String states, loops and effective actions in noncommutative field theory and matrix models

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Abstract

Refining previous work by Iso, Kawai and Kitazawa, we discuss bi-local string states as a tool for loop computations in noncommutative field theory and matrix models. Defined in terms of coherent states, they exhibit the stringy features of noncommutative field theory. This leads to a closed form for the 1-loop effective action in position space, capturing the long-range non-local UV/IR mixing for scalar fields. The formalism applies to generic fuzzy spaces. The non-locality is tamed in the maximally supersymmetric IKKT or IIB model, where it gives rise to supergravity. The linearized supergravity interactions are obtained directly in position space at one loop using string states on generic noncommutative branes.
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1. Introduction

Noncommutative field theory (NCFT) was conceived as a generalization of (quantum) field theory to noncommutative or quantized spaces. One of the early hopes was that the intrinsic uncertainty scale of the geometry would lead to a UV regularization of the corresponding field theory. However, it turned out that this is not the case. Rather, the phenomenon of UV/IR mixing [1] leads to an unexpected behavior of the quantum effective action at low energies, and IR divergences arise due to UV contributions in the loops. This phenomenon was partially understood from various points of view, see e.g. [2–9] and references therein. The realization of noncommu-

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tative field theory in string theory [10] suggested an interpretation in terms of a closed string exchange [5], a geometric understanding in terms of emergent gravity was found [11], and a relation with non-locality was exhibited [3,4,12,13]. In any case, UV/IR mixing means that noncommutative field theory is not simply a deformation of ordinary field theory, but is qualitatively different.

In this paper, we consider a powerful tool in the framework of noncommutative field theory given by string states, refining and developing the ideas introduced in [12]. These states make the string-like character of NC field theory manifest, they provide a clear understanding of UV/IR mixing, and an efficient way to compute loop integrals. String states are defined as $|x\rangle \langle y| \in End(\mathcal{H})$, in terms of coherent states $|x\rangle$ on the noncommutative space under consideration. They are elements of the noncommutative algebra of functions on the space, but they have no classical analog in field theory. They play a dominant role in the loop integrals, which explains the stringy nature of NCFT.

One of the technical results of this paper is a representation of one-loop integrals on fuzzy spaces in terms of integrals over string states rather than group-theoretical harmonics. This was developed to find a practical way of evaluating loop corrections on such backgrounds in Yang–Mills matrix models. The standard way of evaluating these loop integrals is to use a group-theoretical basis of functions (such as spherical harmonics on the fuzzy sphere). However, this leads to unreasonable difficulties, requiring the asymptotics of various group-theoretical objects such as 6J symbols and their higher analogs. Moreover on generic spaces without symmetry, such a computation was practically impossible outside of the semi-classical regime. Most importantly, the group-theoretical approach hides the physical meaning of the results. Although the main ideas of the present approach are contained in [12], we improve their results by replacing the ad-hoc lattices by an integration in position space, which yields a simple closed formulas for the effective action in position space.

We first review the basic facts about coherent states on the fuzzy sphere, which generalize to any quantized compact coadjoint orbit. In particular, the separation of the space of function into the semi-classical IR regime and the – much larger – UV regime is carefully discussed. The latter is best described by the string states, which are interpreted as strings whose energy and momentum is given by their length. These states have the remarkable property that they (approximately) diagonalize the Laplacian, and are "bi-local" in configuration space. The corresponding propagator takes a very simple form, which makes them ideally suited for quantization. An over-completeness relation leads to an exact representation of the trace in the one-loop effective action. We apply this in the basic one-loop integrals, and obtain a closed form for the (quadratic) 1-loop effective action in position space. This works for any quantized coadjoint orbit, and reproduces the known results for the fuzzy sphere which were obtained originally in a more complicated and less transparent way. On the Moyal–Weyl quantum plane, the origin of the non-local UV/IR mixing is clarified. The generalization to generic fuzzy spaces and to higher-loop computations is also discussed.

The results clearly exhibit the non-local nature of generic noncommutative field theories at the quantum level, making the previous observations in [3,4,13] more explicit and manifest. Hence attempts to directly use generic (non-supersymmetric) NC field theories as a replacement for ordinary local QFT are doomed,[1] and only the maximally supersymmetric model(s) remain as candidates for a fundamental, "UV-complete" quantum theory.

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[1] One may however consider various limits of noncommutative field theories, which may again become local, see e.g. [14].

This is the subject of the second part of this paper, where the formalism of string states is used to elaborate the 1-loop effective action of the supersymmetric IKKT or IIB model. In this case, the residual non-locality is mild and can be understood as a manifestation of the 10-dimensional supergravity in target space, which leads to a short-range r^{-8} interaction. It is indeed expected that this model is closely related to IIB supergravity and string theory. Up to now, this could be verified from the matrix model side (mostly for the BFSS model) only for simple configurations such as parallel or spherical branes or for separate objects represented by block-matrices [15–23], possibly with some higher multipole moments. However a derivation for generic (non-commutative) branes was missing and quite out of reach so far. The present formalism allows to generalize the old arguments to a much more general setting, and gives explicitly the 10D supergravity interactions in position space. This is very important in the on-going effort to analytically understand the physics of branes in this model, which is a candidate for a theory of fundamental interactions including gravity.

In particular, the present paper provides the necessary techniques for 1-loop computations on the fuzzy 4-sphere in the IKKT model. This is presented in a separate paper, demonstrating the emergence of 4-dimensional gravity [24].

2. Coherent states and string states

2.1. Coherent states on the fuzzy sphere

The fuzzy 2-sphere S_N^2 [25,26] is defined in terms of 3 hermitian matrices X^a, $a = 1, 2, 3$ which satisfy the algebra

$$[X^a, X^b] = i\varepsilon^{abc}X^c, \qquad X^a X_a = \frac{1}{4}(N^2 - 1) =: R_N^2 . \tag{2.1}$$

Hence $X^a = J_{(N)}^a$ generate the irreducible representation of $SU(2)$ on $\mathcal{H} = \mathbb{C}^N$. Functions on S_N^2 are given by (possibly hermitian) elements of the algebra $\mathcal{A} = End(\mathcal{H})$, which decomposes as $SU(2)$-module into fuzzy spherical harmonics $\hat{Y}_m^l$ according to $\mathcal{A} = \oplus_{l=0}^{N-1}(2l + 1)$. Here (n) denotes the $SO(3)$ irrep with dimension n. The matrix Laplacian is defined as

$$\Box\phi = [X^a, [X_a, \phi]], \qquad \phi \in End(\mathcal{H}) \tag{2.2}$$

and it is easy to see that it has the same spectrum $l(l + 1)$ for $l = 0, 1, 2, .., N - 1$ as the classical Laplacian on the sphere, and $\hat{Y}_m^l$ are the eigenfunctions. The commutation relations (2.1) state that fuzzy S_N^2 is a quantization of $\mathcal{M} = S^2$ with the $SO(3)$-invariant symplectic form ω (or Poisson structure) satisfying the quantization condition

$$\int_{\mathcal{M}} \omega = 2\pi \dim(\mathcal{H}). \tag{2.3}$$

This construction generalizes to any (quantized) coadjoint orbit $\mathcal{M}$ of a compact Lie group, see e.g. [27,28].

As for all quantized coadjoint orbits, coherent states on $\mathcal{M} = S^2 = SU(2)/U(1)$ are given by highest weight states $|\Lambda\rangle \in \mathcal{H}$ and their $SU(2)$ orbits [29],

$$|x\rangle = g_x \cdot |\Lambda\rangle, \qquad g_x \in SU(2)$$

$$x^a = \langle x|X^a|x\rangle \equiv \langle X^a\rangle \qquad x^a x_a = \frac{1}{4}(N - 1)^2 =: r_N^2 . \tag{2.4}$$

Here r_N^2 is the radius of the coherent state orbit. Up to a $U(1)$ phase factor, they are in one-to-one correspondence to points x on $\mathcal{M}$. We therefore label them locally by $x \in \mathcal{M}$, where the "north pole" $p \in \mathcal{M}$ corresponds to the highest weight state $|\Lambda\rangle$. They are optimally localized as follows

$$\Delta^2 = \sum_a \langle (X^a)^2 \rangle - \langle X^a \rangle^2 = R_N^2 - r_N^2 = \frac{N-1}{2}$$

$$=: L_{NC}^2 \ll R_N^2, \qquad N \gg 1 . \tag{2.5}$$

Δ^2 is a measure for the uncertainty in position space, which defines the noncommutativity scale L_{NC}. Upon rescaling $X \to rX$, the sphere can have any desired radius R, and $L_{NC} \sim \frac{R}{\sqrt{N}} \to 0$ as $N \to \infty$ for fixed R. It is easy to see that the uncertainty is minimized for the coherent states; for more details and illustrations see e.g. [30]. Furthermore, the coherent states $|x\rangle$ on S_N^2 form an over-complete basis, with

$$\mathbb{1}_{\mathcal{H}} = c_N \int dx |x\rangle\langle x|, \qquad c_N = \frac{\dim \mathcal{H}}{\mathrm{Vol}\mathcal{M}} . \tag{2.6}$$

Indeed the operator defined on the rhs is invariant under the adjoint action of $SU(2)$, and the only operator with this property is $\sim \mathbb{1}$ (because $\mathcal{H}$ is irreducible). This gives the following representation of the trace of any operator $\mathcal{O} \in End(\mathcal{H})$

$$\mathrm{tr}\mathcal{O} = \frac{\dim \mathcal{H}}{\mathrm{Vol}\mathcal{M}} \int dx \langle x|\mathcal{O}|x\rangle . \tag{2.7}$$

Here tr denotes the trace on $\mathcal{H}$. The overlap of the coherent states decays rapidly with the distance between x and y,

$$|\langle x|y\rangle|^2 = \frac{1}{c_N} \delta_N(x, y) \qquad \to 0 \ \text{for} \ x \neq y, \ N \to \infty \tag{2.8}$$

which defines a regularized delta function $\int dx \delta_N(x, y) = 1$ on $\mathcal{M}$. On the fuzzy sphere, there is an explicit formula [29]

$$|\langle x|y\rangle|^2 = \left(\frac{1 + x \cdot y}{2}\right)^{N-1} \approx \exp\left(-\frac{1}{4}\phi^2(N-1)\right), \qquad \phi^2 \ll 1 \tag{2.9}$$

where ϕ is the angle between x and y. Hence $\delta_N(x, y)$ is localized on an area $\frac{4\pi}{N}$, which reflects the quantization of the sphere in terms of N quantum cells.

The phase of $\langle x|y\rangle$ also contains interesting information. Since the coherent states are determined only up to a $U(1)$ phase, they form a $U(1)$ bundle $\mathcal{B}$ over S^2. Near some point $p \in S^2$ (the north pole, say), we can define a local section $|x\rangle = e^{i\phi_i J_i}|0\rangle$ parametrized by 2 angles ϕ_i, $i = 1, 2$ relative to p. The group action defines a connection ∇, with curvature[2] given by the symplectic form underlying the quantum space, just like in quantum mechanics. Then one finds

$$\langle x|y\rangle = e^{iA(x,y)}\left(\frac{1 + x \cdot y}{2}\right)^{(N-1)/2} =: \frac{1}{c_N}\tilde{\delta}_N(x, y) \tag{2.10}$$

where $\tilde{\delta}_N(x, y)$ is again a (now complex-valued) approximate delta function which satisfies

$$\int dx \, \tilde{\delta}_N(x, y) |x\rangle = |y\rangle \tag{2.11}$$

with similar localization properties. Here $A(x, y)$ is the symplectic area of the spherical triangle spanned by x, y, p.

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[2] This is the line bundle with monopole number $N - 1$.

Operators and symbols   Coherent states provide a useful and explicit link between functions on $\mathcal{M}$ and operators. For an arbitrary operator $\mathcal{O} \in End(\mathcal{H})$, we define the *symbol* of $\mathcal{O}$ to be

$$\mathcal{O}(x) = \langle x | \mathcal{O} | x \rangle \ . \tag{2.12}$$

This should be viewed as de-quantization of $\mathcal{O} \sim \mathcal{O}(x)$. In particular, $x^a = \langle x | X^a | x \rangle$. Combining this with (2.7), we can write the trace in the familiar form

$$\mathrm{tr}\mathcal{O} = c_N \int dx \, \mathcal{O}(x) \ . \tag{2.13}$$

Conversely, one can certainly represent every fuzzy function as

$$\mathcal{O} = c_N^2 \int dx dy \langle x | \mathcal{O} | y \rangle | x \rangle \langle y | \tag{2.14}$$

however this is far from unique. At least on quantized homogeneous spaces one can even find a diagonal representation

$$\mathcal{O} = c_N \int dx \, \tilde{O}(x) | x \rangle \langle x | \ , \tag{2.15}$$

however $\tilde{O}(x) \neq \mathcal{O}(x)$ in general. For example on S_N^2, we can write

$$\hat{Y}_m^l = c_N \int\limits_{S^2} dx Y_m^l(x) | x \rangle \langle x | \tag{2.16}$$

because both sides transform in the same way under $SO(3)$. Similarly, plane waves on the Moyal–Weyl plane $\mathbb{R}_\theta^n$ can be written as

$$e^{ikX} = c \int dx e^{ikx} | x \rangle \langle x | \ . \tag{2.17}$$

Hence all functions on fuzzy spaces can be represented in this diagonal way, however this is very delicate for large momenta and may be completely misleading[3] as we will see. It should only be used in the semi-classical low-energy sector, which is defined as follows:

IR sector   The important property which characterizes the semi-classical or low energy regime for functions on fuzzy spaces is their approximate *locality*. An operator or fuzzy function is in the semi-classical low energy (IR) regime if the *non-local* matrix elements decay at distance scales $|x - y| \sim L_{NC}$, so that

$$\langle x | \mathcal{O} | y \rangle \approx \langle x | \mathcal{O} | x \rangle \, \delta_N(x, y) \ . \tag{2.18}$$

This is the crucial property for external fields which will justify the following methods for computing the effective action. In particular, we will need

$$\langle x | f(X) | y \rangle \approx f(x) \langle x | y \rangle \approx f(y) \langle x | y \rangle \ , \tag{2.19}$$

which holds for functions $f(x)$ which are approximately constant on the scale L_{NC}. For (low) polynomials in X^a, this follows from the fact that the dispersion of X^a is given by Δ^2, which is precisely the NC scale. The maximal angular momentum compatible with this requirement is $l \leq \sqrt{N}$, which is precisely the scale of the fuzzy delta function localized e.g. at the north pole,

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[3] In the same vein, using a star product for loop computations in NC field theory is misleading.

$$|p\rangle\langle p| =: \frac{1}{c_N}\,\delta_N(X; p) \tag{2.20}$$

with symbol $\frac{1}{c_N}\delta_N(x, y)$ (2.8). This is optimally localized with uncertainty $\Delta^2 \sim L_{NC}^2$ (2.5), and has angular momentum $l_{NC} \sim \|[X_i, .]\| \sim \sqrt{N}$.

UV sector In contrast, most of the operators $\mathcal{O} \in End(\mathcal{H})$ have $l > \sqrt{N}$, and are therefore not in the semi-classical IR sector. These are best described in terms of the *non-local* string states

$$\psi_{x,y} := |x\rangle\,\langle y| \qquad \in End(\mathcal{H}) \tag{2.21}$$

introduced in [12] and discussed in detail below. These form the core of the fuzzy or "quantum" geometry, yet they are often neglected. The most extreme example on S_N^2 is the state with maximal J_3 eigenvalue,

$$Y_{N-1}^{N-1} = |p\rangle\langle -p|, \tag{2.22}$$

where $|p\rangle$ is the highest weight state, cf. [31]. This is in the far UV region of the algebra, it is maximally de-localized and has maximal angular momentum $l_{UV} = 2j \sim N \gg \sqrt{N} = l_{NC}$.

Since these string states comprise the bulk of the algebra of functions, it should not be surprising that they lead to significant non-local contributions in the effective action. The resulting string-like theory will be elaborated below. In the context of quantum mechanics, the analogous types of de-localized density matrices lead to the well-known non-local entanglement and EPR-type considerations, which are characteristic for the "deep quantum" regime. Clearly such states are not well-described by deformation quantization or in any semi-classical picture, yet they form the core of noncommutative (or fuzzy) field theory.

Rescaling and planar limit So far, the radius of S^2 was fixed to be R_N^2. Now introduce a scaling factor so that

$$X^a X_a = R^2 \tag{2.23}$$

with any desired radius R. Then for $R = 1$ one obtains the classical sphere as $N \to \infty$, and the dispersion of the coherent states is

$$\Delta^2(|x\rangle) = \langle x| \sum_a (X^a - \langle X^a \rangle)^2 |x\rangle = \frac{2}{N+1} = O\left(\frac{1}{N}\right). \tag{2.24}$$

Hence the quantum cells become small as $N \to \infty$. On the other hand if we scale the radius as

$$R^2 = N\theta/2, \qquad \theta = const, \ N \to \infty, \tag{2.25}$$

the generators X^1, X^2 generate the Moyal–Weyl quantum plane $\mathbb{R}_\theta^2$ near the north pole [32]

$$[X^i, X^j] = i\theta\epsilon^{ij} \quad + \mathcal{O}\left(\frac{1}{N}\right), \tag{2.26}$$

dropping the X^3 generator. This is valid for states localized near the origin (i.e. the north pole). We can then recover the standard coherent states on the Moyal–Weyl plane as $|x\rangle = U_x|0\rangle$ where $U_x = \exp(i\phi_i J^i)$ for $x^i = R\,\epsilon^{ij}\phi_j$, which gives

$$\langle x'|x\rangle = \langle 0|e^{-i\phi' J}e^{i\phi J}|0\rangle = e^{-\frac{i}{2\theta}x^i\varepsilon_{ij}x'^j}e^{-\frac{|x-x'|^2}{4\theta}}. \tag{2.27}$$

Hence the overlap between coherent states is confined to regions of size θ. This is the scale of noncommutativity L_{NC}, which marks the boundary between the IR regime and the UV regime, as discussed above.

Even though we focus on the fuzzy sphere, the construction of coherent states goes through quite literally for any quantized coadjoint orbit such as $\mathbb{C}P_N^n$, and also for the Moyal–Weyl quantum plane $\mathbb{R}_\theta^{2n}$. We refer to [27,29,30] for more details in these cases.

2.2. String states

Now consider some quantized fuzzy space (such as S_N^2, $\mathbb{C}P_N^n$, or even $\mathbb{R}_\theta^{2n}$), with coherent states as above satisfying

$$\langle x|y \rangle = \frac{1}{c_N} \tilde{\delta}_N(x, y), \qquad c_N = \frac{\dim \mathcal{H}}{\mathrm{Vol}\mathcal{M}}, \qquad \int_{\mathcal{M}} dx \, \tilde{\delta}_N(x, y)|x\rangle = |y\rangle . \tag{2.28}$$

We then define the string states as

$$\left| \begin{matrix} x \\ y \end{matrix} \right) := \psi_{x,y} := |x\rangle \langle y| \qquad \in End(\mathcal{H})$$

$$\left(\begin{matrix} x \\ y \end{matrix} \right| := \psi_{x,y}^\dagger := |y\rangle \langle x| \tag{2.29}$$

cf. [12]. We also define the momentum operators acting on $End(\mathcal{H})$

$$\mathcal{P}^a \mathcal{O} := [X^a, \mathcal{O}], \qquad \mathcal{O} \in End(\mathcal{H})$$

$$\Box \mathcal{O} := \mathcal{P}^a \mathcal{P}_a \mathcal{O} \tag{2.30}$$

with expectation values

$$\left(\begin{matrix} x \\ y \end{matrix} \middle| \mathcal{P}^a \middle| \begin{matrix} x \\ y \end{matrix} \right) = \mathrm{tr}\, \psi_{y,x}[X^a, \psi_{x,y}] = \vec{\mathbf{x}}(x) - \vec{\mathbf{x}}(y)$$

$$\left(\begin{matrix} x \\ y \end{matrix} \middle| \mathcal{P}^a \mathcal{P}_a \middle| \begin{matrix} x \\ y \end{matrix} \right) = \mathrm{tr}\, \psi_{y,x}[X^a, [X_a, .]]\psi_{x,y} = E_{xy} . \tag{2.31}$$

Here

$$E_{xy} = (\vec{\mathbf{x}}(x) - \vec{\mathbf{x}}(y))^2 + \Delta_x^2 + \Delta_y^2 \tag{2.32}$$

is the energy of a string state given by its length square plus their intrinsic zero point energy, in units of the noncommutativity scale (note that $\mathcal{P}^a = [X^a, .] \sim \theta^{ab}\partial_b$ has dimension length). $\Delta_{x,y}^2$ denotes the uncertainty at x and y, respectively, which is simply Δ^2 for homogeneous spaces. In particular, the string states $\psi_{x,y}$ have "matrix momentum" $\mathcal{P} = x - y$. This is consistent with previous observations [13,33,34] in noncommutative field theory, which now have a precise mathematical realization in terms of the string states.

We will also need the general matrix elements

$$\left(\begin{matrix} x \\ y \end{matrix} \middle| \mathcal{P}^a \mathcal{P}_a \middle| \begin{matrix} x' \\ y' \end{matrix} \right) = \langle x|X^a X^a|x'\rangle\langle y'|y\rangle + \langle x|x'\rangle\langle y'|X^a X^a|y\rangle - 2\langle x|X^a|x'\rangle\langle y'|X_a|y\rangle$$

$$\approx E_{xy} \langle x|x'\rangle\langle y'|y\rangle \tag{2.33}$$

to a very good approximation. Note that this is nearly diagonal, and E_{xy} is bounded from below by the scale of noncommutativity (2.32).

The remarkable feature of the string states is that they have good localization properties in *both* position and momentum. This makes them very interesting and novel from a QFT point of view.

Even though (or rather because) they are typically in the UV regime far from the semi-classical regime, they are very important for loop computations.

We also note that a non-commutative background in the matrix model can generally be viewed as a condensate of diagonal string states

$$X^a \sim \int d^n x\, x^a |x\rangle\langle x| . \tag{2.34}$$

This is exact for $\mathbb{R}^{2n}_\theta$ and for quantized homogeneous spaces, and holds at least approximately in general, cf. section 2.4.

Propagator Generalizing the over-completeness relation (2.28), we can write

$$c_N^2 \int\limits_{\mathcal{M}\times\mathcal{M}} dx\,dy \left|{x \atop y}\right)\left({x \atop y}\right| = \mathbb{1}_{End(\mathcal{H})} \tag{2.35}$$

which follows again by group invariance. In the same spirit, we can state the central formula of this paper, which is an approximation for the propagator $(\Box + \mu^2)^{-1}$ using the string states. We claim that

$$\boxed{(\bar{\Box} + \mu^2)^{-1} := c_N^2 \int\limits_{\mathcal{M}} dx\,dy \left|{x \atop y}\right) \frac{1}{E_{xy}+\mu^2} \left({x \atop y}\right| \approx (\Box + \mu^2)^{-1}} \tag{2.36}$$

is an excellent approximation to the propagator. Although no rigorous estimates will be given here, this formula can be justified by the following computation

$$(\tilde{\Box} + \mu^2)^{-1}(\Box + \mu^2)\left|{x \atop y}\right) = c_N^2 \int dx'dy' \left|{x' \atop y'}\right) \frac{1}{E_{x'y'}+\mu^2} \left({x' \atop y'}\right|(\Box + \mu^2)\left|{x \atop y}\right)$$

$$\approx c_N^2 \int dx'dy' \left|{x' \atop y'}\right) \frac{1}{E_{x'y'}+\mu^2}(E_{xy}+\mu^2)\langle x'|x\rangle\langle y|y'\rangle$$

$$\approx \left|{x \atop y}\right) \tag{2.37}$$

using (2.33) and (2.28). The approximation here comes from the variation of $E_{x,y}$ on scales of order L_{NC}, which is small since $E_{x,y} \geq 2L^2_{NC}$. We therefore expect (2.36) to be an excellent approximation, even for $\mu^2 = 0$, and there is no problem with any singularities.[4] We will see explicitly that it works very well in the examples discussed below. This justifies replacing $\Box^{-1}$ by $\tilde{\Box}^{-1}$, which is the key proposal.

Finally, we remark that on noncommutative branes, the above "matrix momentum" $\mathcal{P}^a$ is only indirectly related to the usual momentum. E.g. for a semi-classical scalar field we have $\mathcal{P}^a\phi = \theta^{ab}\partial_b\phi$, which leads to a non-trivial relation between the effective metric on a brane ("open string metric") and the induced metric on the target space via θ^{ab} [35]. This is responsible for some of the unusual features of noncommutative field theory.

2.3. One-loop computations using coherent states

The 1-loop effective action can be expressed in terms of the trace of some operator $\mathcal{O}$ acting on the space of wavefunctions. For the case of complex-valued scalar fields on a fuzzy space,

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[4] Note also that the phases of the coherent states cancel out, so there is no hidden phase ambiguity.

this is the space $End(\mathcal{H})$ of operators on the underlying Hilbert space, where $\mathcal{H}$ is an irreducible representation of G. This trace can be written in terms of the string states as follows

$$\mathrm{Tr}_{End(\mathcal{H})}\mathcal{O} = \frac{(\dim\mathcal{H})^2}{(\mathrm{Vol}\mathcal{M})^2} \int\limits_{\mathcal{M}\times\mathcal{M}} dxdy \left(\begin{matrix}x\\y\end{matrix}\middle|\mathcal{O}\middle|\begin{matrix}x\\y\end{matrix}\right) . \tag{2.38}$$

This is an exact formula for any homogeneous quantum space of a (compact) Lie group G with coherent states as discussed above. To prove it, it suffices to note that rhs of (2.38) is a functional which is invariant under $G_L \times G_R$, and by the uniqueness of the singlet in $End(\mathcal{H})$ it must be proportional to the trace. Note that the integral over $\mathcal{M} \times \mathcal{M}$ makes sense even though the spin states ψ_{xy} form a non-trivial bundle over $\mathcal{M} \times \mathcal{M}$, and there is no global section. However any phase factors cancel out in (2.38), and it does not matter whether we integrate over the bundle $\mathcal{B} \times \mathcal{B}$ or over the base.

Now consider the case of hermitian fields $\phi = \phi^\dagger \in End(\mathcal{H})$, which are realized by the string states as follows $e^{i\varphi}\psi_{xy} + e^{-i\varphi}\psi_{yx}$. This suggests that the phase factors might lead to non-trivial interference effects and we should integrate over the entire bundles $\mathcal{B} \times \mathcal{B}$. Nevertheless, these effects cancel and the trace over hermitian operators is simply $\frac{1}{2}\times$ the trace over all operators. Thus

$$\mathrm{Tr}_{Herm(\mathcal{H})}\mathcal{O} = \frac{1}{2}\frac{(\dim\mathcal{H})^2}{(\mathrm{Vol}\mathcal{M})^2} \int\limits_{\mathcal{M}\times\mathcal{M}} dxdy \left(\begin{matrix}x\\y\end{matrix}\middle|\mathcal{O}\middle|\begin{matrix}x\\y\end{matrix}\right) \tag{2.39}$$

where $Herm(\mathcal{H})$ denotes the hermitian operators on $\mathcal{H}$. To evaluate the matrix elements, it is sometimes more transparent to write

$$\left(\begin{matrix}x\\y\end{matrix}\middle|\mathcal{O}\middle|\begin{matrix}x\\y\end{matrix}\right) = \mathrm{tr}\big(|y\rangle\langle x|\mathcal{O}(|x\rangle\langle y|)\big) . \tag{2.40}$$

Using the formalism of quasi-coherent states [30] reviewed in section 2.4, the above formulas should hold also on rather generic quantum spaces to a very good approximation, as long as the operators $\mathcal{O}$ are sufficiently "local". In the present paper, we will focus on the case of quantized coadjoint orbits for simplicity.

As a warm-up, we compute the trace of the Laplacian on the fuzzy sphere S^2_N. Using (2.38) and (2.31) we obtain

$$\mathrm{Tr}_{End(\mathcal{H})}[X^a, [X_a, .]]$$
$$= \frac{N^2}{(\mathrm{Vol}S^2)^2} \int\limits_{S^2\times S^2} dxdx'\mathrm{tr}(|x\rangle\langle x'|)\big(|X^a(x') - X^a(x)|^2 + 2\Delta^2\big)(|x'\rangle\langle x|)$$
$$= \frac{N^2}{(\mathrm{Vol}S^2)^2} \int\limits_{S^2\times S^2} dxdx'(|X^a(x') - X^a(x)|^2 + 2\Delta^2)$$
$$= \frac{N^2}{\mathrm{Vol}S^2} \int\limits_{S^2} dx(|X^a(e) - X^a(x)|^2 + 2\Delta^2) . \tag{2.41}$$

Here $|.|$ is the Euclidean distance in target space $\mathbb{R}^3$, and e is an arbitrary point on S^2, and $\mathcal{H} = \mathbb{C}^N$. We parametrize S^2 with the standard normalization $\mathrm{Vol}(S^2) = 4\pi$. Then $X^a(x) \in \mathbb{R}^3$ are functions on S^2 normalized as

$$R_N^2 = X^a X_a = \frac{1}{4}(N^2 - 1) \tag{2.42}$$

and recalling $\Delta^2 \approx \frac{N}{2}$ (2.5) we obtain

$$\text{Tr}_{End(\mathcal{H})}[X^a, [X_a, .]] \approx \frac{1}{4} \frac{N^2(N^2 - 1)}{4\pi^2} \int\limits_{S^2} dx (|e_3 - x|^2 + O(\frac{1}{N})) . \tag{2.43}$$

where x is now normalized to 1. Evaluating the integral

$$\int\limits_{S^2} |e_3 - x|^2 = 2\pi \int\limits_0^\pi d\theta \sin\theta ((1 - \cos\theta)^2 + \sin^2\theta) = 8\pi \tag{2.44}$$

results in

$$\text{Tr}_{End(\mathcal{H})}[X^a, [X_a, .]] = \frac{1}{2}N^2(N^2 - 1)\left(1 + O(\frac{1}{N})\right) . \tag{2.45}$$

This agrees very well with the exact result

$$\text{Tr}_{End(\mathcal{H})}[X^a, [X_a, .]] = \sum_{j=0}^{N-1} j(j + 1)(2j + 1) = \frac{1}{2}(N^2 - 1)N^2. \tag{2.46}$$

More generally, we can compute for any smooth function f

$$\begin{aligned}
\text{Tr}_{End(\mathcal{H})} f(\square) &= \frac{N^2}{(\text{Vol}S^2)^2} \int\limits_{S^2} dx \int\limits_{S^2} dy f(R_N^2 |x - y|^2 + 2\Delta^2) \\
&= \frac{N^2}{\text{Vol}S^2} \int\limits_{S^2} dx f(R_N^2 |e_3 - x|^2 + 2\Delta^2) \\
&= 2\pi \frac{N^2}{\text{Vol}S^2} \int\limits_0^\pi d\vartheta \sin\vartheta f(R_N^2(1 - \cos\theta)^2 + \sin^2\theta) + 2\Delta^2) \\
&= \frac{N^2}{2} \int\limits_{-1}^1 du f(2R_N^2(1 - u) + 2\Delta^2) \\
&\approx \int\limits_0^N dj \, 2j f(j^2 + 2\Delta^2) \approx \sum_{j=0}^{N-1}(2j + 1)f\left(j(j + 1) + 2\Delta^2\right) \\
&= \text{Tr}_{j_{\max}} f(\square_g + 2\Delta^2) . \tag{2.47}
\end{aligned}$$

Hence the result agrees well with the classical trace over $f(\square_g + 2\Delta^2)$ for any smooth function f with UV cutoff $j_{\max} = N - 1$, and the shift by $2\Delta^2 = N - 1$ is negligible for $N \gg 1$. Here $\square_g$ denote the classical Laplacian on S^2.

Upon closer examination, this computation is actually a bit strange: the contribution of the integral comes from non-classical, UV regime with angular momenta $l^2 \geq \Delta^2 = O(N)$, where we can neglect the shift by $2\Delta^2$. This is the regime where one should in general *not* trust the semi-classical approximation, and this computation only works because the spectrum of the matrix Laplacian $\square$ coincides *exactly* with that of the classical Laplacian $\square_g$, even in the far UV

regime. Thus even though the string states $|x\rangle\langle y|$ cannot be approximated by any classical functions, they allow to compute e.g. the classical heat kernel expansions, as long as the operators under consideration (such as $\Box$) have the classical spectrum even in the UV regime. We will see below that the method works also in other cases, but then the result does not always correspond to the naive semi-classical expectation.

2.3.1. One-loop propagator on S_N^2

As an application of this formalism, we want to compute the one-loop correction to the propagator for scalar ϕ^4 theory on S_N^2, with hermitian scalar field $\phi^\dagger = \phi$ and action

$$S[\phi] = \frac{1}{N}\mathrm{tr}\left(\frac{1}{2}\phi(\Box + \mu^2)\phi + \frac{g}{4!}\phi^4\right) = S_0[\phi] + S_{\mathrm{int}}[\phi]. \tag{2.48}$$

The result will agree with the (more complicated and less transparent) original computation in [32]. We use the standard normalization for $\Box$

$$X^a = J_{(N)}^a, \qquad X^a X_a = \frac{1}{4}(N^2 - 1) = R_N^2$$

$$\Box\phi = [X^a, [X_a, \phi]] \tag{2.49}$$

with spectrum $l(l+1)$. Then the effective action including one-loop quantum corrections can be written as

$$\Gamma_{\mathrm{eff}}[\phi] = S[\phi] + \frac{1}{2}\mathrm{Tr}_{End(\mathcal{H})}\log\left(S''[\phi]\right)$$

$$(\psi, S''[\phi]\psi) = \frac{1}{N}\mathrm{tr}\left(\psi(\Box + \mu^2)\psi + \frac{g}{3}\phi^2\psi^2 + \frac{g}{6}\psi\phi\psi\phi\right) \tag{2.50}$$

where $S''[\phi]$ is the quadratic form for fluctuations around the background ϕ. The one-loop contribution can be expanded follows

$$\Gamma_{1-loop}[\phi] = \mathrm{Tr}\log(.(\Box + \mu^2). + \frac{g}{3}.\phi^2. + \frac{g}{6}.\phi.\phi)$$

$$= \mathrm{Tr}\log(\Box + \mu^2) + \mathrm{Tr}\left(.\frac{1}{\Box + \mu^2}(\frac{g}{3}\phi^2. + \frac{g}{6}\phi.\phi)\right) + O(\phi^4). \tag{2.51}$$

We assume that the background field

$$\phi = \phi(X) \approx c_N \int_{\mathcal{M}} dy\, \phi(y)\, |y\rangle\langle y| \tag{2.52}$$

is slowly varying on the scale of noncommutativity. Then ϕ acts nearly-diagonally on the string basis $\psi_{yx} = |y\rangle\langle x|$, and we can replace

$$\phi\psi_{yx} \approx \phi(y)\psi_{yx} \tag{2.53}$$

and similarly for ϕ^2. Since $\dim\mathcal{H} = N$, (2.38) gives e.g.

$$\mathrm{Tr}(.\phi^2.) = \frac{N^2}{\mathrm{Vol}(\mathcal{M})^2}\int_{\mathcal{M}\times\mathcal{M}} dxdy\,\mathrm{tr}(\psi_{y,x}\phi^2\psi_{x,y})$$

$$= \frac{N^2}{\mathrm{Vol}(\mathcal{M})}\int_{\mathcal{M}} dx\,\langle x|\phi^2|x\rangle. \tag{2.54}$$

Similarly, using the property (2.31) or (2.36) of the propagator we find

$$
\begin{aligned}
Tr(.\square^{-1}\phi^2.) &= \frac{N^2}{\mathrm{Vol}(\mathcal{M})^2} \int\limits_{\mathcal{M}\times\mathcal{M}} dxdy \mathrm{tr}(\psi_{y,x}(\square+\mu^2)^{-1}(\phi^2\psi_{x,y})) \\
&\approx \frac{N^2}{\mathrm{Vol}(\mathcal{M})^2} \int\limits_{\mathcal{M}\times\mathcal{M}} dxdy \frac{1}{R_N^2|x-y|^2+2\Delta^2+\mu^2} \mathrm{tr}(\psi_{y,x}\phi^2\psi_{x,y}) \\
&= \frac{N^2}{\mathrm{Vol}(\mathcal{M}^2)} \int\limits_{\mathcal{M}\times\mathcal{M}} dxdy \frac{1}{R_N^2|x-y|^2+\tilde\mu^2} \langle x|\phi^2|x\rangle \\
&= \frac{\mu_N^2}{\mathrm{Vol}(\mathcal{M})} \int\limits_{\mathcal{M}} dx \, \phi^2(x)
\end{aligned}
\tag{2.55}
$$

where

$$
\tilde\mu^2 = \mu^2 + 2\Delta^2 > 0
\tag{2.56}
$$

and μ_N^2 is the 1-loop planar mass renormalization

$$
\begin{aligned}
\mu_N^2 &= \frac{N^2}{\mathrm{Vol}(S^2)} \int\limits_{S^2} dy \frac{1}{R_N^2|e-y|^2+\tilde\mu^2} \\
&= \frac{N^2}{2R_N^2} \int\limits_0^\pi d\vartheta \sin\vartheta \frac{1}{(1-\cos\vartheta)^2+\sin\vartheta^2+\frac{\tilde\mu^2}{R_N^2}} \\
&= 2\int\limits_{-1}^1 du \frac{1}{2-2u+\frac{\tilde\mu^2}{R_N^2}} \\
&\approx \sum\limits_{j=0}^N \frac{2j+1}{j(j+1)+\mu^2} =: I^P
\end{aligned}
\tag{2.57}
$$

where e is again some (arbitrary) reference point on S^2. The approximation in (2.55) consists of replacing $\square^{-1}$ by its diagonal matrix elements. As discussed before (cf. (2.36)), this is justified as long as ϕ^2 is in the IR regime, i.e. it varies only slowly at the NC scale. Note also that $\square^{-1}$ has bounded matrix elements in the string basis, which ensures that there are no IR divergences in this integral. This "planar" contribution is schematically depicted in Fig. 1. It can be interpreted in terms of an open string from x to y propagating in the loop, integrated over y.

Now consider the "non-planar" contribution

$$
\begin{aligned}
\mathrm{Tr}(.(\square+\mu^2)^{-1}\phi.\phi) &= \frac{N^2}{\mathrm{Vol}(\mathcal{M})^2} \int\limits_{\mathcal{M}\times\mathcal{M}} dxdy \mathrm{tr}(\psi_{y,x}(\square+\mu^2)^{-1}(\phi\psi_{x,y}\phi)) \\
&= \frac{N^2}{\mathrm{Vol}(\mathcal{M})^2} \int\limits_{\mathcal{M}\times\mathcal{M}} dxdy \langle x|(\square+\mu^2)^{-1}\phi|x\rangle\langle y|\phi|y\rangle
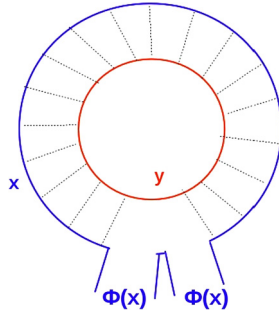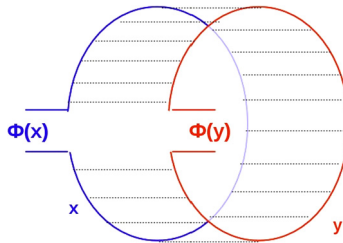\end{aligned}
$$

Fig. 1. Planar 1-loop contribution.

Fig. 2. Non-planar 1-loop contribution.

$$= \frac{N^2}{\text{Vol}(\mathcal{M})^2} \int\limits_{\mathcal{M}\times\mathcal{M}} dx dy \frac{1}{R_N^2 |x-y|^2 + \tilde{\mu}^2} \phi(x)\phi(y) \tag{2.58}$$

depicted in Fig. 2. In contrast to the planar contribution this results in a non-local term (!). This can be interpreted either in terms of an open string loop stretching from x to y, or in terms of a closed string propagating from x to y. Finally we compute the "vacuum energy" contribution

$$
\begin{aligned}
\text{Tr}\log(\Box + \mu^2) &= \frac{N^2}{(\text{Vol}\mathcal{M})^2} \int\limits_{\mathcal{M}} dx \int\limits_{\mathcal{M}} dy \log\left(R_N^2 |x-y|^2 + \tilde{\mu}^2\right) \\
&= \frac{N^2}{4\pi} \int\limits_{S^2} dx \log\left(|e-y|^2 + \frac{\tilde{\mu}^2}{R_N^2}\right) \\
&= \frac{N^2}{2} \int\limits_{0}^{2} du \log\left(2u + \frac{\tilde{\mu}^2}{R_N^2}\right) \\
&= N^2\left(\ln 4 - 1 + O(\frac{\tilde{\mu}^2}{R_N^2})\right) \\
&=: \Gamma_{vac}
\end{aligned}
\tag{2.59}
$$

where e is some point on the unit sphere S^2. Then the one-loop contribution to the effective action up to quadratic order in ϕ is

$$\Gamma_{1-loop} = \Gamma_{vac} + \frac{g}{3} \frac{1}{\text{Vol}(\mathcal{M})} \int\limits_{\mathcal{M}} dx \mu_N^2 \phi(x)^2$$

$$+ \frac{g}{6} \frac{N^2}{\text{Vol}(\mathcal{M})^2 R_N^2} \int\limits_{\mathcal{M} \times \mathcal{M}} dx dy \frac{\phi(x)\phi(y)}{|x-y|^2 + \frac{\tilde{\mu}^2}{R_N^2}} + O(\phi^4) \tag{2.60}$$

cf.[5] [12]. The planar contribution is local and leads to a standard mass renormalization, which agrees with the results in [32] using a traditional mode expansion. We will see that the non-planar loop contribution also agrees with [32], but it is now recognized as a long-range non-local action. This effect has no counterpart in standard quantum field theory. It is of distinctly stringy nature,[6] reflecting the presence of virtual long strings described by the string states. Hence the model describes a non-local theory even on scales much longer than the noncommutativity scale, and should not be considered as approximation to some local QFT. Although similar observation were made in [3,4], the present derivation based on string states is most efficient, and easily generalized. Clearly the higher loop contributions will add even more non-local constrictions [1], and could be obtained explicitly in a similar way (the extension to higher loops will be discussed briefly in section 3).

The above derivation generalizes immediately to other, higher-dimensional fuzzy spaces such as fuzzy $\mathbb{C}P_N^n$, noting that $|x-y|$ is always the Euclidean distance in target space. The one-loop effective action has always the same form (2.60), apart from trivial adaptions. This is already a significant new result, since non-planar loop contributions are very hard using group-theoretical expansions and have not been performed.

Comparison with 1-loop results for fuzzy S_N^2 To check the validity of the approximations in the coherent state approach, we compare (2.60) with the known result for the fuzzy sphere [32]. We evaluate the non-planar contribution for spin l spherical harmonics $\phi = Y_m^l$, which gives

$$\Gamma_{NP} = \frac{g}{6} \frac{N^2}{(4\pi)^2 R_N^2} \int\limits_{S^2 \times S^2} dx dy \frac{1}{|x-y|^2 + \frac{\tilde{\mu}^2}{R_N^2}} \left(\sum_m (-1)^m Y_m^l(x) Y_{-m}^l(y) \right)$$

$$= \frac{g}{6} \frac{2l+1}{4\pi} \frac{N^2}{(4\pi)^2 R_N^2} \int\limits_{S^2 \times S^2} dx dy \frac{1}{|x-y|^2 + \frac{\tilde{\mu}^2}{R_N^2}} P_l(\cos\vartheta)$$

$$= \frac{g}{6} \frac{2l+1}{2} \frac{N^2}{R_N^2 (4\pi)} \int\limits_0^\pi d\vartheta \sin\vartheta \frac{P_l(\cos\vartheta)}{(1-\cos\vartheta)^2 + \sin\vartheta^2 + \frac{\tilde{\mu}^2}{R_N^2}}$$

$$= \frac{g}{6} \frac{2l+1}{4\pi} \int\limits_{-1}^1 du \frac{P_l(u)}{1-u + \frac{\tilde{\mu}^2}{2R_N^2}}$$

---

[5] The same structure was obtained in [12] using partitions into block-matrices. The present approach is more efficient and does not require any ad-hoc partitions of space.

[6] The present low-dimensional model should be viewed as non-critical string theory. The connection to critical string theory will be discussed in section 4.

$$=: \frac{g}{6} \frac{2l+1}{4\pi} I^{NP}(l) \tag{2.61}$$

using the spherical harmonics addition theorem, where ϑ is the angle between x and y. This integral is convergent as $R_N \to \infty$. Taking out the factor $(2l+1)$ from the sum over m, we recover precisely the result which was obtained in [32] using a more complicated group-theoretical computation (which required the asymptotics of the $6J$ symbols). Clearly the present derivation is much more efficient and transparent, it works equally well on higher-dimensional spaces such as $\mathbb{C}P^2$, and – most importantly – it can be applied to more complicated problems such as supersymmetric matrix models.

Planar limit and UV/IR mixing on $\mathbb{R}^2_\theta$ Although the above non-local term is perfectly well-defined on compact fuzzy spaces for finite N, it leads to IR-divergences in the non-compact limit $R \to \infty$, which clearly cannot be canceled by any local counterterms. This is the infamous UV/IR mixing of NC field theory, which is now understood in a completely transparent way. To see this, we recall that the Moyal–Weyl quantum plane $\mathbb{R}^2_\theta$ can be obtained as a scaling limit of the fuzzy sphere (near the north pole) for $X^a = r J^a$ with $R^2 = r^2 R_N^2 = \frac{N\theta}{4}$ and fixed θ. Then the above non-planar contribution to the one-loop effective action takes the form

$$\Gamma_{NP} \approx \frac{gN^2}{6\text{Vol}(\mathcal{M})^2} \int\limits_{\mathcal{M}\times\mathcal{M}} dx dy \frac{\phi(x)\phi(y)}{|x-y|^2 + \mu^2}$$

$$= \frac{g}{6\pi^2\theta^2} \int\limits_{\mathcal{M}\times\mathcal{M}} dx dy \frac{\phi(x)\phi(y)}{|x-y|^2 + \mu^2} \tag{2.62}$$

where $\text{Vol}\mathcal{M} = 4\pi R^2 = \pi N\theta$. Now N has disappeared, and this form can in fact be obtained directly[7] from coherent state representation on $\mathcal{M} = \mathbb{R}^2_\theta$. Even though it is non-local, this term is invariant under translations in the flat limit $N \gg 1$, and we can compute it in a plane wave basis $\phi(x) = \int \frac{d^2k}{2\pi} \phi_k(e^{ixk} + e^{-ikx})$. This leads to

$$\Gamma_{NP} \approx \frac{g}{6\pi^2\theta^2} \int d^2k \phi(k)^2 \int d^2z \frac{1}{|z|_g^2 + \mu^2} e^{ik_i z^i}$$

$$= \frac{g}{6\pi^2\theta^2} \int d^2k \phi(k)^2 \int d^2p \frac{1}{p_i p_j G^{ij} + \mu^2} e^{ik_i \theta^{ij} p_j}, \tag{2.63}$$

replacing $z^i = \theta^{ij} p_j$ in the second step. Here $|.|_g$ is the background (closed string) metric, and

$$G^{ij} = \theta^{ii'}\theta^{jj'}\delta_{i'j'} \tag{2.64}$$

is the "open string" metric which governs noncommutative field theory on $\mathbb{R}^2_\theta$ [10,35]. This is the familiar form[8] for the non-planar contribution to the propagator on $\mathbb{R}^2_\theta$, and the derivation generalizes immediately to the case of $\mathbb{R}^{2n}_\theta$. In this form, the non-locality leads to an IR divergence as

---

[7] Starting with the noncompact case makes IR issues even more tricky, while the present derivation is very clean.

[8] In many papers on NC field theory [6,7], the kinetic term is defined as $\partial_i\phi\partial_i\phi$ rather than $[X^i, \phi][X_i, \phi]$. Then the closed string metric rather than the open string one appears in the last line of (2.63), reconciling it with the literature. Conceptually, the present matrix model approach seems more natural.

$k \to 0$, and the well-known failure[9] of the standard renormalization procedure with local coun-
terterms is obvious given the non-local nature of the theory. In particular, the loop variables p are
now properly understood as position variables x, y, z. The IR divergence in (2.63) suggests that
the standard translation-invariant vacuum is inappropriate, and the non-local equation of motion

$$0 = \left(\Box + \mu^2 + \frac{g}{3}\mu_N^2\right)\phi(x) + \frac{g}{6}\frac{N^2}{\mathrm{Vol}(\mathcal{M})R_N^2}\int_{\mathcal{M}} dy \, \frac{\phi(y)}{|x-y|^2 + \frac{\tilde{\mu}^2}{R_N^2}} \qquad (2.65)$$

suggest the presence of phase transitions and non-trivial "striped" vacua [36–38]. Analogous
remarks apply to NC gauge theory.

While this non-local nature of generic NC field theories excludes their application as funda-
mental theories, they may still be useful e.g. as effective description of physics in strong magnetic
fields, and possibly other contexts. However, there is an important exception to this conclusion,
given by the maximally supersymmetric IKKT and BFSS matrix models. We will see that the
nonlocality is much milder in the IKKT model, given by 10-dimensional supergravity coupled
to the brane. More sophisticated backgrounds such as fuzzy S_N^4 in this model are promising can-
didates for the quantum nature of space–time at short distances. The present methods are also
applicable in these backgrounds, as shown in [24].

2.4. (Quasi-) Coherent states on generic fuzzy spaces

To show the applicability of the above coherent state methods to generic quantum geome-
tries,[10] we recall the general concept of quasi-coherent states introduced in [30], cf. [39,40].
Given any background defined in terms of D hermitean matrices $X^a \in End(\mathcal{H})$, they are de-
fined to be the ground states $|x\rangle$ of the point probe Hamiltonian

$$H_x = (X^a - x^a)(X_a - x_a), \qquad H_x|x\rangle = E(x)|x\rangle \qquad (2.66)$$

for arbitrary $x \in \mathbb{R}^D$. It follows that

$$\langle H_y \rangle_x = \Delta^2(x) + |\vec{\mathbf{x}}(x) - \vec{y}|^2 \qquad (2.67)$$

where $\langle . \rangle_x = \langle x|.|x\rangle$ and $\vec{\mathbf{x}}(x) = \langle X^a \rangle_x$ and

$$\Delta^2(x) := \sum_{a=1}^{d} \langle \left(X^a - \mathbf{x}^{\mathbf{a}}(x)\right)\left(X_a - \mathbf{x}_{\mathbf{a}}(x)\right)\rangle_x \qquad (2.68)$$

is the dispersion. We assume that this defines a "brane" i.e. a sub-variety $\mathcal{M} \subset \mathbb{R}^D$ where $\Delta^2(x)$
is small, and $E(x)$ grows quadratically in the directions transversal to the brane. We assume for
simplicity that these ground states are non-degenerate, defining a rank one projector

$$|x\rangle\langle x| = P_0(H_x). \qquad (2.69)$$

Hence the $|x\rangle$ form a $U(1)$ bundle $\mathcal{B}$ over $\mathcal{M}$. As in section 2.1, we can then map operators in
$\phi \in End(\mathcal{H})$ to functions via

---

[9] This may be circumvented by adding additional terms to the action which strongly modify the noncommutative
geometry, cf. [14].

[10] We only have in mind here the case where the algebra of functions on a compact space is finite-dimensional. There
are many examples where this is not satisfied, and these are expected to have a very different intrinsic nature.

$$\phi(x) = \langle x | \phi | x \rangle \tag{2.70}$$

and

$$\langle x | [\phi, \psi] | x \rangle \approx i\{\phi, \psi\} \tag{2.71}$$

defines a bracket on the classical functions which approximately satisfies the Leibniz rule and the Jacobi identity for large N. This recovers the Poisson bracket for functions. The corresponding (NC) symplectomorphisms $U = e^{i\Lambda(x)}$ define a connection ∇ on $\mathcal{B}$, whose curvature should be the symplectic form ω associated to the Poisson structure. In particular, the symplectic form will satisfy the quantization condition

$$\dim \mathcal{H} = \text{Vol}_\Omega(\mathcal{M}) = \int_{\mathcal{M}} \Omega \tag{2.72}$$

where $\Omega = \frac{1}{(2\pi)^n n!} \omega^{\wedge n}$ is the symplectic volume form on $\mathcal{M}$. We can then write down the following formula for a resolution of the unit in terms of the coherent states, generalizing (2.6):

$$\mathbb{1} = \frac{\dim \mathcal{H}}{\text{Vol}\mathcal{M}} \int_{\mathcal{M}} \Omega |x\rangle \langle x| . \tag{2.73}$$

As a heuristic justification, we note that the expression on the rhs should be invariant under the connection ∇ (since ω is the curvature of ∇), and therefore invariant under symplectomorphisms. This means that it should commute with the generators (at least to a very good approximation), and therefore it should be proportional to $\mathbb{1}$. A rigorous proof or qualification of the overcompleteness relation (2.73) in the generic case is left as a challenge to future work. As before, the localization property of the coherent states can then be written as

$$\langle y | z \rangle = \frac{1}{c_N} \tilde{\delta}(x, y), \qquad c_N = \frac{\dim \mathcal{H}}{\text{Vol}(\mathcal{M})} . \tag{2.74}$$

If the above assumptions are satisfied, then all the formulae for the loop integrals developed in section 2.3 are applicable also in this general case.

3. Higher loops and 't Hooft approach to NC QFT

Given these powerful techniques, one would like to go beyond the one-loop approximation. We will briefly discuss the generalization to higher loops in the spirit of 't Hooft's double line representation, and possible non-perturbative setups.

Suppose we want to compute the n-point functions of a scalar field ϕ on some fuzzy space at higher loops. One approach is to first write down the perturbative contributions as usual using a Gaussian integration using some arbitrary but **ordinary** basis for the matrix modes, and then to rewrite these Feynman rules in terms of the over-complete coherent state representations. This will result in a 't Hooft-type double line representation with the simple propagators (2.36). To make this more explicit, consider complex scalar fields on the fuzzy sphere with action $S[\phi] = S_0[\phi] + S_{\text{int}}[\phi]$ with free part S_0 as in (2.48), and S_{int} could contain any terms of the form $\frac{1}{N} \text{tr}(\phi^*\phi)^n$. We can expand ϕ in an arbitrary basis

$$\phi^i_j = \sum_A (\hat{Y}^A)^i_j \, \varphi_A, \qquad Y^A \in \mathcal{A} = End(\mathcal{H}) \tag{3.1}$$

where $i, j = 1, ..., N$ labels a basis of $\mathcal{H}$. The correlators are obtained as usual from

$$Z[J] = \int_{\mathcal{A}} D\phi e^{-S[\phi] + \mathrm{tr}\phi J} = e^{-W[J]} . \tag{3.2}$$

Then the perturbative expansion of a correlator is given be the sum of contractions. This is most transparent in the matrix basis using ϕ_j^i rather than φ_A. Then the free propagator

$$\langle \phi_j^i \phi^{*k}_{\ l} \rangle_0 \quad \in \mathcal{A} \otimes \mathcal{A}^* \tag{3.3}$$

is viewed as an element in $\mathcal{A} \otimes \mathcal{A}^* \cong End(\mathcal{A})$, and represented by a double line starting at $\binom{i}{j}$ and ending at $\binom{k}{l}$. Since the vertices have the form of a matrix product, the Feynman rules are obtained directly in the 't Hooft double line organization, where the labels i, j of the lines are preserved in the vertices. The diagrams are then be viewed naturally as ribbon graphs on a Riemann surface. However, the labels are not preserved by the propagator, which makes the computations difficult.

The key is now to translate these Feynman–'t Hooft rules into the coherent state representation, for each given diagram. All we have to do is use the form (2.36) for the (approximate) propagator,

$$(\bar{\Box} + \mu^2)^{-1} = c_N^2 \int_{\mathcal{M} \times \mathcal{M}} dx dy \left| \begin{matrix} x \\ y \end{matrix} \right) \frac{1}{|x - y|^2 + \tilde{\mu}^2} \left(\begin{matrix} x \\ y \end{matrix} \right| \quad \in \mathcal{A} \otimes \mathcal{A}^* \tag{3.4}$$

which is explicitly written as an element in $\mathcal{A} \otimes \mathcal{A}^*$. Since these propagators are connected by canonically contracting the indices i.e. evaluating $\mathcal{A} = \mathcal{H} \otimes \mathcal{H}^*$ and $\mathcal{A}^* = \mathcal{H}^* \otimes \mathcal{H}$, this gives immediately the Feynman–'t Hooft rules where the lines of the propagator are now labeled by *positions* $x, y \in \mathcal{M}$ on the fuzzy space which are preserved by the propagators, and trivially connected at the vertices to form ribbon graphs. The sums over the internal lines become position integrals over $\mathcal{M}$. The key feature is that both the propagators and the vertices are now diagonal in position space. The resulting Feynman rules are very simple and natural, and their evaluation is much easier (!) than in ordinary QFT. The one-loop diagrams lead to the diagrams 1 and 2, and the Feynman rules reproduce directly our results in section 2.3, even quicker than using the trace-log formula. It is then easy to compute higher loop corrections; some explicit computations will be presented elsewhere [41] This simplification also leads to the hope that one may devise new techniques to extract their asymptotics, analogous to those in matrix models [42].

Since the resulting formalism is so simple, it is tempting to skip the intermediate steps and to use directly the coherent state representation of general operators $\phi \in End(\mathcal{H})$, e.g. as

$$\phi = \int_{\mathcal{M} \times \mathcal{M}} dx dy |x\rangle \phi(x, y) \langle y| \tag{3.5}$$

where ϕ is represented as a function on $\mathcal{M} \times \mathcal{M}$. Although this representation is not unique, one might try to define a path integral for such functions leading to the same type 2-line diagrams as before. On the other hand, it seems more reasonably to replace the functions $\phi(x, y)$ by finite matrices

$$\phi(x, y) \leftrightarrow \Phi_{x,y} \tag{3.6}$$

on some equidistributed lattice on $\mathcal{M}$ consisting of $\dim \mathcal{H}$ points x_i, interpreted as (matrix) string with energy

$$\tilde{\Box} \Phi = [E, [E, \Phi]] = |x - y|^2, \qquad E_{x,y} = x - y . \tag{3.7}$$

This is the picture of bi-local fields introduced in [12]. Then the original model could be replaced by the simplified "string" matrix model

$$S_{red}[\Phi] = tr([E, \Phi]^2 + \Delta^2 \Phi^2 + g \Phi^4) \qquad (3.8)$$

which can be treated by the usual methods. Now the kinetic term is simplified, and now represented by a single matrix E instead of the set of matrices $\{X^a\}$. In this form, non-perturbative approaches should be applicable. A similar simplification should apply for Yang–Mills matrix models.

The present techniques may also be useful in an exact RG approach to NC field theory, noting that the modes with highest energy are the longest string modes. In the case of a fuzzy sphere, these are the strings connecting opposite points, which should explain the origin of the antipodal terms found in [43]. We leave these topics for future investigations.

Finally, these ideas should also provide an efficient way to compute quantum corrections for *ordinary $SU(N)$* Yang–Mills theory with large N around non-trivial (Higgs) vacua corresponding to fuzzy extra dimensions [44,45]. Again the Feynman rules can be rewritten in a string basis for $\mathfrak{u}(N)$ as above, and the propagators acquire weight factors corresponding to their distance in internal space as above. Then the computations should be comparable to large N gauge theory computations in the trivial vacuum.

Minkowski signature   The present paper is focused on the case of Euclidean signature. In the case of Minkowski signature, $\Box$ has a non-trivial kernel, corresponding to time-like string states $\psi_{x,y}$ with

$$(y - x)^2 + 2\Delta^2 + \mu^2 = 0 \qquad (3.9)$$

One might worry if such a model can ever be well-defined, but numerical simulations [46] demonstrate that this can be achieved by adding suitable IR-regulator terms to the action. We expect that the present techniques provide a useful tool also in the case of Minkowski signature, which is left for future work.

4. The 1-loop effective potential for the IKKT model

The above formalism is clearly also applicable to gauge theories, which are defined by matrix models of Yang–Mills type. In this section, we will use this to study the one-loop effective actions for the maximally supersymmetric IKKT matrix model, on some noncommutative brane background with the required properties as described above. Again there is considerable overlap with [12], but we develop a formalism applicable to generic fuzzy spaces, thus preparing the ground for the application on S_N^4 in [24]. The background is defined in terms of 10 hermitian matrices

$$X^a \sim x^a : \mathcal{M} \hookrightarrow \mathbb{R}^{10} \qquad (4.1)$$

interpreted as quantized embedding function of some quantized symplectic manifold $\mathcal{M}$ in $\mathbb{R}^{10}$. They define the flux

$$[X^a, X^b] = i \Theta^{ab} \qquad (4.2)$$

which corresponds to the quantized Poisson brackets of the x^a. The IKKT or IIB matrix model is defined by the action

$$S_0[X] = \frac{1}{g^2} \text{Tr}\left(-[X_a, X_b][X^a, X^b] + 2\mu^2 X^a X_a + \overline{\Psi}\gamma_a[X^a, \Psi]\right). \tag{4.3}$$

To regularize possible IR singularities, we added a (small) mass μ^2. The equations of motion for the bosonic matrices are

$$(\Box + \frac{1}{2}\mu^2)X_a = 0, \qquad \Box = [X^a, [X_a, .]] . \tag{4.4}$$

Now consider fluctuations around some (not necessarily on-shell) background $X^a \to X^a + \mathcal{A}^a(X^a)$. Then the quadratic action for $\mathcal{A}^a$ is given by

$$S[X + \mathcal{A}] = S[X] + \frac{2}{g^2} \text{Tr}\Big(2\mathcal{A}^a(\Box + \mu^2)X_a$$
$$+ \mathcal{A}_a\big((\Box + \mu^2)\delta_b^a + 2i[\Theta^{ab}, .] - [X^a, [X^b, .]]\big)\mathcal{A}_b\Big) .$$

Hence the quadratic fluctuations $\mathcal{A}^a$ are governed by the quadratic form

$$\text{Tr}\mathcal{A}_a\Big((\Box + \mu^2)\delta_b^a + 2i[\Theta^{ab}, .] - [X^a, [X^b, .]]\Big)\mathcal{A}_b . \tag{4.5}$$

The last term can be canceled by adding a suitable Faddeev–Popov gauge-fixing term for $f = [\mathcal{A}^a, X_a] = 0$ [47]. The one-loop effective action on a matrix background is defined by the Gaussian integration around the background

$$Z[X] = \int_{1 \text{ loop}} d\mathcal{A} d\Psi e^{-S[X+\mathcal{A}, \Psi]} = e^{-\Gamma_{\text{eff}}[X]} \tag{4.6}$$

and we will denote the bare and one-loop contributions as

$$\Gamma_{\text{eff}}[X] = S_0[X] + \Gamma_{1\text{loop}}[X] . \tag{4.7}$$

We recall the following form of the one-loop effective action in the IKKT model [15,47,48]

$$\Gamma_{1\text{loop}}[X] = \frac{1}{2}\text{Tr}\Big(\log(\Box + \mu^2 - M_{ab}^{(\mathcal{A})}[\Theta^{ab}, .]) - \frac{1}{2}\log(\Box - M_{ab}^{(\psi)}[\Theta^{ab}, .]) - 2\log(\Box)\Big)$$

$$= \frac{1}{2}\text{Tr}\left(\sum_{n>0}\frac{1}{n}\Big((\Box^{-1}(-M_{ab}^{(\mathcal{A})}[\Theta^{ab}, .] + \mu^2))^n - \frac{1}{2}(-\Box^{-1}M_{ab}^{(\psi)}[\Theta^{ab}, .])^n\Big)\right)$$

$$= \frac{1}{2}\text{Tr}\left(\frac{1}{4}(\Box^{-1}(M_{ab}^{(\mathcal{A})}[\Theta^{ab}, .])^4 - \frac{1}{8}(\Box^{-1}M_{ab}^{(\psi)}[\Theta^{ab}, .])^4 + \mathcal{O}(\Box^{-1}[\Theta^{ab}, .])^5\right)$$

$$+ \frac{1}{2}\mu^2\text{Tr}\Box^{-1} + O(\mu^4) \tag{4.8}$$

with $a, b = 1, ..., 10$, where

$$\begin{aligned}(M_{ab}^{(\psi)})_\beta^\alpha &= \frac{1}{4i}[\gamma_a, \gamma_b]_\beta^\alpha \\ (M_{ab}^{(\mathcal{A})})_d^c &= i(\delta_b^c\delta_{ad} - \delta_a^c\delta_{bd}),\end{aligned} \tag{4.9}$$

and the $2\log\Box$ term arises from the ghost contribution. Here $\Box$ and Θ^{ab} refer to the operators defined for the background X_i as in the previous sections. Note that the coupling constant g drops out from $\Gamma_{1\text{loop}}$. For $\mu = 0$, the first non-vanishing term in this expansion is $n = 4$ due to maximal supersymmetry. However for soft SUSY breaking with $\mu^2 \neq 0$, there are contributions with $n = 1$, starting with the above μ^2 term.

This 4th order term plus the leading μ^2 contribution is given by the following expression [47]:

$$\Gamma_{1\text{loop};4}[X] = \frac{1}{8}\text{Tr}\left((\Box^{-1}(M_{ab}^{(\mathcal{A})}[\Theta^{ab}, .])^4 - \frac{1}{2}(\Box^{-1}M_{ab}^{(\psi)}[\Theta^{ab}, .])^4\right)$$

$$= \frac{1}{4}\text{Tr}\left(\Box^{-1}[\Theta^{a_1 b_1}, \dots \Box^{-1}[\Theta^{a_4 b_4}, .]]]\right)$$

$$\left(-4g_{b_1 a_2}g_{b_2 a_3}g_{b_3 a_4}g_{b_4 a_1} - 4g_{b_1 a_2}g_{b_2 a_4}g_{b_4 a_3}g_{b_3 a_1} - 4g_{b_1 a_3}g_{b_3 a_2}g_{b_2 a_4}g_{b_4 a_1}\right.$$

$$\left. + g_{b_1 a_2}g_{b_2 a_1}g_{b_3 a_4}g_{b_4 a_3} + g_{b_1 a_3}g_{b_3 a_1}g_{b_2 a_4}g_{b_4 a_2} + g_{b_1 a_4}g_{b_4 a_1}g_{b_2 a_3}g_{b_3 a_2}\right)$$

$$(4.10)$$

and the leading term in μ^2 is

$$\Gamma_{1\text{loop};\mu^2}[X] = -\frac{1}{4}\mu^2\text{Tr}\left(\Box^{-1}\right). \tag{4.11}$$

To explain the new technique for evaluating the trace using string states, we focus on the case of an irreducible fuzzy space of brane[11] given by the quantization of a symplectic manifold $\mathcal{M}$; stacks of branes will not be discussed here. We assume that there is an over-complete set of coherent states $|x\rangle$ on $\mathcal{M}$, with the associated string states $|y\rangle\langle x|$ spanning $End(\mathcal{H})$. According to the results in the previous sections, we can then write

$$\Box^{-1}(|x\rangle\langle y|) \sim \frac{1}{|x-y|^2 + 2\Delta^2}|x\rangle\langle y|$$

$$\Box^{-1}[\Theta^{ab}, .](|x\rangle\langle y|) \sim \frac{1}{|x-y|^2 + 2\Delta^2}\delta\Theta^{ab}(x, y)|x\rangle\langle y| \tag{4.12}$$

on a sufficiently slowly-varying background, where

$$\delta\Theta^{ab}(x, y) := \Theta^{ab}(x) - \Theta^{ab}(y) \tag{4.13}$$

are now ordinary, commutative functions rather than operators. We assume that the dispersion $\Delta_x^2 \approx \Delta^2$ is independent of x for simplicity. Then the traces over $End(\mathcal{H})$ can be evaluated as

$$\Gamma_{1\text{loop};4}[X] \sim \frac{1}{4}\frac{(\dim\mathcal{H})^2}{(\text{Vol}\mathcal{M})^2}$$

$$\int\limits_{\mathcal{M}\times\mathcal{M}} \Omega_x\Omega_y \frac{\delta\Theta^{a_1 b_1}(x, y)\delta\Theta^{a_2 b_2}(x, y)\delta\Theta^{a_3 b_3}(x, y)\delta\Theta^{a_4 b_4}(x, y)}{(|x-y|^2 + 2\Delta^2)^4}$$

$$3\left(-4g_{b_1 a_2}g_{b_2 a_3}g_{b_3 a_4}g_{b_4 a_1} + g_{b_1 a_2}g_{b_2 a_1}g_{b_3 a_4}g_{b_4 a_3}\right)$$

$$= \frac{3}{4}\int\limits_{\mathcal{M}\times\mathcal{M}} dxdy\rho(x)\rho(y)\frac{S_4[\delta\Theta(x, y)]}{(|x-y|^2 + 2\Delta^2)^4}$$

$$\Gamma_{1\text{loop};\mu^2}[X] \sim \frac{5}{2}\int\limits_{\mathcal{M}\times\mathcal{M}} dxdy\rho(x)\rho(y)\frac{\mu^2}{|x-y|^2 + 2\Delta^2} \tag{4.14}$$

suppressing the target space metric g_{ab}. Here $\Omega_x = \rho(x)dx$ is the symplectic volume form on $\mathcal{M}$ such that $\dim\mathcal{H} = \text{Vol}\mathcal{M}$. We denote accordingly

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[11] The string theoretical picture is that N D-instantons bound to and "dissolved" on a D-brane $\mathcal{M}$.

$$S_4[\delta\Theta] = -4tr\,\delta\Theta^4 + (tr\,\delta\Theta^2)^2 \ . \tag{4.15}$$

An important observation [49] is the following: If $\delta\Theta^{ab}(x, y)$ has rank ≤ 4, then

$$\begin{aligned}
-S_4[\delta\Theta] &= 4tr(\delta\Theta g\,\delta\Theta g\,\delta\Theta g\,\delta\Theta g) - (tr\,\delta\Theta g\,\delta\Theta g)^2 \\
&= 4(\delta\Theta_+^{ab}\,\delta\Theta_{+ba})\,(\delta\Theta_-^{cd}\,\delta\Theta_{-dc}), \qquad \delta\Theta_\pm = \delta\Theta \pm \star_g \delta\Theta \\
&\geq 0
\end{aligned} \tag{4.16}$$

where $\star_g$ denotes the 4-dimensional Hodge star with respect to $g_{\mu\nu}$. This leads to an attractive interaction, which vanishes precisely in the (anti-) selfdual case $\delta\Theta = \pm \star_g \delta\Theta$. Thus parallel 4-dimensional branes with flux Θ_A^{ab} and Θ_B^{ab} are attracted to each other with an attractive $-\frac{1}{r^4}$ potential [15,48] and are unstable, unless $\Theta_A^{ab} - \Theta_B^{ab}$ is (anti-)selfdual. For fluxes with rank ≥ 6, the interaction is in general not attractive. $\Gamma_{1\text{loop};4}$ vanishes identically for a single branes with constant flux such as $\mathbb{R}_\theta^4$, which reflects their BPS property.

For slowly varying backgrounds, $\Gamma_{1\text{loop};4}[X]$ describes interactions which decay like $|x - y|^{-8}$, but are bounded for short distances by the NC cutoff Δ^2. In the next section, we will identify these interactions with linearized IIB supergravity on $\mathcal{M}$, generalizing previous results for block-matrix configurations and simple backgrounds [15–23].

As discussed in section 2.3, possible UV divergences are associated with large eigenvalues of $\square$, which corresponds to widely separated points $x, y \in \mathcal{M}$, or longs strings $|y\rangle\langle x|$. This is the essence of UV/IR mixing. Due to the short-range interaction in the supersymmetric model, this does not lead to any problems (at one loop) on manifolds with dimension less than 8, in contrast to non-supersymmetric models.

4.1. Induced interactions and linearized IIB supergravity

Now we want to understand the physics of the above one-loop interactions. It was conjectured in [48] that the IKKT matrix model provides a non-perturbative definition of IIB string theory on $\mathbb{R}^{10}$. The main direct evidence (i.e. based solely on the matrix model itself) are loop computations as above for the interactions of simple branes in target space, which can be computed in the matrix model and compared with string theory or rather IIB supergravity. The relevant (bosonic) degrees of freedom in IIB supergravity mediating such interactions are the graviton, the dilaton, and the anti-symmetric 2-form and 4-form fields.[12] It was indeed found in [48], and corroborated in subsequent works [15,18–20,22,23] that the interaction in the matrix model matches with supergravity at least in the long-distance limit. However, the methods were limited to highly symmetric branes or "D-particles" represented by block-matrices. Using the above techniques, we can extend this to rather generic branes, as long as they admit coherent states as discussed above.

First, consider the self-interaction of an irreducible brane background $\mathcal{M}$. Expanding the above action using the short-hand notation

$$\delta\Theta(x, y) = \Theta_x - \Theta_y \tag{4.17}$$

we get

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[12] The separation between the NSNS and the RR form fields is not clearly visible from the matrix model point of view.

$$S_4[\delta\Theta(x,y)] = -4tr\delta\Theta(x,y)^4 + (tr\delta\Theta(x,y)^2)^2$$
$$= -4tr(T_x^2) + tr(T_x)^2 + (x \leftrightarrow y)$$
$$+ 4\big(4tr(\Theta_x\Theta_x\Theta_x\Theta_y) - tr(\Theta_x\Theta_y)tr(\Theta_x\Theta_x) + (x \leftrightarrow y)\big)$$
$$- 16tr(T_xT_y) + 2trT_xtrT_y$$
$$- 8tr(\Theta_x\Theta_y\Theta_x\Theta_y) + 4(tr(\Theta_x\Theta_y))^2 \tag{4.18}$$

which disappears for $x = y$ as it must. Here we identify the matrix–energy–momentum tensor of the (background) brane in target space as

$$T^{ab}[\Theta] = \Theta^{ac}\Theta^{cb}. \tag{4.19}$$

This is the "closed string" e–m tensor, in agreement[13] with related results in the literature, cf. [19,22]. Furthermore, we denote the effective propagator on $\mathbb{R}^{10}$ as

$$D(x-y) = \frac{3}{2\pi^5}\frac{1}{(|x-y|^2+\Delta^2)^4} \sim \frac{3}{2\pi^5}\frac{1}{|x-y|^8}, \tag{4.20}$$

which for distances $|x-y|^2 \gg \Delta^2$ coincides with the 10-dimensional (Euclidean) propagator in 10 dimensions, but is regularized in the UV by Δ^2. Then the effective interaction induced at one loop is

$$\Gamma_{1\text{loop};4}[X] \sim \frac{\pi^5}{2}\int_{\mathcal{M}\times\mathcal{M}} dxdy\rho(x)\rho(y)\Big(2S_4(\Theta(x))\,D(x-y)$$
$$+ 16\big(\Theta^{ae}(x)\Theta_{ef}(x)\Theta^{fb}(x)$$
$$+ \frac{1}{4}\Theta^{ab}(x)\Theta^{ef}(x)\Theta_{ef}(x)\big)D^{(AS)}_{ab;cd}(x,y)\,\Theta^{dc}(y)$$
$$- 8T^{ab}(x)D^{(S)}_{ab;cd}(x,y)\,T^{cd}(y)$$
$$+ 4\Theta^{aab}(x)\Theta^{ef}(x)D^{(AS)}_{abef;cdgh}(x,y)\,\Theta^{cd}(y)\Theta^{gh}(y)\Big). \tag{4.21}$$

Here

$$D^{(S)}_{ab;cd}(x,y) = \big(g_{ac}g_{bd} + g_{ad}g_{bc} - \frac{1}{4}g_{ab}g_{cd}\big)D(x-y)$$
$$D^{(AS)}_{abef;cdgh}(x,y) = \big(g_{ac}g_{bd}g_{eg}g_{fh} + g_{ac}g_{bh}g_{ed}g_{fg} - g_{ac}g_{bg}g_{ed}g_{fh}\big)D(x-y)$$
$$D^{(AS)}_{ab;cd}(x,y) = \big(g_{ac}g_{bd} - g_{ad}g_{bc}\big)D(x-y) \tag{4.22}$$

For $|x-y|^2 \gg \Delta^2$, the fourth line in (4.21) can clearly be interpreted in terms of a graviton exchange in $\mathbb{R}^{10}$, and $D^{(S)}_{\mu\nu;\alpha\beta}(x,y)$ is indeed the graviton propagator in de Donder gauge. The last line is due to the exchange of a rank four antisymmetric tensor, and $D^{(S)}_{\mu\nu;\alpha\beta}(x,y)$ is the propagator for a rank four antisymmetric tensor. The first line can be interpreted in terms of a dilaton exchange [23,50] coupling the background density $\rho(y)$ to

$$S_4(\Theta) = -4T^{ab}(x)T_{ab}(x) + T(x)T(x), \qquad T = T^{ab}g_{ab}. \tag{4.23}$$

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[13] This should be contrasted with the effective ("open-string") energy momentum tensor which arises in the effective gauge theory on the brane with the open string metric, which has the standard form as in classical Yang–Mills gauge theory.

Finally the second and third lines can be interpreted as exchange of an antisymmetric rank 2 tensor field B_{ab}, which couples to branes[14] via terms of the form [50]

$$\int B_{ab}(\Theta^{ab} + \Theta^{ac}\Theta_{cd}\Theta^{db} + \frac{1}{4}\Theta^{ab}\Theta_{cd}\Theta^{cd} + ...). \tag{4.24}$$

Hence all these terms can be interpreted as interaction mediated by an exchange of the basic fields in 10-dimensional IIB supergravity, coupled to a brane described by the matrix background Θ^{ab}. The specific form of the interaction ensures that it cancels identically for constant, flat backgrounds. Even though this mechanism has in principle been known for a long time [48], the derivations in the literature are based on separate block-matrices, and can be trusted only for large separations between localized branes. The present coherent state representation captures the detailed x-dependence for generic curved branes within the matrix model. With these tools at hand, it should be possible to derive also the (analog of the) Dirac–Born–Infeld effective action starting from the matrix model beyond the one-loop order, incorporating the full quantum effective action for slowly varying fields.

It is quite interesting that this interaction has a UV cutoff scale Δ^2 in $D(x - y)$, reflecting the quantum structure of the brane. This is as expected on branes with B-field, corresponding to noncommutative spaces. Moreover, the above derivation is easily adapted also to branes with vanishing 2-form flux, such as the fuzzy 4-sphere S_N^4 [51]. This is properly understood as a degenerate higher-dimensional quantized symplectic space, where the B-field is averaged over the degenerate S^2-fiber over S^4 [52–54]. The present method allows to compute the one-loop effective action also on this background, in a much simpler and more transparent way than the group-theoretical approach in [54]. This will be published elsewhere [24].

4.2. Fluctuations on a background

Since the 1-loop interaction vanishes for flat backgrounds, the above action becomes more intuitive for fluctuations $X^a = \bar{X}^a + \mathcal{A}^a$ on some background brane $\mathcal{M}$ described by $\bar{X}^a$. Then

$$\Theta^{ij} = \bar{\Theta}^{ij} + \mathcal{F}^{ij} \tag{4.25}$$

where $i\mathcal{F}^{ij} = [X^i, \mathcal{A}^j] - [X^i, \mathcal{A}^j] + [\mathcal{A}^i, \mathcal{A}^j]$ is an excitation on the background $\bar{\Theta}^{ij}$. To organize the various contributions, we note again that S_4 depends only on the combination

$$\delta\Theta^{ab}(x, y) := (\bar{\Theta}^{ab}(x) - \bar{\Theta}^{ab}(y)) + (\mathcal{F}^{ab}(x)) - \mathcal{F}^{ab}(y))$$
$$=: \delta\bar{\Theta}^{ab}(x, y) + \delta\mathcal{F}^{ab}(x, y) . \tag{4.26}$$

Assume that the background $\bar{\Theta}(x)$ is almost constant, while $\mathcal{F}(x)$ is varying on much shorter scales (but still long compared to Δ^2). Then $\delta\mathcal{F}(x, y) \gg \delta\bar{\Theta}(x, y)$, and we can organize $S_4(\mathcal{F})$ as follows

$$S_4(\delta\Theta) = S_4(\delta\mathcal{F}) + \mathcal{O}(\delta\mathcal{F}^3\delta\bar{\Theta}) + \mathcal{O}(\delta\mathcal{F}^2\delta\bar{\Theta}^2) + \mathcal{O}(\delta\mathcal{F}\delta\bar{\Theta}^3) + S_4(\delta\bar{\Theta}^4) \tag{4.27}$$

in decreasing order of significance. We mainly focus on the leading $O(\delta\mathcal{F}^4)$ terms. The mixed terms describes interactions of $\mathcal{F}$ with the background flux $\bar{\Theta}$; they may e.g. modify the propagator for the fluctuations $\mathcal{F}$. Finally $S_4(\delta\bar{\Theta}^4)$ corresponds to the self-interaction of the background as discussed in the previous section.

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[14] As explained in [50], the coupling of the brane to all the supergravity fields such as $B_{\mu\nu}$ etc. follows via T-duality from the well-known results that fundamental strings and lower-dimensional branes can be described in terms of the field strength of the $U(N)$ gauge field in the world-volume of a Dp-brane.

$O(\mathcal{F}^4)$ *term*   The $O(\mathcal{F}^4)$ term on a background $\mathcal{M}$ has the same structure as the $S_4(\Theta)$ term, with the propagator defined by the background $\mathcal{M}$. It is most transparent for widely separated field configurations $\mathcal{F}(x) = \mathcal{F}_A(x) + \mathcal{F}_B(x)$ where $\mathcal{F}_{A,B}(x)$ have non-overlapping support. Then the interaction terms for such configurations have the by now familiar form

$$
\begin{aligned}
\Gamma_{1\text{loop};4}[\mathcal{F}_A, \mathcal{F}_B] \sim \pi^5 \int\limits_{\mathcal{M}\times\mathcal{M}} dx\, dy\, \rho(x)\rho(y) \Big(& 2S_4(\mathcal{F}_A(x))\, D(x-y) \\
& + 16\big(\mathcal{F}_A^{ae}(x)\mathcal{F}_{Aef}(x)\mathcal{F}_A^{fb}(x) \\
& + \frac{1}{4}\mathcal{F}_A^{ab}(x)\mathcal{F}_A^{ef}(x)\mathcal{F}_{Aef}(x)\big)\, D_{ab;cd}^{(AS)}(x,y)\, \mathcal{F}_B^{dc}(y) \\
& - 8T_A^{ab}(x) D_{ab;cd}^{(S)}(x,y)\, T_B^{cd}(y) \\
& + 4\mathcal{F}_A^{aab}(x)\mathcal{F}_A^{ef}(x) D_{abef;cdgh}^{(AS)}(x,y)\, \mathcal{F}_B^{cd}(y)\mathcal{F}_B^{gh}(y) \Big).
\end{aligned}
\tag{4.28}
$$

There is a factor 2 which arises from the two possible associations $x \leftrightarrow A$, $y \leftrightarrow B$ and vice versa. This can again be interpreted as interactions due to the exchange of IIB sugra modes between the excitations A and B. Specifically, the first line is associated with a dilaton exchange with the background, the 2nd and 3rd lines with the exchange of an antisymmetric rank 2 field, the 4th line with a graviton exchange, and the last line with exchange of a rank 4 tensor field.

We could also obtain a derivative expansion of the above interaction by expanding the $\delta\mathcal{F} = \mathcal{F}(x) - \mathcal{F}(y)$ into powers of $(x - y)$. Then the effective action for $\mathcal{F}$ becomes a 4-th order derivative interaction with interaction strength given by Δ^{-4}, which was elaborated directly in [47]. Hence the above form provides a closed form for its long-distance behavior. For the nonabelian case, this one-loop action is known to provide the leading $\mathcal{F}^4$ term in an expansion of the DBI action (cf. [47]), and the present technique should allow to corroborate this connection in more detail.

4.3. Non-supersymmetric matrix models

Finally consider briefly the case of generic (non-supersymmetric) matrix models and their relation with NC gauge theory. As long as all fields are in the adjoint, the one-loop effective action can still be expressed in a similar way as (4.8), however starting at $O(\delta\mathcal{F}^2)$ rather than $O(\delta\mathcal{F}^4)$. At short distances, this leads to a derivative expansion starting with 2 derivatives of $\mathcal{F}$. At long distances, the propagators lead to a non-local interaction decaying like $(|x - y|^2 + \Delta^2)^{-2}$.

We can now make contact with the emergent gravity picture of NC gauge theory [35]: The $U(1)$ sector of such a NC gauge theory defines (in the local, semi-classical limit) a non-trivial effective ("open string") metric for the remaining fields. In accord with the mechanism of induced gravity, the 1-loop integrals of any fields on such a background induces an Einstein–Hilbert-type action in the effective action (among others). In the case of NC field theory this arises due to IR modes in the loops as verified in [8,9], corresponding to the leading term in the above derivative expansion of $\mathcal{F}$. The new insights in the present paper complement this picture by an explicit form for the induced long-distance interaction, which is due to the UV modes in the loops. In the case of maximal SUSY, this leads to 10D supergravity as shown above. In generic non-SUSY models this interaction will in general not lead to 4-dimensional Einstein gravity, but to a different type of shorter-range gravitational interaction. However as shown in a companion paper [24], the linearized 4D Einstein equations *do* emerge in the IKKT model, but only on more sophisticated "covariant" noncommutative backgrounds and by a different mechanism.

5. Conclusion

One message of this paper is that noncommutative field theory is very different from local field theory, and is more appropriately viewed as a theory of open strings ending on branes. Although this insight is not new [10], the formalism of bi-local string states makes this interpretation manifest and compelling from the noncommutative point of view. The bulk of the kinematic phase space consists of an UV sector whose degrees of freedom are described by string states $|x\rangle\langle y| \in End(\mathcal{H})$, introduced previously in [12]. These are naturally interpreted as open strings, and behave completely differently from classical fields. We develop a formalism based on integrals over string states which greatly simplifies the computation of the loop integrals. This leads to a simple closed expression for the one-loop effective action in position space for generic fuzzy spaces, and provides a clear picture of the non-locality encoded in the UV/IR mixing, which arises from long strings with high energy. The extension to higher loops is also indicated. A rigorous proof or qualification of the overcompleteness relation (2.73) in the generic case is left as a challenge to future work.

In the maximally supersymmetric IKKT matrix model, the present formalism allows to derive directly the position space interactions which arise from quantum effects on fuzzy brane backgrounds, confirming the interpretation in terms of IIB supergravity. This should provide an analytical tool to address the stabilization of 4-dimensional space–time in the matrix model, cf. [54]. It should also be possible now to derive directly the DBI action for branes in the matrix model. Finally, the techniques developed here are applied in [24] to the fuzzy 4-sphere, which exhibits 4-dimensional emergent gravity.

Even though generic non-commutative field theories defined by non-supersymmetric matrix models are non-local, this does not exclude applications in suitable contexts such as condensed matter physics with strong magnetic fields. Some of these models exhibit interesting phase structures [36–38,55–59], and the 't Hooft-like formalism proposed here should allow to greatly improve the analytic understanding of these models. Furthermore, suitable limits of these models may lead to non-trivial and interesting applications [14]. Therefore the development of these powerful techniques should be useful also in these contexts.

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