An extended Heath–Jarrow–Morton risk-neutral drift

Leonard Tchuindjo*
School of Engineering and Applied Science, George Washington University, 1776 G Street NW, Washington DC 20052, USA

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Using a finite dimensional Hilbert space framework, this work proposes a new derivation of the HJM [D. Heath, R. Jarrow, A. Morton, Bond pricing and the term structure of interest rates: A new methodology for contingent claims valuation, Econometrica 60 (1992) 77–105] risk-neutral drift that takes into account nonzero instantaneous correlations between factors. The results obtained generalize the original HJM risk-neutral drift and provide an approach by which interest rate derivatives can be priced using functions of directly observable factors.

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1. Introduction

The methodology of [6] (HJM hereafter) for the term structure modeling provides a unified framework for pricing and hedging interest rate derivative securities. This methodology assumes that all factors driving the interest rate are orthogonal, uses the equivalent martingale measure in the pricing procedure, and calibrates the model to the current yield curve without considering the market price of risk. In this framework, the term structure is described by the forward rates, which follow stochastic differential equations with specific drift and volatility terms. Under the risk-neutral measure the market no-arbitrage condition imposes a relationship, called the risk-neutral drift, between the drift and the volatility terms.

According to the original HJM model, the volatility terms are driven by uncorrelated Brownian motions. Although correlated factors can be appropriately orthogonalized before being used, such orthogonalized factors are mathematical constructs that lack financial meanings. Therefore, in a multiple-factor environment, the connection between interest rate derivative prices and some market observable variables is unclear. In an attempt to address this issue, this work derives a more general risk-neutral drift, assuming there are nonzero instantaneous correlations between Brownian motions driving the forward rates. This generalization of the HJM model with correlations has the appealing feature of preserving the economic meaning of factors driving the forward rates.

The rest of this work is organized as follows. The next section is an overview of the HJM model. Section 3 presents the derivation of an extended HJM risk-neutral drift in a finite dimensional Hilbert space. Section 4 is a brief discussion about the main result of the work.

2. An overview of the HJM model

To provide an overview of the HJM model on a finite time horizon $T \in [0, \infty)$, we consider a complete filtered probability space $(\Omega, \mathcal{F}_T, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})$, where: $\Omega$ is the set of all possible states of nature; $\mathbb{P}$ is a physical measure that attaches objective probabilities to subsets of $\Omega$; $\mathcal{F}_T$ is the $\sigma$-algebra representing measurable events; and

* Tel.: +1 202 994 7905.
E-mail address: tleonard@gwu.edu.

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Let the Cameron–Martin–Girsanov theorem,

\[ Q \]

Proof. The correlation matrix at time \( t \).

\[ n \]

considered to be a martingale measure, \( Q \).

\[ \sum_{i=1}^{n} \sigma_i(t, T, \omega_t) dW_n^i(t, \omega_t), \]

\[ df(t, T, \omega_t) = \mu(t, T, \omega_t) dt + \sum_{i=1}^{n} \sigma_i(t, T, \omega_t) dW_n^i(t, \omega_t), \]

where: \( \mu(t, T, \omega_t) : \{(t, T) : t \in [0, T]\} \times \Omega \rightarrow \mathbb{R} \) is the drift, which is \( \mathbb{P} \)-almost everywhere absolutely integrable on any finite time horizon; and the volatilities \( \sigma_i(t, \cdot, \omega_t) \) belong to \( \tilde{\mathfrak{g}} \), the set of all functions defined from \( \{(t, T) : t \in [0, T]\} \times \Omega \) onto \( \mathbb{R} \), that are \( \mathbb{P} \)-almost everywhere non-negative, bounded, square integrable on any finite time horizon, and Lipschitz continuous with respect to the second variable. Further the \( \sigma_i(\cdot, \cdot, \omega_t) \) are jointly measurable from \( \mathcal{B} \{(t, T) : t \in [0, T]\} \times \mathcal{F}_T \rightarrow \mathcal{B} \), where \( \mathcal{B} \) is the Borel \( \sigma \)-algebra restricted to \( [0, T] \).

Using the asset pricing theory of [4,5], HJM show that the absence of arbitrage implies that under the equivalent martingale measure, \( Q \), the stochastic differential equation of the instantaneous forward rate is given by

\[ df(t, T, \omega_t) = \sum_{i=1}^{n} \sigma_i(t, T, \omega_t) \int_{t}^{T} \sigma_i(t, s, \omega_t) ds + \sum_{i=1}^{n} \sigma_i(t, T, \omega_t) dW_n^i(t, \omega_t), \]

\[ \omega \text{ for the case where the Brownian motions driving the forward rate dynamics are correlated. Hereafter, for simplicity of notation, the argument \( \omega_t \) representing the path dependence of the process will be omitted.} \]

3. An extension of the HJM risk-neutral drift

In this section we assumed that all factors driving the forward rate are correlated. Thus we are going to derive a more general risk-neutral drift under the equivalent measure, such that the original HJM result becomes a particular case. A finite dimensional Hilbert space is used to prepare a framework for introducing a time dependent correlation matrix in the model.

Let \( H_n \) be the vector space of real column vectors of nonzero dimension \( n \) (hereafter all vectors are column vectors unless otherwise specified). We equip \( H_n \) with a real inner product \( \langle \cdot, \cdot \rangle : H_n \times H_n \rightarrow \mathbb{R} \), defined by \( (A, B) = A^T B \) for all \( (A, B) \in H_n^2 \), where \( A^T \) represents the transpose of \( A \). Thus, \( H_n \) becomes a pre-Hilbert space. Further, \( H_n \) can be considered to be a \( n \)-dimensional Hilbert space, as it is isometric to \( \mathbb{R}^n \).

Let \( \{W^i(t) : t \in [0, T]\} \) be a vector of \( n \) correlated \( (\mathcal{F}_t)_{0 \leq t \leq T} \)-adapted standard Brownian motions on \( (\Omega, \mathcal{F}, \mathbb{P}) \), such that almost everywhere with respect to the physical probability measure \( \mathbb{P} \) and the sigma algebra \( \mathcal{F}_t \), we have

\[ dW^i(t) = K(t) dt, \]

where \( K(t) \) is the \( n \times n \) correlation coefficient matrix, at time \( t \), between the Brownian motions in the vector.

Let \( \{\Gamma^i(t) : t \in [0, T]\} \) be a vector of \( n \) \( (\mathcal{F}_t)_{0 \leq t \leq T} \)-predictable stochastic processes satisfying Novikov’s condition. A new probability measure \( Q \), equivalent to \( \mathbb{P} \), can be defined on \( \mathcal{F}_t \), such that \( dQ/d\mathbb{P}_{\mathcal{F}_t} = \exp\left(\int_{0}^{t} \Gamma^i(s) dW^i(s) - (1/2) \int_{0}^{t} \Gamma^i(s) \Gamma^i(s) ds\right) \) is the Radon–Nikodym derivative, at time \( t \), of the measure \( Q \) with respect to the measure \( \mathbb{P} \).

Let \( \{W^Q(t) : t \in [0, T]\} \), such that \( W^Q(t) = W^i(t) - \int_{0}^{t} \Gamma^i(s) ds \), be a vector of \( n \) \( (\mathcal{F}_t)_{0 \leq t \leq T} \)-adapted stochastic processes.

**Theorem 1.** \( \{W^Q(t) : t \in [0, T]\} \) is a vector of \( n \) correlated \( (\mathcal{F}_t)_{0 \leq t \leq T} \)-adapted \( Q \)-standard Brownian motions having \( K(t) \) as their correlation matrix at time \( t \).

**Proof.** By the Cameron–Martin–Girsanov theorem, \( \{W^Q(t) : t \in [0, T]\} \) as defined in the above equation is an \( n \)-dimensional \( Q \)-standard Brownian motion. It remains to prove that \( dW^Q(t) (dW^Q(t))^T = K(t) dt \) almost everywhere with respect to the measure \( Q \) and the \( \sigma \)-algebra \( \mathcal{F}_t \).

As the Brownian motion is a square integrable martingale (see e.g., [7, p. 48]), \( \mathbb{E}^Q \left[ dW^Q(t) (dW^Q(t))^T \mathcal{F}_t \right] < \infty \). By the change of measure theorem,

\[ \mathbb{E}^Q \left[ dW^Q(t) (dW^Q(t))^T \mathcal{F}_t \right] = \mathbb{E}^P \left[ dQ/d\mathbb{P}_{\mathcal{F}_t} dW^Q(t) (dW^Q(t))^T \mathcal{F}_t \right]. \]
As \((\mathcal{F}_t)_{0\leq t\leq T}\) is a filtration, \(\mathcal{F}_t \subseteq \mathcal{F}_T\). Hence, by the law of iterated expectation
\[
E^Q \left[ dW^Q(t) \left( dW^Q(t) \right)^T \mid \mathcal{F}_t \right] = E^P \left[ E^Q \left[ dW^Q(t) \left( dW^Q(t) \right)^T \mid \mathcal{F}_t \right] \mid \mathcal{F}_T \right].
\]

Applying Fubini’s theorem to change the order of the two expectations with respect to the physical probability measure \(P\), we have
\[
E^Q \left[ dW^Q(t) \left( dW^Q(t) \right)^T \mid \mathcal{F}_t \right] = E^P \left[ E^Q \left[ dW^Q(t) \left( dW^Q(t) \right)^T \mid \mathcal{F}_t \right] \mid \mathcal{F}_T \right].
\]

As \(W^Q(t)\) is \(\mathcal{F}_t\)-measurable, \(dW^Q(t) \left( dW^Q(t) \right)^T\) can be pulled out from the inner expectation. Thus, the preceding equation can be rewritten as
\[
E^Q \left[ dW^Q(t) \left( dW^Q(t) \right)^T \mid \mathcal{F}_t \right] = E^P \left[ dW^Q(t) \left( dW^Q(t) \right)^T \mid \mathcal{F}_T \right].
\]

Hence,
\[
E^Q \left[ dW^Q(t) \left( dW^Q(t) \right)^T \mid \mathcal{F}_t \right] = E^P \left[ dW^P(t) \left( dW^P(t) \right)^T \mid \mathcal{F}_T \right].
\]

By substituting the expression for \(dW^Q(t)\) into the right hand side of the above equation, we have
\[
E^Q \left[ dW^Q(t) \left( dW^Q(t) \right)^T \mid \mathcal{F}_t \right] = E^P \left[ dW^P(t) \left( dW^P(t) \right)^T \mid \mathcal{F}_T \right] + E^P \left[ \Gamma(t) \left( \Gamma(t) \right)^T (dt)^2 - \Gamma(t) \left( \Gamma(t) \right)^T dt \right.
\]
\[
- \left. \left. dW^P(t) \left( \Gamma(t) \right)^T dt \right| \mathcal{F}_T \right].
\]

According to the box algebra multiplication table, \(dW^P(t)dt = (dt)^2 = 0\) almost everywhere with respect to \(P\) (see, e.g., [9, p. 124]). Hence
\[
E^Q \left[ dW^Q(t) \left( dW^Q(t) \right)^T \mid \mathcal{F}_t \right] = E^P \left[ dW^P(t) \left( dW^P(t) \right)^T \mid \mathcal{F}_T \right] = K(t) dt. \quad \square
\]

**Theorem 1** claims that the change of measure preserves the correlation structure, between standard Brownian motions. In what follows, we use the result this theorem to generalize the HJM risk-neutral drift restriction. For this purpose, let us assume that the stochastic differential equation of the forward rate is given by
\[
df(t, T) = \alpha(t, T) dt + \left\{ \Gamma(t) \right\} \cdot \left( \sigma(t) \right),
\]
where: \(\alpha(t, T)\) is the drift satisfying the properties of the drift of Eq. (1); \(\left\{ \Gamma(t) \right\} \) is a vector of \(n\) correlated \(P\)-standard Brownian motions defined as above; and \(\Gamma(t, T)\) is a vector of \(n\) volatility functions defined from \(\{\left( t, T \right) : t \in [0, T] \} \times \Omega\) onto \(\mathbb{R}^n\), and satisfying the properties of the volatilities of Eq. (1).

In stochastic integral form, Eq. (4) can be rewritten as
\[
f(t, T) = f(0, T) + \int_0^T \alpha(s, T) ds + \int_0^T \left\{ \Gamma(s, T) / \sigma(s) \right\} ds.
\]

Let \(B(t, T) = \exp \left\{ - \int_0^t f(s, T) ds \right\}\) be the price at time \(t\) of a zero-coupon bond maturing at time \(T\). By Leibnitz’s rule of differentiation,
\[
\frac{d}{dt} \ln B(t, T) = \left[ \frac{d}{dt} \left( \int_0^T f(t, s) ds \right) \right] dt = f(t, T) dt - \int_0^T df(t, s) ds.
\]

Substituting \(f(t, T)\) from Eq. (5) and \(df(t, s)\) from Eq. (4) into Eq. (6), we obtain
\[
\frac{d}{dt} \ln B(t, T) = f(0, T) dt + \int_0^T \alpha(t, s) ds dt + \left( \int_0^T \left\{ V(s, t) / \sigma(s) \right\} ds \right) dt
\]
\[
- \int_0^T \alpha(t, s) ds dt - \int_0^T \left\{ V(t, s) / \sigma(s) \right\} ds
\]
\[
= \left[ \int_0^T \alpha(t, s) ds + \int_0^T \left\{ V(s, t) / \sigma(s) \right\} ds \right] dt - \left( \int_0^T V(t, s) ds \right) .
\]

(7)
Applying Ito’s lemma to the preceding equation, the stochastic differential equation of the zero-coupon bond price can be written as

\[
\text{dB}(t, T)/B(t, T) = \left[ f(0, t) + \int_0^t \alpha(s, t) \, ds + \int_0^t \langle V(s, t), \, dW^q(s) \rangle - \int_0^T \alpha(t, s) \, ds \right] dt \\
- \left\{ \int_t^T V(t, s) \, ds, \, dW^p(t) \right\} + \frac{1}{2} \left[ \left( \int_t^T \langle V(t, s), \, dW^p(t) \rangle \right)^2 \right].
\] (8)

As \( \left\{ \int_t^T V(t, s) \, ds, \, dW^p(t) \right\} \) is a scalar and \( \langle \cdot, \cdot \rangle \) is a symmetric form on \( H_n \), we have

\[
\left[ \left( \int_t^T V(t, s) \, ds, \, dW^p(t) \right)^2 \right] = \left[ \left( \int_t^T V(t, s) \, ds \right) dW^p(t) \right] \left[ \left( dW^p(t) \right)^T \left( \int_t^T V(t, s) \, ds \right) \right]
\]

\[
= \left( \int_t^T V(t, s) \, ds \right) \left[ dW^p(t) \left( dW^p(t) \right)^T \right] \int_t^T V(t, s) \, ds.
\]

Using Eq. (3), the preceding equation can be rewritten as

\[
\left[ \left( \int_t^T V(t, s) \, ds, \, dW^p(t) \right)^2 \right] = \left( \int_t^T V(t, s) \, ds, \, K(t) \int_t^T V(t, s) \, ds \right) dt.
\] (9)

Hence, Eq. (8) becomes

\[
\text{dB}(t, T)/B(t, T) = \left[ f(0, t) + \int_0^t \alpha(s, t) \, ds + \int_0^t \langle V(s, t), \, dW^q(s) \rangle - \int_0^T \alpha(t, s) \, ds \right] dt \\
+ \left\{ \int_t^T V(t, s) \, ds, \, K(t) \int_t^T V(t, s) \, ds \right\} dt - \left\{ \int_t^T V(t, s) \, ds, \, dW^p(t) \right\}.
\] (10)

Let us define a nonlinear operator, \( D \), on \( \mathbb{R}^n \) such that for all \( V(\cdot, \cdot) \in \mathbb{R}^n \),

\[
D[V(t, T)] = f(0, t) + \int_0^t \alpha(s, t) \, ds - \int_0^T \alpha(t, s) \, ds + \int_0^t \langle V(s, t), \, dW^q(s) \rangle \\
+ \frac{1}{2} \left\{ \int_t^T V(t, s) \, ds, \, K(t) \int_t^T V(t, s) \, ds \right\}.
\] (11)

By the Hilbert space representation theorem, we can find an \( n \)-dimensional bounded \( (\mathcal{F}_t)_{0 \leq t \leq T} \)-predictable stochastic process, \( \{\Phi(t) : t \in [0, T]\} \), such that

\[
D[V(t, T)] = \left\{ \int_t^T V(t, s) \, ds, \, \Phi(t) \right\}.
\] (12)

Eqs. (11) and (12) imply that

\[
\left\{ \int_t^T V(t, s) \, ds, \, \Phi(t) \right\} = f(0, t) + \int_0^t \alpha(s, t) \, ds - \int_0^T \alpha(t, s) \, ds + \int_0^t \langle V(s, t), \, dW^q(s) \rangle \\
+ \frac{1}{2} \left\{ \int_t^T V(t, s) \, ds, \, K(t) \int_t^T V(t, s) \, ds \right\}.
\] (13)

\( \Phi(t) \) provides a proportional relationship between the drift rate of change of prices and the amount of risk in price changes stemming from each Brownian motion. \( \Phi(t) \) represents the \( n \)-dimensional market price of risk at time \( t \). Provided that \( \Phi(t) \) satisfies Novikov’s condition, by Theorem 1 the process \( \{W^q(t) : t \in [0, T]\} \) defined by

\[
dW^q(t) = dW^p(t) - \Phi(t) \, dt
\] (14)

is an \( n \)-dimensional vector of correlated \( Q \)-standard Brownian motions, having \( K(t) \) as their correlation matrix.

By differentiating both sides of Eq. (14), with respect to \( T \), we obtain

\[
\alpha(t, T) = \left\{ V(t, T), \, K(t) \int_t^T V(t, s) \, ds \right\} - \langle V(t, T), \, \Phi(t) \rangle.
\] (15)
Substituting Eq. (15) into Eq. (4), the forward rate dynamics under the equivalent probability measure \( \mathbb{Q} \) is given by the following equation:

\[
\begin{align*}
\frac{df(t, T)}{dt} &= \left[ V(t, T) \cdot K(t) \int_t^T V(t, s)ds \right] - \langle V(t, T), \Phi(t) \rangle \ dt + \langle V(t, T), dW^\mathbb{Q}(t) \rangle.
\end{align*}
\]

As \( \langle \cdot, \cdot \rangle \) is a bilinear form on \( H_0 \), the preceding equation becomes

\[
\begin{align*}
\frac{df(t, T)}{dt} &= \left[ V(t, T) \cdot K(t) \int_t^T V(t, s)ds \right] dt + \langle V(t, T), (dW^\mathbb{Q}(t) - \Phi(t)dt) \rangle.
\end{align*}
\]

Using Eq. (14), the preceding equation becomes

\[
\begin{align*}
\frac{df(t, T)}{dt} &= \left[ V(t, T) \cdot K(t) \int_t^T V(t, s)ds \right] dt + \langle V(t, T), dW^\mathbb{Q}(t) \rangle. \quad (16)
\end{align*}
\]

### 4. Discussion

In Eq. (16) if the vector of Brownian motions, the vector of volatilities and the correlation matrix are, respectively, explicitly given by

\[
\begin{align*}
W^\mathbb{Q}(t) &= \left[ W^\mathbb{Q}_1(t), W^\mathbb{Q}_2(t), \ldots, W^\mathbb{Q}_n(t) \right]^T, \\
V(t, T) &= [\sigma_1(t, T), \sigma_2(t, T), \ldots, \sigma_n(t, T)]^T,
\end{align*}
\]

and

\[
K(t) = \left[ \rho_{ij}(t) \right]_{(i,j = 1,2,\ldots,n)}, \quad \text{such that} -1 \leq \rho_{ij}(t) = \rho_{ji}(t) \leq 1 \quad \text{and} \quad \rho_{ii}(t) = 1,
\]

then Eq. (16) can be rewritten as

\[
\begin{align*}
\frac{df(t, T)}{dt} &= \sum_{i=1}^n \left\{ \sum_{j=1}^n \left[ \rho_{ij} \sigma_i(t, T) \int_t^T \sigma_j(t, s)ds \right] \right\} + \sum_{i=1}^n \sigma_i(t, T) dW^\mathbb{Q}_i(t) \\
&= \sum_{i=1}^n \left\{ \sum_{j=1}^n \left[ \rho_{ij} \sigma_i(t, T) \int_t^T \sigma_j(t, s)ds \right] \right\} + \sum_{i=1}^n \left[ \sigma_i(t, T) \int_t^T \sigma_i(t, s)ds \right] + \sum_{i=1}^n \sigma_i(t, T) dW^\mathbb{Q}_i(t).
\end{align*}
\]

Comparing the previous equation to Eq. (2), it can be noted that when there are nonzero correlations between the factors driving the forward rates, additional terms are added to the regular HJM risk-neutral drift.

The derivation of the risk-neutral drift when the factors are correlated enables the pricing of interest rate derivatives using easy-to-interpret factors. For example, following the empirical studies of [2,3] that find significant negative correlations between the two main factors driving the prices of corporate bonds – the risk-free interest rate and the credit spread – it is more intuitive to model the term structure of corporate bonds by these two observable correlated factors than to use two factors that are the results of an orthogonalization and do not have any economic meaning.

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### References