\times n}$ and if $D\in C^{n\times n}$ and $E\in C^{n\times n}$ are diagonal, then*

$$D(A\circ B)E=(DAE)\circ B=(DA)\circ(BE)=(AE)\circ(DB)=A\circ(DBE).$$

Theorem 4. *If $A,B\in R^{n\times n}$, $A\geqslant O$, and $B\geqslant O$, then*

$$\begin{aligned}\rho(A\circ B)\leqslant&\max_{i\neq j}\frac{1}{2}\Big\{a_{i,i}b_{i,i}+a_{j,j}b_{j,j}+\Big[(a_{i,i}b_{i,i}-a_{j,j}b_{j,j})^2\\ &+4(\rho(A)-a_{i,i})(\rho(B)-b_{i,i})(\rho(A)-a_{j,j})(\rho(B)-b_{j,j})\Big]^{\frac{1}{2}}\Big\}.\end{aligned}\tag{4}$$

Proof. It is clear that (4) holds with equality for $n=1$.

We next assume that $n\geqslant 2$.

If $A\circ B$ is irreducible, then A and B are irreducible. From Lemma 2, we have

$$\rho(A)-a_{i,i}>0\quad\forall i\in N\tag{5}$$

and

$$\rho(B)-b_{i,i}>0\quad\forall i\in N.\tag{6}$$

Since $A=(a_{i,j})$, $B=(b_{i,j})$ are nonnegative irreducible, then there exists two positive vectors u, v such that $Au=\rho(A)u$, $Bv=\rho(B)v$. Thus, we have

$$a_{i,i}+\sum_{j\neq i}\frac{a_{i,j}u_j}{u_i}=\rho(A)\tag{7}$$

and

$$b_{i,i}+\sum_{j\neq i}\frac{b_{i,j}v_j}{v_i}=\rho(B). \tag{8}$$

Define $U=diag(u_1,\ldots,u_n), V=diag(v_1,\ldots,v_n)$. We know that U and V are nonsingular diagonal matrices. Let $\widehat{A}=(\hat{a}_{i,j})=U^{-1}AU$ and $\widehat{B}=(\hat{b}_{i,j})=V^{-1}BV$, then we have

$$\widehat{A}=(\hat{a}_{i,j})=U^{-1}AU=\begin{bmatrix} a_{1,1} & \frac{a_{1,2}u_2}{u_1} & \cdots & \frac{a_{1,n}u_n}{u_1}\\ \frac{a_{2,1}u_1}{u_2} & a_{2,2} & \cdots & \frac{a_{2,n}u_n}{u_2}\\ \vdots & \vdots & \ddots & \vdots\\ \frac{a_{n,1}u_1}{u_n} & \frac{a_{n,2}u_2}{u_n} & \cdots & a_{n,n}\end{bmatrix},$$

$$\widehat{B}=(\hat{b}_{i,j})=V^{-1}BV=\begin{bmatrix} b_{1,1} & \frac{b_{1,2}v_2}{v_1} & \cdots & \frac{b_{1,n}v_n}{v_1}\\ \frac{b_{2,1}v_1}{v_2} & b_{2,2} & \cdots & \frac{b_{2,n}v_n}{v_2}\\ \vdots & \vdots & \ddots & \vdots\\ \frac{b_{n,1}v_1}{v_n} & \frac{b_{n,2}v_2}{v_n} & \cdots & b_{n,n}\end{bmatrix}.$$

It is easy to show that $\widehat{A}$ and $\widehat{B}$ are nonnegative irreducible matrices, and all the row sums of $\widehat{A}$ are equal to $\rho(A)$ and all the row sums of $\widehat{B}$ are equal to $\rho(B)$.

Also let $W=VU$, then W is nonsingular. From Lemma 4, we have

$$\begin{aligned}(VU)^{-1}(A\circ B)(VU) &= U^{-1}V^{-1}(A\circ B)VU=U^{-1}(A\circ(V^{-1}BV))U\\ &= (U^{-1}AU)\circ(V^{-1}BV)=\widehat{A}\circ\widehat{B}.\end{aligned}$$

Thus, we have that $\rho(A\circ B)=\rho(\widehat{A}\circ\widehat{B})$.

We next consider the spectral radius $\rho(\widehat{A}\circ\widehat{B})$ of $\widehat{A}\circ\widehat{B}$. For nonnegative irreducible matrices $\widehat{A}$, $\widehat{B}$, from Definition of the *Hadamard product* of $\widehat{A}$ and $\widehat{B}$, (5)–(8) and Lemma 3, we have

$$\begin{aligned}\rho(\widehat{A}\circ\widehat{B}) &\leqslant \max_{i\neq j}\frac{1}{2}\Bigg\{\hat{a}_{i,i}\hat{b}_{i,i}+\hat{a}_{j,j}\hat{b}_{j,j}\\ &\quad+\left[(\hat{a}_{i,i}\hat{b}_{i,i}-\hat{a}_{j,j}\hat{b}_{j,j})^2+4\sum_{k\neq i}\hat{a}_{i,k}\hat{b}_{i,k}\sum_{k\neq j}\hat{a}_{j,k}\hat{b}_{j,k}\right]^{\frac{1}{2}}\Bigg\}\\ &= \max_{i\neq j}\frac{1}{2}\Bigg\{a_{i,i}b_{i,i}+a_{j,j}b_{j,j}\\ &\quad+\left[(a_{i,i}b_{i,i}-a_{j,j}b_{j,j})^2+4\sum_{k\neq i}\frac{a_{i,k}u_k}{u_i}\frac{b_{i,k}v_k}{v_i}\sum_{k\neq j}\frac{a_{j,k}u_k}{u_j}\frac{b_{j,k}v_k}{v_j}\right]^{\frac{1}{2}}\Bigg\}\\ &\leqslant \max_{i\neq j}\frac{1}{2}\Bigg\{a_{i,i}b_{i,i}+a_{j,j}b_{j,j}+\Bigg[(a_{i,i}b_{i,i}-a_{j,j}b_{j,j})^2\\ &\quad+4\left(\sum_{k\neq i}\frac{a_{i,k}u_k}{u_i}\sum_{k\neq i}\frac{b_{i,k}v_k}{v_i}\right)\left(\sum_{k\neq j}\frac{a_{j,k}u_k}{u_j}\sum_{k\neq j}\frac{b_{j,k}v_k}{v_j}\right)\Bigg]^{\frac{1}{2}}\Bigg\}\\ &= \max_{i\neq j}\frac{1}{2}\Bigg\{a_{i,i}b_{i,i}+a_{j,j}b_{j,j}+\Bigg[(a_{i,i}b_{i,i}-a_{j,j}b_{j,j})^2\end{aligned}$$

$$+4(\rho(A)-a_{i,i})(\rho(B)-b_{i,i})(\rho(A)-a_{j,j})(\rho(B)-b_{j,j})]^{\frac{1}{2}}\Big\}. \quad (9)$$

If $A \circ B$ is reducible. We denote by $D = (d_{i,j})$ the $n \times n$ permutation matrix with $d_{1,2} = d_{2,3} = \cdots = d_{n-1,n} = d_{n,1} = 1$, the remaining $d_{i,j}$ zero, then both $A + tD$ and $B + tD$ are nonnegative irreducible matrices for any chosen positive real number t. Now we substitute $A + tD$ and $B + tD$ for A and B, respectively in the previous case, and then letting $t \to 0$, the result follows by continuity.

Using ideas of the proof of Theorem 4, we give new proofs of inequality in [3, Observation 5.7.4] and inequality (1) in [4].

For inequality $\rho(A \circ B) \leqslant \rho(A)\rho(B)$.

From the proof of Theorem 4, we know that $\rho(A \circ B) = \rho(\widehat{A} \circ \widehat{B})$. Then we have

$$\rho(A\circ B)=\rho(\widehat{A}\circ\widehat{B})\leqslant \max_{1\leqslant i\leqslant n}\sum_{j=1}^{n}\hat{a}_{i,j}\hat{b}_{i,j}\leqslant \max_{1\leqslant i\leqslant n}\sum_{j=1}^{n}\hat{a}_{i,j}\max_{1\leqslant i\leqslant n}\sum_{j=1}^{n}\hat{b}_{i,j}=\rho(A)\rho(B).$$

For inequality $\rho(A \circ B) \leqslant \max_{1\leqslant i\leqslant n}\{2a_{i,i}b_{i,i} + \rho(A)\rho(B) - a_{i,i}\rho(B) - b_{i,i}\rho(A)\}$.

Similar to the proof of Theorem 4, we have

$$\rho(A\circ B)=\rho(\widehat{A}\circ\widehat{B})\leqslant \max_{1\leqslant i\leqslant n}\sum_{j=1}^{n}\hat{a}_{i,j}\hat{b}_{i,j}=\max_{1\leqslant i\leqslant n}\left(a_{i,i}b_{i,i}+\sum_{j\neq i}\hat{a}_{i,j}\hat{b}_{i,j}\right).$$

From (7) and (8), we have

$$\begin{aligned}\rho(A\circ B)&\leqslant \max_{1\leqslant i\leqslant n}\left(a_{i,i}b_{i,i}+\textstyle\sum_{j\neq i}\hat{a}_{i,j}\hat{b}_{i,j}\right)\\&\leqslant \max_{1\leqslant i\leqslant n}\left(a_{i,i}b_{i,i}+\textstyle\sum_{j\neq i}\hat{a}_{i,j}\sum_{j\neq i}\hat{b}_{i,j}\right)\\&= \max_{1\leqslant i\leqslant n}(a_{i,i}b_{i,i}+(\rho(A)-a_{i,i})(\rho(B)-b_{i,i}))\\&= \max_{1\leqslant i\leqslant n}\{2a_{i,i}b_{i,i}+\rho(A)\rho(B)-a_{i,i}\rho(B)-b_{i,i}\rho(A)\}.\qquad\square\end{aligned}$$

Remark 2. Fang [4] has shown that the upper bound in (1) for $\rho(A \circ B)$ is sharper than the bound $\rho(A)\rho(B)$. We next give a simple comparison between the upper bound in (1) and the upper bound in (4). Without loss of generality, for $i \neq j$, assume that

$$2a_{i,i}b_{i,i}+\rho(A)\rho(B)-a_{i,i}\rho(B)-b_{i,i}\rho(A)\geqslant 2a_{j,j}b_{j,j}+\rho(A)\rho(B)-a_{j,j}\rho(B)-b_{j,j}\rho(A). \quad (10)$$

Thus, we can write (10) equivalently as

$$a_{i,i}b_{i,i}+(\rho(A)-a_{i,i})(\rho(B)-b_{i,i})\geqslant a_{j,j}b_{j,j}+(\rho(A)-a_{j,j})(\rho(B)-b_{j,j}). \quad (11)$$

From (4), we have

$$\begin{aligned}a_{i,i}b_{i,i}&+a_{j,j}b_{j,j}+\left[(a_{i,i}b_{i,i}-a_{j,j}b_{j,j})^2+4(\rho(A)-a_{i,i})(\rho(B)-b_{i,i})(\rho(A)-a_{j,j})(\rho(B)-b_{j,j})\right]^{\frac{1}{2}}\\&\leqslant a_{i,i}b_{i,i}+a_{j,j}b_{j,j}+\Big[(a_{i,i}b_{i,i}-a_{j,j}b_{j,j})^2+4(\rho(A)\\&-a_{i,i})(\rho(B)-b_{i,i})\left((\rho(A)-a_{i,i})(\rho(B)-b_{i,i})+a_{i,i}b_{i,i}-a_{j,j}b_{j,j}\right)\Big]^{\frac{1}{2}}\\&= a_{i,i}b_{i,i}+a_{j,j}b_{j,j}+\Big[(a_{i,i}b_{i,i}-a_{j,j}b_{j,j})^2+4(\rho(A)-a_{i,i})^2(\rho(B)-b_{i,i})^2\\&+4(\rho(A)-a_{i,i})(\rho(B)-b_{i,i})(a_{i,i}b_{i,i}-a_{j,j}b_{j,j})\Big]^{\frac{1}{2}}\\&= a_{i,i}b_{i,i}+a_{j,j}b_{j,j}+\left[(a_{i,i}b_{i,i}-a_{j,j}b_{j,j}+2(\rho(A)-a_{i,i})(\rho(B)-b_{i,i}))^2\right]^{\frac{1}{2}}\end{aligned}$$

$$= a_{i,i}b_{i,i} + a_{j,j}b_{j,j} + a_{i,i}b_{i,i} - a_{j,j}b_{j,j} + 2(\rho(A) - a_{i,i})(\rho(B) - b_{i,i})$$
$$= 2a_{i,i}b_{i,i} + 2(\rho(A) - a_{i,i})(\rho(B) - b_{i,i}), \tag{12}$$

Thus, from (4) and (12), we have

$$\begin{aligned}\rho(A \circ B) &\leqslant \max_{i \neq j} \frac{1}{2}\Bigg\{a_{i,i}b_{i,i} + a_{j,j}b_{j,j} + \Bigg[(a_{i,i}b_{i,i} - a_{j,j}b_{j,j})^2 \\ &\quad + 4(\rho(A) - a_{i,i})(\rho(B) - b_{i,i})(\rho(A) - a_{j,j})(\rho(B) - b_{j,j})\Bigg]^{\frac{1}{2}}\Bigg\} \\ &\leqslant \max_{1 \leqslant i \leqslant n} \frac{1}{2}\{2a_{i,i}b_{i,i} + 2(\rho(A) - a_{i,i})(\rho(B) - b_{i,i})\} \\ &= \max_{1 \leqslant i \leqslant n}\{a_{i,i}b_{i,i} + (\rho(A) - a_{i,i})(\rho(B) - b_{i,i})\} \\ &= \max_{1 \leqslant i \leqslant n}\{2a_{i,i}b_{i,i} + \rho(A)\rho(B) - a_{i,i}\rho(B) - b_{i,i}\rho(A)\}.\end{aligned}$$

Hence, the bound in (4) is sharper than the known one $\rho(A)\rho(B)$ in [3] and the bound $\max_{1 \leqslant i \leqslant n}$ $\{2a_{i,i}b_{i,i} + \rho(A)\rho(B) - a_{i,i}\rho(B) - b_{i,i}\rho(A)\}$ in [4].

Consider the example in Introduction. Let $A = I$, $B = J$, it is easy to show that $\rho(A \circ B) = 1$ and

$$\begin{aligned}\rho(A \circ B) &\leqslant \max_{i \neq j} \frac{1}{2}\Bigg\{a_{i,i}b_{i,i} + a_{j,j}b_{j,j} + \Big[(a_{i,i}b_{i,i} - a_{j,j}b_{j,j})^2 \\ &\quad + 4(\rho(A) - a_{i,i})(\rho(B) - b_{i,i})(\rho(A) - a_{j,j})(\rho(B) - b_{j,j})\Big]^{\frac{1}{2}}\Bigg\} = 1.\end{aligned}$$

We next give another example to validate our results.

Example 1. Consider two 4×4 nonnegative matrices

$$A = \begin{bmatrix} 4 & 1 & 0 & 2 \\ 1 & 0.05 & 1 & 1 \\ 0 & 1 & 4 & 0.5 \\ 1 & 0.5 & 0 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

It is easy to show that $\rho(A \circ B) = \rho(A) = 5.7339$. By calculation, we have that $\rho(A)\rho(B) = 22.9336$. According to inequalities (1) and (4), we have

$$\rho(A \circ B) \leqslant \max_{1 \leqslant i \leqslant 4}\{2a_{i,i}b_{i,i} + \rho(A)\rho(B) - a_{i,i}\rho(B) - b_{i,i}\rho(A)\} = 17.1017,$$

and

$$\begin{aligned}\rho(A \circ B) &\leqslant \max_{i \neq j} \frac{1}{2}\Bigg\{a_{i,i}b_{i,i} + a_{j,j}b_{j,j} + \Big[(a_{i,i}b_{i,i} - a_{j,j}b_{j,j})^2 \\ &\quad + 4(\rho(A) - a_{i,i})(\rho(B) - b_{i,i})(\rho(A) - a_{j,j})(\rho(B) - b_{j,j})\Big]^{\frac{1}{2}}\Bigg\} = 11.6478.\end{aligned}$$

From Theorem 4 we can obtain the following corollary:

Corollary 5. *Let A, B be two $n \times n$ nonnegative matrices. Then we have*

$$|det(A \circ B)| \leqslant [\rho(A \circ B)]^n$$

$$\leqslant \max_{i\neq j} \frac{1}{2^n}\left\{a_{i,i}b_{i,i}+a_{j,j}b_{j,j}+\left[(a_{i,i}b_{i,i}-a_{j,j}b_{j,j})^2\right.\right.$$
$$\left.\left.+4(\rho(A)-a_{i,i})(\rho(B)-b_{i,i})(\rho(A)-a_{j,j})(\rho(B)-b_{j,j})\right]^{\frac{1}{2}}\right\}^n$$
$$\leqslant \max_{1\leqslant i\leqslant n}\{2a_{i,i}b_{i,i}+\rho(A)\rho(B)-a_{i,i}\rho(B)-b_{i,i}\rho(A)\}^n \leqslant (\rho(A)\rho(B))^n.$$

3. Inequalities for the Fan product of M-matrices

In this Section, we will give a lower bound for $\tau(A\bigstar B)$.

Lemma 5. *Let A, B be two nonsingular M-matrices and if D and E are two positive diagonal matrices, then*
$$D(A\bigstar B)E=(DAE)\bigstar B=(DA)\bigstar(BE)=(AE)\bigstar(DB)=A\bigstar(DBE).$$

Proof. Lemma 5 follows from Definition of *Fan product*. □

Theorem 7. *Let $A=(a_{i,j}), B=(b_{i,j})\in R^{n\times n}$ be two nonsingular M-matrices. Then*

$$\tau(A\bigstar B) \geqslant \min_{i\neq j} \frac{1}{2}\left\{a_{i,i}b_{i,i}+a_{j,j}b_{j,j}-\left[(a_{i,i}b_{i,i}-a_{j,j}b_{j,j})^2\right.\right.$$
$$\left.\left.+4(a_{i,i}-\tau(A))(b_{i,i}-\tau(B))(a_{j,j}-\tau(A))(b_{j,j}-\tau(B))\right]^{\frac{1}{2}}\right\}. \tag{13}$$

Proof. It is quite evident that (13) holds with equality for $n=1$.

We next assume that $n\geqslant 2$.

If $A\bigstar B$ is irreducible, then A and B are irreducible. Since $A-\tau(A)I$ and $B-\tau(B)I$ are singular irreducible M-matrices, Theorem 6.4.16 of [1] yields that

$$a_{i,i}-\tau(A)>0\quad \forall i\in N \tag{14}$$

and

$$b_{i,i}-\tau(B)>0\quad \forall i\in N. \tag{15}$$

Since $A=(a_{i,j}), B=(b_{i,j})$ are irreducible nonsingular M-matrices, then there exists two positive vectors u, v such that $Au=\tau(A)u, Bv=\tau(B)v$. Thus, we have

$$a_{i,i}-\sum_{j\neq i}\frac{|a_{i,j}|u_j}{u_i}=\tau(A) \tag{16}$$

and

$$b_{i,i}-\sum_{j\neq i}\frac{|b_{i,j}|v_j}{v_i}=\tau(B). \tag{17}$$

Define $\widetilde{U}=diag(u_1,\ldots,u_n)$, $\widetilde{V}=diag(v_1,\ldots,v_n)$. Then we have that $\widetilde{U}$ and $\widetilde{V}$ are nonsingular diagonal matrices. Let $\widetilde{A}=(\tilde{a}_{i,j})=\widetilde{U}^{-1}A\widetilde{U}$ and $\widetilde{B}=(\tilde{b}_{i,j})=\widetilde{V}^{-1}B\widetilde{V}$, then we have

$$\widetilde{A}=(\tilde{a}_{i,j})=\widetilde{U}^{-1}A\widetilde{U}=\begin{bmatrix} a_{1,1} & \frac{a_{1,2}u_2}{u_1} & \cdots & \frac{a_{1,n}u_n}{u_1}\\ \frac{a_{2,1}u_1}{u_2} & a_{2,2} & \cdots & \frac{a_{2,n}u_n}{u_2}\\ \vdots & \vdots & \ddots & \vdots\\ \frac{a_{n,1}u_1}{u_n} & \frac{a_{n,2}u_2}{u_n} & \cdots & a_{n,n}\end{bmatrix},$$

$$\widetilde{B} = (\tilde{b}_{i,j}) = \widetilde{V}^{-1}A\widetilde{V} = \begin{bmatrix} b_{1,1} & \frac{b_{1,2}v_2}{v_1} & \cdots & \frac{b_{1,n}v_n}{v_1} \\ \frac{b_{2,1}v_1}{v_2} & b_{2,2} & \cdots & \frac{b_{2,n}v_n}{v_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{b_{n,1}v_1}{v_n} & \frac{b_{n,2}v_2}{v_n} & \cdots & b_{n,n} \end{bmatrix}.$$

It is easy to show that $\widetilde{A}$ and $\widetilde{B}$ are also irreducible nonsingular M-matrices.

Also let $\widetilde{W} = \widetilde{V}\widetilde{U}$, then $\widetilde{W}$ is nonsingular. From Lemma 6, we have

$$\begin{aligned}(\widetilde{V}\widetilde{U})^{-1}(A\bigstar B)(\widetilde{V}\widetilde{U}) &= \widetilde{U}^{-1}\widetilde{V}^{-1}(A\bigstar B)\widetilde{V}\widetilde{U} = \widetilde{U}^{-1}(A\bigstar(\widetilde{V}^{-1}B\widetilde{V}))\widetilde{U} \\ &= (\widetilde{U}^{-1}A\widetilde{U})\bigstar(\widetilde{V}^{-1}B\widetilde{V}) = \widetilde{A}\bigstar\widetilde{B}.\end{aligned}$$

Thus, we have that $\tau(\widetilde{A}\bigstar\widetilde{B}) = \tau(A\bigstar B)$.

We next consider the minimum eigenvalue $\tau(\widetilde{A}\bigstar\widetilde{B})$ of $\widetilde{A}\bigstar\widetilde{B}$. For irreducible nonsingular M-matrices $\widetilde{A}, \widetilde{B}$, let $\lambda \in \sigma(\widetilde{A}\bigstar\widetilde{B})$ satisfy $\tau(\widetilde{A}\bigstar\widetilde{B}) = \lambda$, then we have that $0 < \lambda < a_{i,i}b_{i,i}, \forall i \in N$. From Definition of the *Fan product* of $\widetilde{A}$ and $\widetilde{B}$, (14)–(17) and Theorem 1.23 of [5], there is a pair (i,j) of positive integers with $i \neq j$ such that

$$|\lambda - a_{i,i}b_{i,i}||\lambda - a_{j,j}b_{j,j}| \leqslant \sum_{k\neq i} |-\tilde{a}_{i,k}\tilde{b}_{i,k}| \sum_{k\neq j} |-\tilde{a}_{j,k}\tilde{b}_{j,k}|.$$

Thus, for $i \neq j$, we have

$$\begin{aligned}|(\lambda - a_{i,i}b_{i,i})(\lambda - a_{j,j}b_{j,j})| &\leqslant \sum_{k\neq i} |\tilde{a}_{i,k}\tilde{b}_{i,k}| \sum_{k\neq j} |\tilde{a}_{j,k}\tilde{b}_{j,k}| \\ &\leqslant \sum_{k\neq i} \frac{|a_{i,k}|u_k}{u_i} \sum_{k\neq i} \frac{|b_{i,k}|v_k}{v_i} \sum_{k\neq j} \frac{|a_{j,k}|u_k}{u_j} \sum_{k\neq j} \frac{|b_{j,k}|v_k}{v_j} \\ &= (a_{i,i} - \tau(A))(b_{i,i} - \tau(B))(a_{j,j} - \tau(A))(b_{j,j} - \tau(B)).\end{aligned} \tag{18}$$

From inequality (18) and $0 < \lambda < a_{i,i}b_{i,i}, \forall i \in N$, we have

$$(\lambda - a_{i,i}b_{i,i})(\lambda - a_{j,j}b_{j,j}) \leqslant (a_{i,i} - \tau(A))(b_{i,i} - \tau(B))(a_{j,j} - \tau(A))(b_{j,j} - \tau(B)). \tag{19}$$

Thus, from inequality (19), we have

$$\lambda \geqslant \frac{1}{2}\left\{a_{i,i}b_{i,i} + a_{j,j}b_{j,j} - \left[(a_{i,i}b_{i,i} - a_{j,j}b_{j,j})^2 + 4(a_{i,i} - \tau(A))(b_{i,i} - \tau(B))(a_{j,j} - \tau(A))(b_{j,j} - \tau(B))\right]^{\frac{1}{2}}\right\}.$$

That is

$$\begin{aligned}\tau(A\bigstar B) &\geqslant \frac{1}{2}\left\{a_{i,i}b_{i,i} + a_{j,j}b_{j,j} - \left[(a_{i,i}b_{i,i} - a_{j,j}b_{j,j})^2 + 4(a_{i,i} - \tau(A))(b_{i,i} - \tau(B))(a_{j,j} - \tau(A))(b_{j,j} - \tau(B))\right]^{\frac{1}{2}}\right\} \\ &\geqslant \min_{i\neq j} \frac{1}{2}\left\{a_{i,i}b_{i,i} + a_{j,j}b_{j,j} - \left[(a_{i,i}b_{i,i} - a_{j,j}b_{j,j})^2 + 4(a_{i,i} - \tau(A))(b_{i,i} - \tau(B))(a_{j,j} - \tau(A))(b_{j,j} - \tau(B))\right]^{\frac{1}{2}}\right\}.\end{aligned}$$

If $A\bigstar B$ is reducible. It is well known that a matrix in Z_n is a nonsingular M-matrix if and only if all its leading principal minors are positive (see condition (E17) of Theorem 6.2.3 of [1]). If we denote by

$D=(d_{i,j})$ the $n\times n$ permutation matrix with $d_{1,2}=d_{2,3}=\cdots=d_{n-1,n}=d_{n,1}=1$, the remaining $d_{i,j}$ zero, then both $A-tD$ and $B-tD$ are irreducible nonsingular M-matrices for any chosen positive real number t, sufficiently small such that all the leading principal minors of both $A-tD$ and $B-tD$ are positive. Now we substitute $A-tD$ and $B-tD$ for A and B, respectively in the previous case, and then letting $t\to 0$, the result follows by continuity.

Using ideas of the proof of Theorem 7, we next give a new proof of inequality (2) in [4].

Let $\lambda\in\sigma(\widetilde{A}\bigstar\widetilde{B})$ satisfy $\tau(\widetilde{A}\bigstar\widetilde{B})=\lambda$. Similar to the proof of Theorem 7, by theorem of Gerschgorin, we have

$$|\lambda-a_{i,i}b_{i,i}|\leqslant\sum_{k\neq i}|-\frac{a_{i,k}u_k}{u_i}\frac{b_{i,k}v_k}{v_i}|.$$

Thus, we have

$$a_{i,i}b_{i,i}-\lambda\leqslant\sum_{k\neq i}\frac{|a_{i,k}|u_k}{u_i}\sum_{k\neq i}\frac{|b_{i,k}|v_k}{v_i}=(a_{i,i}-\tau(A))(b_{i,i}-\tau(B)).$$

Hence, we have

$$\begin{aligned}\lambda&\geqslant a_{i,i}b_{i,i}-(a_{i,i}-\tau(A))(b_{i,i}-\tau(B))\\&=a_{i,i}\tau(B)+b_{i,i}\tau(A)-\tau(A)\tau(B)\geqslant\min_{1\leqslant i\leqslant n}\left\{a_{i,i}\tau(B)+b_{i,i}\tau(A)-\tau(A)\tau(B)\right\}.\qquad\square\end{aligned}$$

Remark 3. Fang [4] has shown that the lower bound in (2) for $\tau(A\bigstar B)$ is sharper than the bound $\tau(A)\tau(B)$. We next give a simple comparison between the lower bound in (2) and the lower bound in (13). Without loss of generality, for $i\neq j$, assume that

$$a_{i,i}\tau(B)+b_{i,i}\tau(A)-\tau(A)\tau(B)\geqslant a_{j,j}\tau(B)+b_{j,j}\tau(A)-\tau(A)\tau(B).\tag{20}$$

Thus, we can write (20) equivalently as

$$-a_{i,i}\tau(B)-b_{i,i}\tau(A)+\tau(A)\tau(B)\leqslant -a_{j,j}\tau(B)-b_{j,j}\tau(A)+\tau(A)\tau(B).\tag{21}$$

That is

$$(a_{i,i}-\tau(A))(b_{i,i}-\tau(B))-a_{i,i}b_{i,i}\leqslant(a_{j,j}-\tau(A))(b_{j,j}-\tau(B))-a_{j,j}b_{j,j}.\tag{22}$$

Thus, from (22), we have

$$(a_{i,i}-\tau(A))(b_{i,i}-\tau(B))\leqslant(a_{j,j}-\tau(A))(b_{j,j}-\tau(B))+a_{i,i}b_{i,i}-a_{j,j}b_{j,j}.\tag{23}$$

From (13) and (23), we have

$$\begin{aligned}&\frac{1}{2}\Bigg\{a_{i,i}b_{i,i}+a_{j,j}b_{j,j}-\Big[(a_{i,i}b_{i,i}-a_{j,j}b_{j,j})^2\\&\quad+4(a_{i,i}-\tau(A))(b_{i,i}-\tau(B))(a_{j,j}-\tau(A))(b_{j,j}-\tau(B))\Big]^{\frac{1}{2}}\Bigg\}\\&\geqslant\frac{1}{2}\Bigg\{a_{i,i}b_{i,i}+a_{j,j}b_{j,j}-\Big[(a_{i,i}b_{i,i}-a_{j,j}b_{j,j})^2\\&\quad+4(a_{j,j}-\tau(A))(b_{j,j}-\tau(B))\Big((a_{j,j}-\tau(A))(b_{j,j}-\tau(B))+a_{i,i}b_{i,i}-a_{j,j}b_{j,j}\Big)\Big]^{\frac{1}{2}}\Bigg\}\\&=\frac{1}{2}\Bigg\{a_{i,i}b_{i,i}+a_{j,j}b_{j,j}-\Big[(a_{i,i}b_{i,i}-a_{j,j}b_{j,j})^2\\&\quad+4(a_{j,j}-\tau(A))^2(b_{j,j}-\tau(B))^2+4(a_{j,j}-\tau(A))(b_{j,j}-\tau(B))(a_{i,i}b_{i,i}-a_{j,j}b_{j,j})\Big]^{\frac{1}{2}}\Bigg\}\end{aligned}$$

$$= \frac{1}{2}\left\{a_{i,i}b_{i,i} + a_{j,j}b_{j,j} - \left[\left(a_{i,i}b_{i,i} - a_{j,j}b_{j,j} + 2(a_{j,j} - \tau(A))(b_{j,j} - \tau(B))\right)^2\right]^{\frac{1}{2}}\right\}$$
$$= \frac{1}{2}\left\{a_{i,i}b_{i,i} + a_{j,j}b_{j,j} - \left(a_{i,i}b_{i,i} + a_{j,j}b_{j,j} - 2a_{j,j}\tau(B) - 2b_{j,j}\tau(A) + 2\tau(A)\tau(B)\right)\right\}$$
$$= a_{j,j}\tau(B) + b_{j,j}\tau(A) - \tau(A)\tau(B). \tag{24}$$

Thus, from (13) and (24), we get

$$\tau(A\bigstar B) \geqslant \min_{i\neq j} \frac{1}{2}\left\{a_{i,i}b_{i,i} + a_{j,j}b_{j,j} - \left[(a_{i,i}b_{i,i} - a_{j,j}b_{j,j})^2 + 4(a_{i,i} - \tau(A))(b_{i,i} - \tau(B))(a_{j,j} - \tau(A))(b_{j,j} - \tau(B))\right]^{\frac{1}{2}}\right\}$$
$$\geqslant \min_{1\leqslant i\leqslant n}\left\{a_{i,i}\tau(B) + b_{i,i}\tau(A) - \tau(A)\tau(B)\right\}.$$

From Theorem 7 and [1, p. 380] we can obtain the following corollary:

Corollary 8. *Let A, B be two nonsingular M-matrices. Then we have*

$$|det(A\bigstar B)| \geqslant [\tau(A\bigstar B)]^n$$
$$\geqslant \min_{i\neq j} \frac{1}{2^n}\left\{a_{i,i}b_{i,i} + a_{j,j}b_{j,j} + \left[(a_{i,i}b_{i,i} - a_{j,j}b_{j,j})^2 + 4(a_{i,i} - \tau(A))(b_{i,i} - \tau(B))(a_{j,j} - \tau(A))(b_{j,j} - \tau(B))\right]^{\frac{1}{2}}\right\}^n$$
$$\geqslant \min_{1\leqslant i\leqslant n}\{a_{i,i}\tau(B) - b_{i,i}\tau(A) - \tau(A)\tau(B)\}^n \geqslant (\tau(A)\tau(B))^n.$$

Example 2. Consider two 3×3 M-matrices

$$A = \begin{bmatrix} 2 & -1 & 0 \\ 0 & 1 & -0.5 \\ -0.5 & -1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -0.25 & -0.25 \\ -0.5 & 1 & -0.25 \\ -0.25 & -0.5 & 1 \end{bmatrix}.$$

It is easy to show that $\tau(A) = 0.5402$, $\tau(B) = 0.3432$ and $\tau(A\bigstar B) = 0.8819$. By calculation, we have that $\tau(A)\tau(B) = 0.1854$. According to inequalities (2) and (13), we have

$$\tau(A\bigstar B) \geqslant \min_{1\leqslant i\leqslant 3}\{a_{i,i}\tau(B) + b_{i,i}\tau(A) - \tau(A)\tau(B)\} = 0.6980,$$

and

$$\tau(A\bigstar B) \geqslant \min_{i\neq j} \frac{1}{2}\left\{a_{i,i}b_{i,i} + a_{j,j}b_{j,j} - \left[(a_{i,i}b_{i,i} - a_{j,j}b_{j,j})^2 + 4(a_{i,i} - \tau(A))(b_{i,i} - \tau(B))(a_{j,j} - \tau(A))(b_{j,j} - \tau(B))\right]^{\frac{1}{2}}\right\} = 0.7655.$$

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