Discrete Conjugate Boundary Value Problems

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(Received and accepted October 1998)

Abstract—This paper discusses higher-order discrete conjugate boundary value problems of singular and nonsingular type. © 2000 Elsevier Science Ltd. All rights reserved.

Keywords—Conjugate discrete boundary value problems, Singular problems, Nonsingular problems, Nonnegative solutions.

1. INTRODUCTION

This paper discusses the \( n \)th \((n \geq 2)\) order discrete conjugate boundary value problem

\[
(-1)^{n-p} \Delta^n y(i-p) = f(i, y(i)), \quad i \in J_p, \\
\Delta^i y(0) = 0, \quad 0 \leq i \leq p-1 \text{ (i.e., } y(0) = \cdots = y(p-1) = 0), \\
\Delta^i y(T+n-i) = 0, \quad 0 < i < n-p-1 \text{ (i.e., } y(T+p+1) = \cdots = y(T+n) = 0) ;
\]

(1.1)

here \( T \in \{1,2,\ldots\}, J_p = \{p,p+1,\ldots,T+p\}, 1 \leq p \leq n-1, \) and \( y : I_n = \{0,1,\ldots,T+n\} \rightarrow \mathbb{R} \).

We will let \( C(I_n) \) denote the class of maps \( w \) continuous on \( I_n \) (discrete topology) with norm \( ||w|| = \max_{k \in I_n} |w(k)| \). By a solution to (1.1), we mean a \( w \in C(I_n) \) such that \( w \) satisfies the difference equation in (1.1) for \( i \in J_p \) and \( w \) satisfies the conjugate boundary data. In this paper, we discuss separately the cases when \( f \) is nonsingular and when \( f \) is singular (i.e., \( f(i,u) \) singular at \( u = 0 \)). All the results are new and they extend and complement the theory in the literature [1-5]. Indeed this paper is the first time the singular higher-order discrete conjugate boundary value problem has been discussed (see [6] for results when \( n = 2 \)).

For the remainder of this introduction, we gather together some results which will be used in Sections 2 and 3. In [1-3,5], it was shown that if \( y \) satisfies

\[
\Delta^n y(k) = \phi(k), \quad k \in I_0 = \{0,1,\ldots,T\}, \\
\Delta^i y(0) = 0, \quad 0 \leq i \leq p-1, \\
\Delta^i y(T+n-i) = 0, \quad 0 \leq i \leq n-p-1,
\]

(1.2)
then

\[ y(k) = \sum_{j=0}^{T} G_3(k,j) \phi(j), \quad \text{for } k \in I_n; \]

here

\[ G_3(k,j) = \sum_{l=0}^{p-1} \left[ \sum_{i=0}^{p-l-1} \binom{n-p+i-1}{i} \frac{k^{(l+i)}}{(T+n-l)(n-p+i)} \right] \frac{(-j-1)^{(n-l-1)}}{l!(n-l-1)!} (T+n-k)^{(n-p)} \]

if \( j \in \{0,1,\ldots,k-1\} \), and

\[ G_3(k,j) = -\sum_{l=0}^{n-p-1} \left[ \sum_{i=0}^{n-p-l-1} \binom{p+i-1}{i} \frac{(T+p+l+i-k)^{(l+i)}}{(T+p+1+l+i)^{(p+i)}} \right] \frac{(-1)^l (T+p-j)^{(n-l+1)} k^p}{l!(n-l-1)!} \]

if \( j \in \{k,k+1,\ldots,T\} \). Next consider

\[ \Delta^n y(k-p) = \phi(k), \quad k \in J_p, \]
\[ \Delta^i y(0) = 0, \quad 0 \leq i \leq p-1, \]
\[ \Delta^i y(T+n-i) = 0, \quad 0 \leq i \leq n-p-1. \]  \hspace{1cm} (1.3)

Notice (1.3) is the same as

\[ \Delta^n y(k) = \phi(k+p), \quad k \in I_0, \]
\[ \Delta^i y(0) = 0, \quad 0 \leq i \leq p-1, \]
\[ \Delta^i y(T+n-i) = 0, \quad 0 \leq i \leq n-p-1, \]  \hspace{1cm} (1.4)

and so

\[ y(k) = \sum_{j=0}^{T} G_3(k,j) \phi(j+p), \quad \text{for } k \in I_n. \]  \hspace{1cm} (1.5)

This is the same as

\[ y(k) = \sum_{j=p}^{T+p} G_3(k,j-p) \phi(j), \quad \text{for } k \in I_n. \]  \hspace{1cm} (1.6)

We write

\[ y(k) = \sum_{j=p}^{T+p} G(k,j) \phi(j), \quad \text{for } k \in I_n, \]  \hspace{1cm} (1.7)

where

\[ G(k,j) = G_3(k,j-p). \]

Also in [2,3], the following result was established.

**Theorem 1.1.** Suppose \( y : I_n \to \mathbb{R} \) is such that

\[ (-1)^{n-p} \Delta^n y(k) \geq 0, \quad k \in I_0, \]
\[ \Delta^i y(0) = 0, \quad 0 \leq i \leq p-1, \]
\[ \Delta^i y(T+n-i) = 0, \quad 0 \leq i \leq n-p-1. \]  \hspace{1cm} (1.8)

Then

\[ y(k) \geq \theta \max_{j \in I_n} |y(j)| = \theta \|y\|, \quad \text{for } k \in J_p, \]  \hspace{1cm} (1.9)
where \(0 < \theta < 1\) is a constant given by

\[
\theta = \min\{b(p), b(p+1)\};
\]  

(1.10)

here \(b\) is given by

\[
b(x) = \frac{\min\{g(x, p), g(x, T + p)\}}{\min\{g(x, \lceil x \rceil), g(x, \lceil x \rceil + 1), g(x, p), g(x, T + p)\}}
\]  

(1.11)

with

\[
g(x, k) = k(x-1) (T + n - k)^{n-2} \quad \text{and} \quad \theta(x) = \frac{(x-1)T + (x-2)n + x}{n-1}
\]  

(1.12)

(note \([.\] denotes the greatest integer function).

Now suppose \(y : I_n \to \mathbb{R}\) satisfies

\[
(-1)^{n-p} \Delta^n y(k - p) \geq 0, \quad k \in J_p,
\]

\[
\Delta^i y(0) = 0, \quad 0 \leq i \leq p - 1,
\]

\[
\Delta^i y(T + n - i) = 0, \quad 0 \leq i \leq n - p - 1.
\]  

(1.13)

Of course, \((-1)^{n-p} \Delta^n y(k - p) \geq 0\) for \(k \in J_p\) is exactly the same as \((-1)^{n-p} \Delta^n y(k) \geq 0\) for \(k \in I_0\) and so

\[
y(k) \geq \theta \|y\| = \theta \max_{j \in I_n} |y(j)|, \quad \text{for } k \in J_p.
\]  

(1.14)

Next, we present an existence principle for the discrete conjugate boundary value problem

\[
(-1)^{n-p} \Delta^n y(k - p) = f(k, y(k)), \quad k \in J_p,
\]

\[
y(0) = a,
\]

\[
\Delta^i y(0) = 0, \quad 1 \leq i \leq p - 1,
\]

\[
y(T + n) = a,
\]

\[
\Delta^i y(T + n - i) = 0, \quad 1 \leq i \leq n - p - 1.
\]  

(1.15)

**Theorem 1.2.** Suppose \(f : J_p \times \mathbb{R} \to \mathbb{R}\) is continuous (i.e., continuous as a map from the topological space \(J_p \times \mathbb{R}\) into the topological space \(\mathbb{R}\) (of course, the topology on \(J_p\) will be the discrete topology)). Assume there is a constant \(M > \|a\|\), independent of \(\lambda\), with

\[
\|y\| = \max_{j \in I_n} |y(j)| \neq M
\]

for any solution \(y \in C(I_n)\) to

\[
(-1)^{n-p} \Delta^n y(k - p) = \lambda f(k, y(k)), \quad k \in J_p,
\]

\[
y(0) = a,
\]

\[
\Delta^i y(0) = 0, \quad 1 \leq i \leq p - 1,
\]

\[
y(T + n) = a,
\]

\[
\Delta^i y(T + n - i) = 0, \quad 1 \leq i \leq n - p - 1,
\]  

(1.16)\(\lambda\)

for each \(\lambda \in (0, 1)\). Then (1.15)\(\lambda\) has a solution.

**Proof.** Solving (1.16)\(\lambda\) is equivalent to finding a \(y \in C(I_n)\) which satisfies

\[
y(k) = a + \lambda \sum_{j=p}^{T+p} G(k, j) f(j, y(j)), \quad \text{for } k \in I_n;
\]  

(1.17)\(\lambda\)
here $G$ is as in (1.7). Define the operator $S: C(I_n) \to C(I_n)$ by setting

$$Sy(k) = a + \sum_{j=p}^{T+p} G(k,j) f(j,y(j)).$$

Now (1.17)

$$y = (1 - \lambda)a + \lambda Sy.$$

It is easy to see [3,7] that $S: C(I_n) \to C(I_n)$ is continuous and completely continuous. Let

$$U = \{ u \in C(I_n) : \| u \| < M \} \quad \text{and} \quad E = C(I_n).$$

The nonlinear alternative of Leray-Schauder [4] guarantees that $S$ has a fixed point in $U$, i.e., (1.15) has a solution. \[ \square \]

2. NONSINGULAR PROBLEMS

In this section, we are interested in establishing the existence of nonnegative solutions to discrete conjugate higher-order boundary value problems of nonsingular type. For convenience we discuss (1.1). However, we note that the ideas in this section could be used to discuss other higher-order discrete problems; for example, the $(n,p)$, focal, and conjugate problems in [3].

THEOREM 2.1. Suppose the following conditions are satisfied:

$$f: J_p \times [0, \infty) \to [0, \infty)$$

is continuous, (2.1)

there exists a continuous, nondecreasing function $\psi: [0, \infty) \to [0, \infty)$

with $\psi > 0$ on $(0, \infty)$ and a function $q: J_p \to [0, \infty)$ with $f(k,u) \leq q(k) \psi(u)$, for all $u \geq 0$ and $k \in J_p$ (2.2)

and

$$\sup_{c \in (0,\infty)} \left( \frac{c}{\psi(c)} \right) > Q; \quad \text{here} \quad Q = \max_{k \in I_n} \sum_{j=p}^{T+p} q(j) (-1)^{n-p} G(k,j).$$

Then (1.1) has a nonnegative solution.

PROOF. Consider the family of problems

$$(-1)^{n-p} \Delta^n y(k-p) = \lambda f^*(k,y(k)), \quad k \in J_p,$$

$$\Delta^i y(0) = 0, \quad 0 \leq i \leq p-1,$$

$$\Delta^i y(T+n-i) = 0, \quad 0 \leq i \leq n-p-1,$$

for $0 < \lambda < 1$; here

$$f^*(k,u) = \begin{cases} f(k,u), & u \geq 0, \\ f(k,0), & u \leq 0. \end{cases}$$

Let $y$ be any solution of (2.4) for $0 < \lambda < 1$. Then

$$y(k) = \lambda \sum_{j=p}^{T+p} (-1)^{n-p} G(k,j) f(j,y(j)) \geq 0, \quad \text{for} \quad k \in I_n.$$ (2.5)

For notational purposes let $y_0 = \sup_{k \in I_n} y(k)$. Let $M > 0$ satisfy

$$\frac{M}{\psi(M)} > Q.$$ (2.6)
From (2.5) we have for \( k \in I_n \),
\[
y(k) \leq \sum_{j=p}^{T+p} (-1)^{n-p} G(k, j) q(j) \psi(y(j)) \leq \psi(y_0) \sum_{j=p}^{T+p} (-1)^{n-p} G(k, j) q(j) \leq Q \psi(y_0)
\]
and so
\[
\frac{y_0}{\psi(y_0)} \leq Q. \tag{2.7}
\]
Now (2.6) together with (2.7) implies \( y_0 \neq M \). Thus, any solution \( y \) of (2.4), satisfies \( \|y\| \neq M \), i.e., \( y_0 \neq M \). Now Theorem 1.2 implies (2.4) has a solution \( y \) and, of course, \( y(k) \geq 0 \) for \( k \in I_n \). Thus, \( y \) is a solution of (1.1).

### 3. SINGULAR PROBLEMS

Next we discuss (1.1) when our nonlinearity \( f(i, y) \) may be singular at \( y = 0 \).

**Theorem 3.1.** Suppose the following conditions are satisfied:

\[
f : J_p \times (0, \infty) \rightarrow (0, \infty)
\]

is continuous,

\[
f(k, u) \leq g(u) + h(u) \text{ on } J_p \times (0, \infty) \text{ with } g > 0 \text{ continuous and nonincreasing on } (0, \infty), \ h \geq 0 \text{ continuous on } [0, \infty) \text{ and } h/g \text{ nondecreasing on } (0, \infty)
\]

for each constant \( H > 0 \) there exists a continuous function

\[
\psi_H : J_p \rightarrow (0, \infty) \text{ with } f(k, u) \geq \psi_H(k) \text{ on } J_p \times (0, H]
\]

there exists a constant \( K_\theta > 0 \) with \( g(\theta u) \leq K_\theta g(u) \) for all \( u \geq 0 \); here \( \theta \) is as in (1.10)

and

\[
\sup_{c \in (0, \infty)} \left( \frac{c}{g(c) + h(c)} \right) > K_\theta Q; \tag{3.5}
\]

here

\[
Q = \max_{k \in J_p} \sum_{j=p}^{T+p} (-1)^{n-p} G(k, j) \text{ and } G \text{ is as in (1.7)}.
\]

Then (1.1) has a solution \( y \in C(I_n) \) with \( y(i) > 0 \) for \( i \in J_p \).

**Proof.** Choose \( M > 0 \) with

\[
\frac{M}{Q K_\theta [g(M) + h(M)]} > 1. \tag{3.6}
\]

Next choose \( \epsilon > 0 \) and \( \epsilon < M \) with

\[
\frac{M}{Q K_\theta [g(M) + h(M)] + \epsilon} > 1. \tag{3.7}
\]

Let \( n_0 \in \{1, 2, \ldots\} \) be chosen so that \( 1/n_0 < \epsilon \) and let \( N_0 = \{n_0, n_0 + 1, \ldots\} \). We first show

\[
(-1)^{n-p} \Delta^n y(k - p) = f^**(k, y(k)), \quad k \in J_p,
\]

\[
y(0) = \frac{1}{m},
\]

\[
\Delta^i y(0) = 0, \quad 1 \leq i \leq p - 1,
\]

\[
y(T + n) = \frac{1}{m},
\]

\[
\Delta^i y(T + n - i) = 0, \quad 1 \leq i \leq n - p - 1,
\]

\[
(3.8)^m
\]
has a solution for each $m \in N_0$; here

$$f^{**}(k, u) = \begin{cases} f(k, u), & u \geq \frac{1}{m}, \\ f\left(k, \frac{1}{m}\right), & u < \frac{1}{m}. \end{cases}$$

To show (3.8)$^m$ has a solution for each $m \in N_0$, we will apply Theorem 1.2. Consider the family of problems

$$(-1)^{n-p} \Delta^n y(k - p) = \lambda f^{**}(k, y(k)), \quad k \in J_p,$$

$$y(0) = \frac{1}{m},$$

$$\Delta^i y(0) = 0, \quad 1 \leq i \leq p - 1,$$  \hspace{1cm} (3.9)$^m$

$$y(T + n) = \frac{1}{m},$$

$$\Delta^i y(T + n - i) = 0, \quad 1 \leq i \leq n - p - 1,$$

for $0 < \lambda < 1$. Let $y \in C(I_n)$ be any solution of (3.9)$^m$. Then

$$y(k) = \frac{1}{m} + \lambda \sum_{j=p}^{T+p} (-1)^{n-p} G(k, j) f^{**}(j, y(j)), \quad k \in I_n,$$  \hspace{1cm} (3.10)

and so $y(k) \geq 1/m$ for $k \in I_n$.

**Remark 3.1.** Any solution $u$ of (3.9)$_\lambda$ satisfies $u(k) \geq 1/m$ for $k \in I_n$ also.

We next claim that

$$\|y\| = \sup_{j \in I_n} y(j) \neq M, \quad (\text{here } M \text{ is as in (3.6))}$$  \hspace{1cm} (3.11)

for any solution $y$ to (3.9)$^\lambda$. To see this, let $y$ be any solution of (3.9)$_\lambda$ and let the absolute maximum of $y(k)$ be at say $i_0 \in J_p$. Then (3.10), (3.2), (1.14), and (3.4) (with $y(k) \geq 1/m$ for $k \in I_n$) imply

$$y(i_0) \leq \frac{1}{m} + \left(1 + \frac{h(y(i_0))}{g(y(i_0))}\right) \sum_{j=p}^{T+p} (-1)^{n-p} G(i_0, j) g(y(j))$$

$$\leq \epsilon + \left(1 + \frac{h(y(i_0))}{g(y(i_0))}\right) \sum_{j=p}^{T+p} (-1)^{n-p} G(i_0, j) g(\theta y(i_0))$$

$$\leq \epsilon + [g(y(i_0)) + h(y(i_0))] K_0 \sum_{j=p}^{T+p} (-1)^{n-p} G(i_0, j)$$

$$\leq \epsilon + [g(y(i_0)) + h(y(i_0))] K_0 Q.$$

Consequently,

$$\frac{y(i_0)}{\epsilon + [g(y(i_0)) + h(y(i_0))] K_0 Q} \leq 1.$$  \hspace{1cm} (3.12)

Now (3.7) and (3.12) imply $y(i_0) \neq M$ and so (3.11) is true. Consequently, Theorem 1.2 guarantees that (3.8)$^m$ has a solution $y_m \in C(I_n)$ with $1/m \leq y_m(i) \leq M$ for $i \in I_n$ and $y_m$ satisfies

$$(-1)^{n-p} \Delta^n y(k - p) = f(k, y(k)), \quad k \in J_p,$$

$$y(0) = \frac{1}{m},$$

$$\Delta^i y(0) = 0, \quad 1 \leq i \leq p - 1,$$

$$y(T + n) = \frac{1}{m},$$

$$\Delta^i y(T + n - i) = 0, \quad 1 \leq i \leq n - p - 1,$$
Next we obtain a sharper lower bound on $y_m$. Notice $y_m$ satisfies

$$y_m(i) = \frac{1}{m} + \sum_{j=p}^{T+p} (-1)^{n-p} G(i, j) f(j, y_m(j)), \quad \text{for } i \in I_n. \quad (3.13)$$

Also (3.3) guarantees the existence of a continuous function $\psi_M : J_p \to (0, \infty)$ with $f(i, u) \geq \psi_M(i)$ for $(i, u) \in J_p \times (0, M]$. This together with (3.13) yields

$$y_m(i) \geq \sum_{j=p}^{T+p} (-1)^{n-p} G(i, j) \psi_M(j) \equiv \Phi_M(i), \quad \text{for } i \in J_p. \quad (3.14)$$

Clearly,

$$\{y_m\}_{m \in N_0} \text{ is a bounded family on } I_n. \quad (3.15)$$

The Arzela-Ascoli Theorem [2] guarantees the existence of a subsequence $N$ of $N_0$ and a function $y \in C(J_n)$ with $y_n \to y$ in $C(I_n)$ as $n \to \infty$ through $N$. Also

$$y(0) = \cdots = y(p - 1) = y(T + p + 1) = \cdots = y(T + n) = 0.$$

Fix $i \in J_p$. Then $y_m, m \in N$, satisfies (3.13). Also

$$\Phi_M = \min_{i \in J_p} \Phi_M(i) \leq y_m(j) \leq M, \quad \text{for } j \in J_p \text{ and } m \in N. \quad (3.16)$$

Let $m \to \infty$ through $N$ in (3.13) to obtain

$$y(i) = \sum_{j=p}^{T+p} (-1)^{n-p} G(i, j) f(j, y(j)), \quad \text{for } i \in J_p$$

and so $(-1)^{n-p} \Delta^n y(i - p) = f(i, y(i))$ for $i \in J_p$. Also notice that (3.16) implies $y(j) \geq \Phi_M > 0$ for $j \in J_p$.

**Example 3.1.** Consider the boundary value problem

$$(-1)^{n-p} \Delta^n y(k - p) = \mu \left( [y(k)]^{-\alpha} + A [y(k)]^\beta + B \right), \quad \text{for } k \in J_p,$$

$$\Delta^i y(0) = 0, \quad 0 \leq i \leq p - 1, \quad (3.17)$$

$$\Delta^i y(T + n - i) = 0, \quad 0 \leq i \leq n - p - 1,$$

with $\alpha > 0, \beta \geq 0, A \geq 0, B \geq 0$, and $\mu > 0$. If

$$\mu < \frac{\theta^\alpha}{Q} \sup_{c \in (0, \infty)} \left( \frac{c^{\alpha+1}}{1 + A c^{\alpha+\beta} + B c^\alpha} \right) \quad (3.18)$$

(here $\theta$ is as in (1.10) and $Q$ is as in the statement of Theorem 3.1) then (3.17) has a solution $y \in C(I_n)$ with $y(i) > 0$ for $i \in J_p$.

**Remark 3.2.** If $\beta < 1$, then (3.18) is true for all $\mu > 0$.

The result follows immediately from Theorem 3.1 with $g(u) = \mu u^{-\alpha}$ and $h(u) = \mu [A u^\beta + B]$. Clearly, (3.1), (3.2), (3.3) (with $\psi_H = \mu H^{-\alpha}$), and (3.4) (with $K_\theta = \theta^{-\alpha}$) are satisfied. Also, (3.18) guarantees that (3.5) is true. Existence of a solution is now guaranteed from Theorem 3.1.
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