On two inequalities for the Hadamard product and the Fan product of matrices

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ABSTRACT

If A and B are n × n nonsingular M-matrices, a lower bound on the smallest eigenvalue τ(A⊙B) for the Fan product of A and B is given. In addition, using the estimate on the Perron root of nonnegative matrices, we also obtain an upper bound on the spectral radius ρ(A⊙B) for nonnegative matrices A and B. These bounds improve some existing results.

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1. Introduction

For a positive integer n, N denotes the set {1, 2, . . . , n}. The set of all n × n complex matrices is denoted by Cn×n and Rn×n denotes the set of all n × n real matrices throughout.

Let A = (aij) and B = (bij) be two real n × n matrices. Then, A ⩾ B (> B) if aij ⩾ bij (> bij) for all 1 ⩽ i ⩽ n, 1 ⩽ j ⩽ n. If O is the null matrix and A ⩾ O (> O), we say that A is a nonnegative (positive)
matrix. The spectral radius of $A$ is denoted by $\rho(A)$. If $A$ is a nonnegative matrix, the Perron–Frobenius theorem guarantees that $\rho(A) \in \sigma(A)$, where $\sigma(A)$ denotes the spectrum of $A$.

For $n \geq 2$, an $n \times n$ matrix $A \in \mathbb{C}^{n \times n}$ is reducible if there exists an $n \times n$ permutation matrix $P$ such that

$$p^T A P = \begin{bmatrix} A_{1,1} & A_{1,2} \\ 0 & A_{2,2} \end{bmatrix},$$

where $A_{1,1}$ is an $r \times r$ submatrix and $A_{2,2}$ is an $(n-r) \times (n-r)$ submatrix, where $1 \leq r < n$. If no such permutation matrix exists, then $A$ is irreducible. If $A$ is a $1 \times 1$ complex matrix, then $A$ is irreducible if its single entry is nonzero, and reducible otherwise.

Let $A$ be an irreducible nonnegative matrix. It is well known that there exists a positive vector $u$ such that $Au = \rho(A)u$, $u$ being called right Perron eigenvector of $A$.

The Hadamard product of $A \in \mathbb{C}^{n \times n}$ and $B \in \mathbb{C}^{n \times n}$ is defined by $A \circ B \equiv (a_{ij}b_{ij}) \in \mathbb{C}^{n \times n}$.

In [3, p. 358], there is a simple estimate for $\rho(A \circ B)$: if $A, B \in \mathbb{R}^{n \times n}$, $A \geq 0$, and $B \geq 0$, then $\rho(A \circ B) \leq \rho(A) \rho(B)$. From Exercise [3, p. 358], we know this inequality can be very weak by taking $B = I$, the matrix of all ones. For example, if $A = I$, $B = I$, then we have

$$\rho(A \circ B) - \rho(A) = 1 \leq \rho(A) \rho(B) = n$$

when $n$ is very large. But also clearly show that equality can occur (let $A = I$ and $B = I$).

Recently, Fang [4] gave an upper bound for $\rho(A \circ B)$, that is,

$$\rho(A \circ B) \leq \max_{1 \leq i \leq n} \{2a_{ij}b_{ij} + \rho(A)\rho(B) - a_{ij}\rho(B) - b_{ij}\rho(A)\} \quad (1)$$

which is sharper than the bound $\rho(A) \rho(B)$ in [3, p. 358].

For two nonnegative matrices $A$, $B$, we will give a new upper bound for $\rho(A \circ B)$ in Section 2. The bound is sharper than the bound $\rho(A) \rho(B)$ in [3, p. 358] and the bound $\max_{1 \leq i \leq n} \{2a_{ij}b_{ij} + \rho(A)\rho(B) - a_{ij}\rho(B) - b_{ij}\rho(A)\}$ in [4].

The set $Z_n \subset \mathbb{R}^{n \times n}$ is defined by

$$Z_n = \{A = (a_{ij}) \in \mathbb{R}^{n \times n} : a_{ij} \leq 0 \text{ if } i \neq j, i, j = 1, \ldots, n\}$$

the simple sign pattern of the matrices in $Z_n$ has many striking consequences. Let $A = (a_{ij}) \in Z_n$ and suppose $A = \alpha I - P$ with $\alpha \in \mathbb{R}$ and $P \geq 0$. Then $\alpha - \rho(P)$ is an eigenvalue of $A$, every eigenvalue of $A$ lies in the disc $\{z \in \mathbb{C} : |z - \alpha| \leq \rho(P)\}$, and hence every eigenvalue $\lambda$ of $A$ satisfies $\text{Re} \lambda \geq \alpha - \rho(P)$.

In particular, $A$ is an $M$-matrix if and only if $\alpha > \rho(P)$. If $A$ is an $M$-matrix, one may always write $A = yI - P$ with $y = \max |a_{ij} : i = 1, \ldots, n|$, $P = yI - A \geq 0$; necessarily, $y > \rho(P)$.

If $A = (a_{ij}) \in Z_n$, and if we denote $\min \{\text{Re} \lambda : \lambda \in \sigma(A)\}$ by $\tau(A)$. Basic for our purpose are the following simple facts (see Problem 16, 19 and 28 in Section 2.5 of [3]):

(i) $\tau(A) \in \sigma(A)$; $\tau(A)$ is called the minimum eigenvalue of $A$.
(ii) If $A, B \in Z_n$, and $A \geq B$, then $\tau(A) \geq \tau(B)$.
(iii) If $A \in Z_n$, then $\rho(A^{-1})$ is the Perron eigenvalue of the nonnegative matrix $A^{-1}$, and $\tau(A) = \frac{1}{\rho(A^{-1})}$ is a positive real eigenvalue of $A$.

Let $A$ be an irreducible nonsingular $M$-matrix. It is well known that there exists a positive vector $u$ such that $Au = \tau(A)u$, $u$ being called right Perron eigenvector of $A$.

Let $A \in \mathbb{C}^{n \times n}$, $B \in \mathbb{C}^{n \times n}$. The Fan product of $A$ and $B$ is denoted by $A \bowtie B \equiv C = (c_{ij}) \in \mathbb{C}^{n \times n}$ and is defined by

$$c_{ij} = \begin{cases} -a_{ij}b_{ij}, & \text{if } i \neq j, \\ a_{ij}b_{ij}, & \text{if } i = j. \end{cases}$$

If $A, B \in Z_n$ are $M$-matrices, then so is $A \bowtie B$. In [3, p. 359], a lower bound for $\tau(A \bowtie B)$ was given: Let $A, B \in Z_n$ be $M$-matrices. Then $A^{-1} \circ B^{-1} \geq (A \bowtie B)^{-1}$, and hence $\tau(A \bowtie B) \geq \tau(A) \tau(B)$. Fang [4] gave a sharper lower bound for $\tau(A \bowtie B)$, that is,

$$\tau(A \bowtie B) \geq \min_{1 \leq i \leq n} \{a_{ij} \tau(B) + b_{ij} \tau(A) - \tau(A) \tau(B)\}. \quad (2)$$
For two nonsingular $M$-matrices $A$ and $B$, we will give a new lower bound for $\tau(A \star B)$ in Section 3.

2. Inequalities for the Hadamard product of nonnegative matrices

In this section, we will give an upper bound for $\rho(A \circ B)$. In order to prove our results, we first give some Lemmas.

**Lemma 1** [1]. Let $A \in \mathbb{R}^{n \times n}$ be given. Then either $A$ is irreducible or there exists a permutation $P$ such that

$$p^T AP = \begin{bmatrix}
R_{1,1} & R_{1,2} & \cdots & R_{1,m} \\
O & R_{2,2} & \cdots & R_{2,m} \\
\vdots & \vdots & \ddots & \vdots \\
O & O & \cdots & R_{m,m}
\end{bmatrix},$$

where each square submatrix $R_{ij}$, $1 \leq j \leq m$, is either irreducible or a $1 \times 1$ null matrix.

**Remark 1.** Eq. (3) is said to be the normal form of a reducible matrix $A$. Clearly, the eigenvalues of $A$ are the eigenvalues of the square submatrices $R_{ij}$, $1 \leq j \leq m$ (cf. [5]).

**Lemma 2** [1]. Let $A \in \mathbb{R}^{n \times n}$ be a nonnegative matrix. If $A_k$ is a principal submatrix of $A$, then $\rho(A_k) \leq \rho(A)$. If, in addition, $A$ is irreducible and $A_k \neq A$, then $\rho(A_k) < \rho(A)$.

**Lemma 3** [2]. Let $A = (a_{ij}) \in \mathbb{R}^{n \times n}$ be a nonnegative matrix. Then

$$\rho(A) \leq \max_{i \neq j} \frac{1}{2} \left\{ a_{ii} + a_{jj} + \left( a_{ii} - a_{jj} \right)^2 + 4 \sum_{k \neq i, k \neq j} a_{ik} a_{jk} \right\}^{1/2}.$$  \hspace{1cm} (4)

**Lemma 4** [3]. Let $A, B \in \mathbb{C}^{n \times n}$ and if $D \in \mathbb{C}^{n \times n}$ and $E \in \mathbb{C}^{n \times n}$ are diagonal, then

$$D(A \circ B)E = (DAE) \circ B = (DA) \circ (BE) = (AE) \circ (DBE).$$

**Theorem 4.** If $A, B \in \mathbb{R}^{n \times n}$, $A \geq 0$, and $B \geq 0$, then

$$\rho(A \circ B) \leq \max_{i \neq j} \frac{1}{2} \left\{ a_{ii}b_{ii} + a_{jj}b_{jj} + \left( a_{ij}b_{ij} - a_{jj}b_{jj} \right)^2 + 4 \left( \rho(A) - a_{ii} \right) \left( \rho(B) - b_{jj} \right) \right\}^{1/2}. \hspace{1cm} (4)$$

**Proof.** It is clear that (4) holds with equality for $n = 1$.

We next assume that $n > 2$.

If $A \circ B$ is irreducible, then $A$ and $B$ are irreducible. From Lemma 2, we have

$$\rho(A) - a_{ii} > 0 \ \forall i \in N \hspace{1cm} (5)$$

and

$$\rho(B) - b_{jj} > 0 \ \forall i \in N. \hspace{1cm} (6)$$

Since $A = (a_{ij})$, $B = (b_{ij})$ are nonnegative irreducible, then there exists two positive vectors $u, v$ such that $Au = \rho(A)u$, $Bv = \rho(B)v$. Thus, we have

$$a_{ij} + \sum_{j \neq i} \frac{a_{ij}u_j}{u_i} = \rho(A) \hspace{1cm} (7)$$

and
\[ b_{ii} + \sum_{j \neq i} \frac{b_{ij}v_j}{v_i} = \rho(B). \] (8)

Define \( U = \text{diag}(u_1, \ldots, u_n), V = \text{diag}(v_1, \ldots, v_n) \). We know that \( U \) and \( V \) are nonsingular diagonal matrices. Let \( \hat{A} = (\hat{a}_{ij}) = U^{-1}AU \) and \( \hat{B} = (\hat{b}_{ij}) = V^{-1}BV \), then we have

\[
\hat{A} = (\hat{a}_{ij}) = U^{-1}AU = \begin{bmatrix}
\frac{a_{11}}{u_1} & \frac{a_{1,2}u_2}{u_1} & \cdots & \frac{a_{1,n}u_n}{u_1} \\
\frac{a_{2,1}u_1}{u_2} & \frac{a_{2,2}}{u_2} & \cdots & \frac{a_{2,n}u_n}{u_2} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{a_{n,1}u_1}{u_n} & \frac{a_{n,2}u_2}{u_n} & \cdots & \frac{a_{n,n}}{u_n}
\end{bmatrix},
\]

\[
\hat{B} = (\hat{b}_{ij}) = V^{-1}BV = \begin{bmatrix}
\frac{b_{1,1}}{v_1} & \frac{b_{1,2}v_2}{v_1} & \cdots & \frac{b_{1,n}v_n}{v_1} \\
\frac{b_{2,1}v_1}{v_2} & \frac{b_{2,2}}{v_2} & \cdots & \frac{b_{2,n}v_n}{v_2} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{b_{n,1}v_1}{v_n} & \frac{b_{n,2}v_2}{v_n} & \cdots & \frac{b_{n,n}}{v_n}
\end{bmatrix}.
\]

It is easy to show that \( \hat{A} \) and \( \hat{B} \) are nonnegative irreducible matrices, and all the row sums of \( \hat{A} \) are equal to \( \rho(A) \) and all the row sums of \( \hat{B} \) are equal to \( \rho(B) \).

Also let \( W = VU \), then \( W \) is nonsingular. From Lemma 4, we have

\[
(VU)^{-1}(A \circ B)(VU) = U^{-1}V^{-1}(A \circ B)VU = U^{-1}(A \circ (V^{-1}BV))U = (U^{-1}AU) \circ (V^{-1}BV) = \hat{A} \circ \hat{B}.
\]

Thus, we have that \( \rho(A \circ B) = \rho(\hat{A} \circ \hat{B}) \).

We next consider the spectral radius \( \rho(\hat{A} \circ \hat{B}) \) of \( \hat{A} \circ \hat{B} \). For nonnegative irreducible matrices \( \hat{A}, \hat{B} \), from Definition of the Hadamard product of \( \hat{A} \) and \( \hat{B} \), (5)–(8) and Lemma 3, we have

\[
\rho(\hat{A} \circ \hat{B}) \leq \max_{i \neq j} \frac{1}{2} \left\{ \hat{a}_{ii}\hat{b}_{jj} + \hat{a}_{ij}\hat{b}_{jj} \right. \\
+ \left[ (\hat{a}_{ij}\hat{b}_{ii} - \hat{a}_{ij}\hat{b}_{jj})^2 + 4 \sum_{k \neq i} \hat{a}_{ik}\hat{k}\hat{b}_{1k}\sum_{k \neq j} \hat{a}_{jk}\hat{k}\hat{b}_{jk} \right]^{\frac{1}{2}} \right\}
\]

\[
= \max_{i \neq j} \frac{1}{2} \left\{ a_{ij}b_{ii} + a_{ij}b_{jj} \right. \\
+ \left[ (a_{ij}b_{ii} - a_{ij}b_{jj})^2 + 4 \sum_{k \neq i} a_{ik}u_k b_{1k}v_k \sum_{k \neq j} a_{jk}u_k b_{jk}v_k \right]^{\frac{1}{2}} \right\}
\]

\[
\leq \max_{i \neq j} \frac{1}{2} \left\{ a_{ij}b_{ii} + a_{ij}b_{jj} + \left[ (a_{ij}b_{ii} - a_{ij}b_{jj})^2 \\
+ 4 \left( \sum_{k \neq i} a_{ik}u_k \sum_{k \neq i} b_{1k}v_k \right) \left( \sum_{k \neq j} a_{jk}u_k \sum_{k \neq j} b_{jk}v_k \right) \right]^{\frac{1}{2}} \right\}
\]

\[
= \max_{i \neq j} \frac{1}{2} \left\{ a_{ij}b_{ii} + a_{ij}b_{jj} + \left[ (a_{ij}b_{ii} - a_{ij}b_{jj})^2 \\
+ 4 \left( \sum_{k \neq i} a_{ik}u_k \sum_{k \neq i} b_{1k}v_k \right) \left( \sum_{k \neq j} a_{jk}u_k \sum_{k \neq j} b_{jk}v_k \right) \right]^{\frac{1}{2}} \right\}
\]

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If $A \circ B$ is reducible. We denote by $D = (d_{ij})$ the $n \times n$ permutation matrix with $d_{1,2} = d_{2,3} = \cdots = d_{n-1,n} = d_{n,1} = 1$, the remaining $d_{ij}$ zero, then both $A + tD$ and $B + tD$ are nonnegative irreducible matrices for any chosen positive real number $t$. Now we substitute $A + tD$ and $B + tD$ for $A$ and $B$, respectively in the previous case, and then letting $t \to 0$, the result follows by continuity.

Using ideas of the proof of Theorem 4, we give new proofs of inequality in [3, Observation 5.7.4] and inequality (1) in [4].

For inequality $\rho(A \circ B) \leq \rho(A)\rho(B)$.

From the proof of Theorem 4, we know that $\rho(A \circ B) = \rho(\hat{A} \circ \hat{B})$. Then we have

$$\rho(A \circ B) = \rho(\hat{A} \circ \hat{B}) \leq \max_{1 \leq i \leq n} \sum_{j=1}^{n} \hat{a}_{ij} \hat{b}_{ij} \leq \max_{1 \leq i \leq n} \sum_{j=1}^{n} \hat{a}_{ij} \max_{1 \leq i \leq n} \sum_{j=1}^{n} \hat{b}_{ij} = \rho(A)\rho(B).$$

For inequality $\rho(A \circ B) \leq \max_{1 \leq i \leq n} \{ 2a_{ij}b_{ij} + \rho(A)\rho(B) - a_{ij}\rho(B) - b_{ij}\rho(A) \}$.

Similar to the proof of Theorem 4, we have

$$\rho(A \circ B) = \rho(\hat{A} \circ \hat{B}) \leq \max_{1 \leq i \leq n} \sum_{j=1}^{n} \hat{a}_{ij} \hat{b}_{ij} = \max_{1 \leq i \leq n} \left( a_{ij}b_{ij} + \sum_{j \neq i} \hat{a}_{ij} \hat{b}_{ij} \right).$$

From (7) and (8), we have

$$\rho(A \circ B) \leq \max_{1 \leq i \leq n} \left( a_{ij}b_{ij} + \sum_{j \neq i} \hat{a}_{ij} \hat{b}_{ij} \right)$$

$$\leq \max_{1 \leq i \leq n} \left( a_{ij}b_{ij} + \sum_{j \neq i} \hat{a}_{ij} \sum_{j \neq i} \hat{b}_{ij} \right)$$

$$= \max_{1 \leq i \leq n} \left( a_{ij}b_{ij} + (\rho(A) - a_{ij})(\rho(B) - b_{ij}) \right)$$

$$= \max_{1 \leq i \leq n} \left( 2a_{ij}b_{ij} + \rho(A)\rho(B) - a_{ij}\rho(B) - b_{ij}\rho(A) \right).$$

**Remark 2.** Fang [4] has shown that the upper bound in (1) for $\rho(A \circ B)$ is sharper than the bound $\rho(A)\rho(B)$. We next give a simple comparison between the upper bound in (1) and the upper bound in (4).

Without loss of generality, for $i \neq j$, assume that

$$2a_{ij}b_{ij} + \rho(A)\rho(B) - a_{ij}\rho(B) - b_{ij}\rho(A) \geq 2a_{ij}b_{ij} + \rho(A)\rho(B) - a_{ij}\rho(B) - b_{ij}\rho(A).$$

Thus, we can write (10) equivalently as

$$a_{ij}b_{ij} + (\rho(A) - a_{ij})(\rho(B) - b_{ij}) \geq a_{ij}b_{ij} + (\rho(A) - a_{ij})(\rho(B) - b_{ij}).$$

From (4), we have

$$a_{ij}b_{ij} + a_{ij}b_{ij} + \left[ (a_{ij}b_{ij} - a_{ij}b_{ij})^2 + 4(\rho(A) - a_{ij})(\rho(B) - b_{ij})(\rho(A) - a_{ij})(\rho(B) - b_{ij}) \right]^{1/2}$$

$$\leq a_{ij}b_{ij} + a_{ij}b_{ij} + \left[ (a_{ij}b_{ij} - a_{ij}b_{ij})^2 + 4(\rho(A) - a_{ij})(\rho(B) - b_{ij})(\rho(A) - a_{ij})(\rho(B) - b_{ij}) \right]^{1/2}$$

$$= a_{ij}b_{ij} + a_{ij}b_{ij} + \left[ (a_{ij}b_{ij} - a_{ij}b_{ij})^2 + 4(\rho(A) - a_{ij})(\rho(B) - b_{ij})^2 \right]^{1/2}$$

$$+ 4(\rho(A) - a_{ij})(\rho(B) - b_{ij})(a_{ij}b_{ij} - a_{ij}b_{ij})^{1/2}$$

$$= a_{ij}b_{ij} + a_{ij}b_{ij} + \left[ (a_{ij}b_{ij} - a_{ij}b_{ij})^2 + 2(\rho(A) - a_{ij})(\rho(B) - b_{ij})^2 \right]^{1/2}$$

$$+ 4(\rho(A) - a_{ij})(\rho(B) - b_{ij})(a_{ij}b_{ij} - a_{ij}b_{ij})^{1/2}$$

$$= a_{ij}b_{ij} + a_{ij}b_{ij} + \left[ (a_{ij}b_{ij} - a_{ij}b_{ij})^2 + 2(\rho(A) - a_{ij})(\rho(B) - b_{ij})^2 \right]^{1/2}$$
According to inequalities (1) and (4), we have

\[ \rho(A \circ B) \leq \max_{i \neq j} \frac{1}{2} \left\{ a_{ij}b_{ij} + a_{ji}b_{ji} + \left( a_{ij}b_{ij} - a_{ji}b_{ji} \right)^2 \right. \\
+ 4(\rho(A) - a_{ij})(\rho(B) - b_{ij})(\rho(A) - a_{ji})(\rho(B) - b_{ij}) \left. \right\}^{\frac{1}{2}} \]

Thus, from (4) and (12), we have

\begin{align*}
\rho(A \circ B) & \leq \max_{1 \leq i \leq n} \frac{1}{2} \left\{ 2a_{ii}b_{ii} + 2(\rho(A) - a_{ii})(\rho(B) - b_{ii}) \right. \\
& = \max_{1 \leq i \leq n} \left\{ a_{ii}b_{ii} + (\rho(A) - a_{ii})(\rho(B) - b_{ii}) \right\} \\
& = \max_{1 \leq i \leq n} \left\{ 2a_{ii}b_{ii} + (\rho(A)\rho(B) - a_{ii}\rho(B) - b_{ii}\rho(A)) \right\}.
\end{align*}

Hence, the bound in (4) is sharper than the known one \( \rho(A)\rho(B) \) in [3] and the bound \( \max_{1 \leq i \leq n} \left\{ 2a_{ii}b_{ii} + (\rho(A)\rho(B) - a_{ii}\rho(B) - b_{ii}\rho(A)) \right\} \) in [4].

Consider the example in Introduction. Let \( A = I, B = J \), it is easy to show that \( \rho(A \circ B) = 1 \) and

\begin{align*}
\rho(A \circ B) & \leq \max_{i \neq j} \frac{1}{2} \left\{ a_{ij}b_{ij} + a_{ji}b_{ji} + \left( a_{ij}b_{ij} - a_{ji}b_{ji} \right)^2 \right. \\
& + 4(\rho(A) - a_{ij})(\rho(B) - b_{ij})(\rho(A) - a_{ji})(\rho(B) - b_{ji}) \left. \right\}^{\frac{1}{2}} \right\} = 1.
\end{align*}

We next give another example to validate our results.

**Example 1.** Consider two \( 4 \times 4 \) nonnegative matrices

\[
A = \begin{bmatrix}
4 & 1 & 0 & 2 \\
1 & 0.05 & 1 & 1 \\
0 & 1 & 4 & 0.5 \\
1 & 0.5 & 0 & 4
\end{bmatrix}, \quad B = \begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1
\end{bmatrix}.
\]

It is easy to show that \( \rho(A \circ B) = \rho(A) = 5.7339 \). By calculation, we have that \( \rho(A)\rho(B) = 22.9336 \). According to inequalities (1) and (4), we have

\[
\rho(A \circ B) \leq \max_{1 \leq i \leq 4} \left\{ 2a_{ii}b_{ii} + (\rho(A)\rho(B) - a_{ii}\rho(B) - b_{ii}\rho(A)) \right\} = 17.1017,
\]

and

\[
\rho(A \circ B) \leq \max_{i \neq j} \frac{1}{2} \left\{ a_{ij}b_{ij} + a_{ji}b_{ji} + \left( a_{ij}b_{ij} - a_{ji}b_{ji} \right)^2 \right. \\
+ 4(\rho(A) - a_{ij})(\rho(B) - b_{ij})(\rho(A) - a_{ji})(\rho(B) - b_{ji}) \left. \right\}^{\frac{1}{2}} \right\} = 11.6478.
\]

From Theorem 4 we can obtain the following corollary:

**Corollary 5.** Let \( A, B \) be two \( n \times n \) nonnegative matrices. Then we have

\[ |\det(A \circ B)| \leq [\rho(A \circ B)]^n \]
3. Inequalities for the Fan product of $M$-matrices

In this Section, we will give a lower bound for $\tau (A\#B)$.

Lemma 5. Let $A, B$ be two nonsingular $M$-matrices and if $D$ and $E$ are two positive diagonal matrices, then


Proof. Lemma 5 follows from Definition of Fan product. □

Theorem 7. Let $A = (a_{ij}), B = (b_{ij}) \in \mathbb{R}^{n \times n}$ be two nonsingular $M$-matrices. Then

$$\tau (A\#B) \geq \min_{i \neq j} \frac{1}{2} \left\{ a_{ij}b_{ii} + a_{ij}b_{jj} - \left[ (a_{ij}b_{ii} - a_{ij}b_{jj})^2 + 4(a_{i,j} - \tau (A))(b_{i,j} - \tau (B))((a_{i,j} - \tau (A))(b_{i,j} - \tau (B))) \right]^{\frac{1}{2}} \right\}. \quad (13)$$

Proof. It is quite evident that (13) holds with equality for $n = 1$.

We next assume that $n \geq 2$.

If $A\#B$ is irreducible, then $A$ and $B$ are irreducible. Since $A - \tau (A)I$ and $B - \tau (B)I$ are singular irreducible $M$-matrices, Theorem 6.4.16 of [1] yields that

$$a_{ij} - \tau (A) > 0 \quad \forall i \in N \quad (14)$$

and

$$b_{ij} - \tau (B) > 0 \quad \forall i \in N. \quad (15)$$

Since $A = (a_{ij}), B = (b_{ij})$ are irreducible nonsingular $M$-matrices, then there exists two positive vectors $u, v$ such that $Au = \tau (A)u, Bv = \tau (B)v$. Thus, we have

$$a_{ij} - \sum_{j \neq i} \frac{|a_{ij}|u_j}{u_i} = \tau (A) \quad (16)$$

and

$$b_{ij} - \sum_{j \neq i} \frac{|b_{ij}|v_j}{v_i} = \tau (B). \quad (17)$$

Define $\tilde{U} = \text{diag}(u_1, \ldots, u_n), \tilde{V} = \text{diag}(v_1, \ldots, v_n).$ Then we have that $\tilde{U}$ and $\tilde{V}$ are nonsingular diagonal matrices. Let $\tilde{A} = (\tilde{a}_{ij}) = \tilde{U}^{-1}\tilde{A}\tilde{U}$ and $\tilde{B} = (\tilde{b}_{ij}) = \tilde{V}^{-1}\tilde{B}\tilde{V},$ then we have

$$\tilde{A} = (\tilde{a}_{ij}) = \tilde{U}^{-1}\tilde{A}\tilde{U} = \begin{bmatrix} a_{1,1} & \frac{a_{1,2}u_2}{u_1} & \cdots & \frac{a_{1,n}u_n}{u_1} \\ \frac{a_{2,1}u_1}{u_2} & a_{2,2} & \cdots & \frac{a_{2,n}u_n}{u_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{a_{n,1}u_1}{u_n} & \frac{a_{n,2}u_2}{u_n} & \cdots & a_{n,n} \end{bmatrix}. \quad (18)$$
\[ \tilde{B} = (\tilde{b}_{ij}) = \tilde{V}^{-1}A\tilde{V} = \begin{bmatrix}
    b_{1,1} & b_{1,2}v_2 & \cdots & b_{1,n}v_n \\
    b_{2,1}v_1 & b_{2,2} & \cdots & b_{2,n}v_n \\
    \vdots & \vdots & \ddots & \vdots \\
    b_{n,1}v_1 & b_{n,2}v_2 & \cdots & b_{n,n}
\end{bmatrix}. \]

It is easy to show that \( \tilde{A} \) and \( \tilde{B} \) are also irreducible nonsingular \( M \)-matrices. Also let \( \tilde{W} = \tilde{V}\tilde{U} \), then \( \tilde{W} \) is nonsingular. From Lemma 6, we have

\[
(\tilde{V}\tilde{U})^{-1} (\tilde{A} \boxtimes \tilde{B}) (\tilde{V}\tilde{U}) = \tilde{U}^{-1} \tilde{V}^{-1} (A \boxtimes B) \tilde{U} \tilde{V} = \tilde{U}^{-1} (A \boxtimes (\tilde{V}^{-1}B\tilde{V})) \tilde{U} = (\tilde{U}^{-1}\tilde{A}\tilde{V}) \boxtimes (\tilde{V}^{-1}B\tilde{V}) = \tilde{A} \boxtimes \tilde{B}.
\]

Thus, we have that \( \tau(\tilde{A} \boxtimes \tilde{B}) = \tau(A \boxtimes B) \).

We next consider the minimum eigenvalue \( \tau(\tilde{A} \boxtimes \tilde{B}) \) of \( \tilde{A} \boxtimes \tilde{B} \). For irreducible nonsingular \( M \)-matrices \( \tilde{A}, \tilde{B} \), let \( \lambda \in \sigma(\tilde{A} \boxtimes \tilde{B}) \) satisfy \( \tau(\tilde{A} \boxtimes \tilde{B}) = \lambda \), then we have that \( 0 < \lambda < a_{ij}b_{ij}, \forall i \in N \). From Definition of the Fan product of \( \tilde{A} \) and \( \tilde{B} \), (14)–(17) and Theorem 1.23 of [5], there is a pair \((i, j)\) of positive integers with \( i \neq j \) such that

\[ |\lambda - a_{ij}b_{ij}| |\lambda - a_{jj}b_{jj}| \leq \sum_{k \neq i} | \tilde{a}_{ik} \tilde{b}_{ik} | \sum_{k \neq j} | \tilde{a}_{jk} \tilde{b}_{jk} |. \]

Thus, for \( i \neq j \), we have

\[
| \lambda - a_{ij}b_{ij} | | \lambda - a_{jj}b_{jj} | \leq \sum_{k \neq i} \frac{|a_{ik}| u_k}{u_i} \sum_{k \neq j} \frac{|b_{jk}| v_k}{v_j} \sum_{k \neq j} \frac{|a_{jk}| u_k}{u_j} \sum_{k \neq j} \frac{|b_{jk}| v_k}{v_j} = (a_{ij} - \tau(A))(b_{ij} - \tau(B))(a_{jj} - \tau(A))(b_{jj} - \tau(B)). \tag{18}
\]

From inequality (18) and \( 0 < \lambda < a_{ij}b_{ij}, \forall i \in N \), we have

\[
(\lambda - a_{ij}b_{ij}) (\lambda - a_{jj}b_{jj}) \leq (a_{ij} - \tau(A))(b_{ij} - \tau(B))(a_{jj} - \tau(A))(b_{jj} - \tau(B)). \tag{19}
\]

Thus, from inequality (19), we have

\[
\lambda \geq \frac{1}{2} \left\{ a_{ij}b_{ij} + a_{jj}b_{jj} - \left[ (a_{ij}b_{ij} - a_{jj}b_{jj})^2 + 4(a_{ij} - \tau(A))(b_{ij} - \tau(B))(a_{jj} - \tau(A))(b_{jj} - \tau(B)) \right]^{1/2} \right\}.
\]

That is

\[
\tau(A \boxtimes B) \geq \frac{1}{2} \left\{ a_{ij}b_{ij} + a_{jj}b_{jj} - \left[ (a_{ij}b_{ij} - a_{jj}b_{jj})^2 + 4(a_{ij} - \tau(A))(b_{ij} - \tau(B))(a_{jj} - \tau(A))(b_{jj} - \tau(B)) \right]^{1/2} \right\}
\geq \min_{i \neq j} \frac{1}{2} \left\{ a_{ij}b_{ij} + a_{jj}b_{jj} - \left[ (a_{ij}b_{ij} - a_{jj}b_{jj})^2 + 4(a_{ij} - \tau(A))(b_{ij} - \tau(B))(a_{jj} - \tau(A))(b_{jj} - \tau(B)) \right]^{1/2} \right\}.
\]

If \( A \boxtimes B \) is reducible. It is well known that a matrix in \( Z_n \) is a nonsingular \( M \)-matrix if and only if all its leading principal minors are positive (see condition (E17) of Theorem 6.2.3 of [1]). If we denote by
\[ D = (d_{ij}) \text{ the } n \times n \text{ permutation matrix with } d_{1,2} = d_{2,3} = \cdots = d_{n-1,n} = d_{n,1} = 1, \text{ the remaining } d_{ij} \text{ zero, then both } A - tD \text{ and } B - tD \text{ are irreducible nonsingular } M\text{-matrices for any chosen positive real number } t, \text{ sufficiently small such that all the leading principal minors of both } A - tD \text{ and } B - tD \text{ are positive. Now we substitute } A - tD \text{ and } B - tD \text{ for } A \text{ and } B, \text{ respectively in the previous case, and then letting } t \to 0, \text{ the result follows by continuity.} \]

Using ideas of the proof of Theorem 7, we next give a new proof of inequality (2) in [4].

Let \( \lambda \in \sigma (A \# B) \) satisfy \( \tau (A \# B) = \lambda \). Similar to the proof of Theorem 7, by theorem of Gerschgorin, we have
\[
|\lambda - a_{ij}b_{ij}| \leq \sum_{k \neq i} \left| \frac{a_{ik}u_k}{u_i} b_{jk}v_k \right|.
\]

Thus, we have
\[
a_{ij}b_{ij} - \lambda \leq \sum_{k \neq i} \frac{|a_{ik}|u_k}{u_i} \sum_{k \neq i} \frac{|b_{jk}|v_k}{v_i} = (a_{ij} - \tau (A))(b_{ij} - \tau (B)).
\]

Hence, we have
\[
\lambda \geq a_{ij}b_{ij} - (a_{ij} - \tau (A))(b_{ij} - \tau (B)) = a_{ij}\tau (B) + b_{ij}\tau (A) - (\tau (A)\tau (B)) \geq \min_{1 \leq i \leq n} \{a_{ij}\tau (B) + b_{ij}\tau (A) - (\tau (A)\tau (B)) \}. \quad \square
\]

**Remark 3.** Fang [4] has shown that the lower bound in (2) for \( \tau (A \# B) \) is sharper than the bound \( \tau (A)\tau (B) \). We next give a simple comparison between the lower bound in (2) and the lower bound in (13). Without loss of generality, for \( i \neq j \), assume that
\[
a_{ij}\tau (B) + b_{ij}\tau (A) - (\tau (A)\tau (B)) \geq a_{ij}\tau (B) + b_{ij}\tau (A) - (\tau (A)\tau (B)). \quad (20)
\]

Thus, we can write (20) equivalently as
\[
- a_{ij}\tau (B) - b_{ij}\tau (A) + (\tau (A)\tau (B)) \leq -a_{ij}\tau (B) - b_{ij}\tau (A) + (\tau (A)\tau (B)). \quad (21)
\]

That is
\[
(a_{ij} - \tau (A))(b_{ij} - \tau (B)) - a_{ij}b_{ij} \leq (a_{ij} - \tau (A))(b_{ij} - \tau (B)) - a_{ij}b_{ij}. \quad (22)
\]

Thus, from (22), we have
\[
(a_{ij} - \tau (A))(b_{ij} - \tau (B)) \leq (a_{ij} - \tau (A))(b_{ij} - \tau (B)) + a_{ij}b_{ij} - a_{ij}b_{ij}. \quad (23)
\]

From (13) and (23), we have
\[
\frac{1}{2}\left\{ a_{ij}b_{ij} + a_{jj}b_{jj} - \left[ (a_{ij}b_{iji} - a_{jj}b_{jj})^2 + 4(a_{ij} - \tau (A))(b_{ij} - \tau (B))(a_{jj} - \tau (A))(b_{jj} - \tau (B)) \right]^\frac{1}{2} \right\}
\]
\[
\geq \frac{1}{2}\left\{ a_{ij}b_{ij} + a_{jj}b_{jj} - \left[ (a_{ij}b_{iji} - a_{jj}b_{jj})^2 + 4(a_{jj} - \tau (A))(b_{jj} - \tau (B))\left( (a_{jj} - \tau (A))(b_{jj} - \tau (B)) + a_{ij}b_{ij} - a_{jj}b_{jj} \right) \right]^\frac{1}{2} \right\}
\]
\[
= \frac{1}{2}\left\{ a_{ij}b_{ij} + a_{jj}b_{jj} - \left[ (a_{ij}b_{iji} - a_{jj}b_{jj})^2 + 4(a_{jj} - \tau (A))^2(b_{jj} - \tau (B))^2 + 4(a_{jj} - \tau (A))(b_{jj} - \tau (B))(a_{ij}b_{ij} - a_{jj}b_{jj}) \right]^\frac{1}{2} \right\}
\]
have that helped us to correct some errors and improve the quality of the paper.

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\[
\begin{align*}
\frac{1}{2} \left\{ a_{i,i}b_{i,i} + a_{j,j}b_{j,j} - \left[ (a_{i,i}b_{i,i} - a_{j,j}b_{j,j} + 2(a_{i,j} - \tau(A))b_{j,j} - \tau(B)) \right]^2 \right\} \\
= \frac{1}{2} \left\{ a_{i,i}b_{i,i} + a_{j,j}b_{j,j} - \left[ (a_{i,i}b_{i,i} + a_{j,j}b_{j,j} - 2a_{i,j}\tau(B) - 2b_{j,j}\tau(A) + 2\tau(A)\tau(B)) \right] \right\} \\
= a_{j,j}\tau(B) + b_{j,j}\tau(A) - \tau(A)\tau(B).
\end{align*}
\]

(24)

Thus, from (13) and (24), we get

\[
\tau(A \bigstar B) \geq \min_{i \neq j} \frac{1}{2n} \left\{ a_{i,i}b_{i,i} + a_{j,j}b_{j,j} - \left[ (a_{i,i}b_{i,i} - a_{j,j}b_{j,j})^2 \\
+ 4(a_{i,i} - \tau(A))(b_{i,i} - \tau(B)) + (a_{j,j} - \tau(A))(b_{j,j} - \tau(B)) \right] \right\}^\frac{1}{2} \\
\geq \min_{1 \leq i \leq n} \left\{ a_{i,i}\tau(B) + b_{i,i}\tau(A) - \tau(A)\tau(B) \right\}.
\]

From Theorem 7 and [1, p. 380] we can obtain the following corollary:

**Corollary 8.** Let $A, B$ be two nonsingular $M$-matrices. Then we have

\[
|\det(A \bigstar B)| \geq |\tau(A \bigstar B)|^n
\]

\[
\geq \min_{i \neq j} \frac{1}{2^n} \left\{ a_{i,i}b_{i,i} + a_{j,j}b_{j,j} - \left[ (a_{i,i}b_{i,i} - a_{j,j}b_{j,j})^2 \\
+ 4(a_{i,i} - \tau(A))(b_{i,i} - \tau(B))(a_{j,j} - \tau(A))(b_{j,j} - \tau(B)) \right] \right\}^n \\
\geq \min_{1 \leq i \leq n} \left\{ a_{i,i}\tau(B) - b_{i,i}\tau(A) - \tau(A)\tau(B) \right\}^n \geq (\tau(A)\tau(B))^n.
\]

**Example 2.** Consider two $3 \times 3$ $M$-matrices

\[
A = \begin{bmatrix}
2 & -1 & 0 \\
0 & 1 & -0.5 \\
-0.5 & 1 & 2
\end{bmatrix}, \quad B = \begin{bmatrix}
1 & -0.25 & -0.25 \\
-0.5 & 1 & -0.25 \\
-0.25 & -0.5 & 1
\end{bmatrix}.
\]

It is easy to show that $\tau(A) = 0.5402$, $\tau(B) = 0.3432$ and $\tau(A \bigstar B) = 0.8819$. By calculation, we have that $\tau(A)\tau(B) = 0.1854$. According to inequalities (2) and (13), we have

\[
\tau(A \bigstar B) \geq \min_{1 \leq i \leq 3} \left\{ a_{i,i}\tau(B) - b_{i,i}\tau(A) - \tau(A)\tau(B) \right\} = 0.6980
\]

and

\[
\tau(A \bigstar B) \geq \min_{i \neq j} \frac{1}{2} \left\{ a_{i,i}b_{i,i} + a_{j,j}b_{j,j} - \left[ (a_{i,i}b_{i,i} - a_{j,j}b_{j,j})^2 \\
+ 4(a_{i,i} - \tau(A))(b_{i,i} - \tau(B))(a_{j,j} - \tau(A))(b_{j,j} - \tau(B)) \right] \right\} = 0.7655.
\]
References