On the non-existence of quasi-3 designs

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Received 28 August 2001; received in revised form 25 February 2002; accepted 11 March 2002

Abstract

In this survey article we discuss the existence of quasi-3 designs, and settle the existence question for all $v < 144$. The only such designs are the ones already known. We also consider the question of whether the dual design of a quasi-3 design is a quasi-3 design.

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Keywords: Design; Quasi-3; Quasi-symmetric; Symmetric difference property

1. Introduction

We assume familiarity with the basic terminology of designs; the reader may consult [1,2] or [7] for definitions. Following [7], we use the term “square design” for what is usually called a symmetric design ($b = v$).

A square design is said to be quasi-3 for points if there exist two distinct nonnegative integers $x$ and $y$ such that for any three distinct points, the number of blocks containing all three is either $x$ or $y$. We refer to the integers $x$ and $y$ as the triple containment sizes. Such designs have been considered in Cameron [5] and Cameron and Van Lint [7, Chapter 5]. We shall say that a square design is quasi-3 for blocks if the number of points in the intersection of any three distinct blocks takes only two values. (A square design that is quasi-3 for blocks is called “nearly triply regular” in Raposa [11]. A design that is quasi-3 for points and also quasi-3 for blocks, with $x = 0$ in both cases, is called a “semi-symmetric 3-design” in Hughes [8,9].) A design $D$ is quasi-3 for

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\textsuperscript{1} Work partially supported by NSA Grant MDA904-99-1-0010.

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blocks if and only if the dual design $D^T$ is quasi-3 for points. Throughout this paper, the term “quasi-3” by itself will mean quasi-3 for points.

A 2-design is said to be quasi-symmetric if the number of points in the intersection of two blocks takes only two values. It is clear that the derived design $D_B$ with respect to a block $B$, of a square design $D$ that is quasi-3 for blocks, will be a quasi-symmetric design.

In an earlier paper [3] we discussed a particular family of parameters of quasi-3 designs, arising from regular Hadamard matrices.

In this paper we discuss the general existence question:

For which $(v,k,\lambda)$ does there exist a quasi-3 $(v,k,\lambda)$ design?

This article is mainly a survey article and a collection of the known results. In Section 2 we give all the known examples of quasi-3 designs, and in Section 3 we discuss the question of whether the dual design of a quasi-3 design is a quasi-3 design. In particular, we show that this is true for all known examples. In Section 4 we show that for all $v<144$ (and $\lambda>2$), there are no quasi-3 designs apart from the already known ones. This leaves open the possibility of a classification.

2. Known classes of quasi-3 designs

First we recall the known (to us) examples of quasi-3 designs.

(E1) Any square design with $\lambda\leq 2$ is quasi-3. If $\lambda=1$ (projective plane) this is obvious, and if $\lambda=2$ then three points cannot lie in two blocks, since two blocks meet in exactly two points. So three points lie in 0 or 1 blocks. (Square designs with $\lambda=2$ are also known as biplanes.)

(E2) The point-hyperplane design in $\text{PG}(n,q)$ is quasi-3, because three 1-dimensional subspaces of the corresponding $(n+1)$-dimensional vector space could span a 2-dimensional subspace or a 3-dimensional subspace. (Similarly, the point-$(k$-flat) design is “quasi-3”, but these are not square designs if $k<n−1$.) In this design we have $x=(q^{n−2}−1)/(q−1)$ and $y=\lambda=(q^{r−1}−1)/(q−1)$. In [5] (also Proposition 5.15 of [7]) quasi-3 designs with $y=\lambda$ are characterized as projective geometries.

(E3) This class consists of all SDP designs. The definition of an SDP design is that the sum (mod2) of (the characteristic vectors of) any three blocks $B,B',B''$ is either a block or a block complement. The identity

$$|B+B'+B''|=|B|+|B'|+|B''|−2|B\cap B'|−2|B\cap B''|−2|B\cap B'||+4|B\cap B'\cap B''|$$

shows that there are only two possibilities for $|B\cap B'\cap B''|$, proving that an SDP design is quasi-3 for blocks. In Section 3 we will show that the dual of any SDP design is also an SDP design, and it follows that any SDP design is also quasi-3 for points. Any SDP design must have parameters $v=2^{2m}$, $k=2^{2m−1}−2^{m−1}$, $\lambda=2^{2m−2}−2^{m−1}$, $x=2^{2m−3}−2^{m−1}$ and $y=2^{2m−3}−2^{m−2}$; see [7] or [10] for more on SDP designs.

(E4) By Lemma 4 in [3], the family $\mathcal{F}$ of all quasi-3 designs with parameters $(4u^2$, $2u^2−u$, $u^2−u)$ that correspond to quasi-3 regular Hadamard matrices is closed
under Kronecker products. If we start with $(16,6,2)$ designs and the regular $4 \times 4$ Hadamard matrix, we thereby generate an infinite subfamily $\mathcal{E}$ of $\mathcal{F}$ by taking all possible Kronecker products. This class of examples consists of all designs in family $\mathcal{E}$. As far as we are aware, the designs in $\mathcal{E}$ are the only ones in $\mathcal{F}$ that are known to exist.

There are (up to isomorphism) three designs with parameters $(16,6,2)$. They may be distinguished by their 2-ranks, which are 6, 7 and 8. The design with 2-rank 6 is an SDP design, but the designs with 2-ranks 7 and 8 are not SDP designs, although all are quasi-3 designs by example (E1). It is not hard to show that taking Kronecker products involving non-SDP designs results in non-SDP designs. Therefore, family $\mathcal{E}$ contains infinitely many non-SDP designs.

(E5) The complementary design of a quasi-3 design is quasi-3. For, by inclusion-exclusion, the number of blocks in a quasi-3 design $D$ containing none of three given points is $v - 3k + 3\lambda - \{x \text{ or } y\}$. Thus the complementary design $\bar{D}$ is a quasi-3$(v,v - k,v - 2k + \lambda)$ design with triple containment sizes $\bar{x} = v - 3k + 3\lambda - y$ and $\bar{y} = v - 3k + 3\lambda - x$. This class of examples contains all complementary designs of examples (E1)–(E4).

### 3. On the dual of a quasi-3 design

In [5] and in [7, pp. 75–76], the authors point out that it is not known whether the dual of a quasi-3 square design is necessarily quasi-3 also. In other words, if a design is quasi-3 for points, must it be quasi-3 for blocks? We consider this question in this section, giving some general information for arbitrary quasi-3 designs first.

#### 3.1. Formulas from counting

Suppose there exists a square $(v,k,\lambda)$ design that is quasi-3 for points with triple containment sizes $x$ and $y$. Let $\mathcal{A}$, respectively, $\mathcal{B}$, be the number of triples of distinct points that are together contained in $x$, respectively, $y$, blocks. Similarly, for each $i, \ 0 \leq i \leq \lambda$, let $\sigma_i$ be the number of triples of distinct blocks that intersect in exactly $i$ points. In the following, lower case letters denote points and upper case letters denote blocks.

First, counting triples $(a,b,c)$ and triples $(A,B,C)$ gives the following two equations:

$$\mathcal{A} + \mathcal{B} = v(v - 1)(v - 2) = \sum_{i=0}^{\lambda} \sigma_i.$$

Next, counting incidence flags $(a,b,c,A)$ in two ways, and counting flags $(a,A,B,C)$ in two ways, gives

$$\mathcal{A}x + \mathcal{B}y = vk(k - 1)(k - 2) = \sum_{i} i\sigma_i.$$
Counting flags \((a, b, c, A, B)\) in two ways and flags \((a, b, A, B, C)\) in two ways gives
\[
\mathcal{A}x(x-1) + \mathcal{B}y(y-1) = v(v-1)\lambda(\lambda-1)(\lambda-2) = \sum_i i(i-1)\sigma_i.
\]
Finally, counting incidence flags \((a, b, c, A, B, C)\) in two ways gives the equation:
\[
\mathcal{A}x(x-1)(x-2) + \mathcal{B}y(y-1)(y-2) = \sum_i i(i-1)(i-2)\sigma_i.
\]
These equations imply the following relations:

\begin{align*}
(D1) \quad \sum \sigma_i &= \mathcal{A} + \mathcal{B}, \\
(D2) \quad \sum i\sigma_i &= \mathcal{A}x + \mathcal{B}y, \\
(D3) \quad \sum i^2\sigma_i &= \mathcal{A}x^2 + \mathcal{B}y^2, \\
(D4) \quad \sum i^3\sigma_i &= \mathcal{A}x^3 + \mathcal{B}y^3
\end{align*}

and these also imply the following:

\begin{align*}
(D5) \quad \sum (i-x)(i-y)\sigma_i &= 0, \\
(D6) \quad \sum (i-x)^2(i-y)\sigma_i &= 0, \\
(D7) \quad \sum (i-x)(i-y)^2\sigma_i &= 0.
\end{align*}

We note that if the following equation were true,
\[
\sum_i i^4\sigma_i = \mathcal{A}x^4 + \mathcal{B}y^4,
\]
we could then derive
\[
\sum_i (i-x)^2(i-y)^2\sigma_i = 0,
\]
and vice-versa, and this would answer the question about dual designs in the affirmative. However, we can see no reason why this should be so.
By Eq. (D7) we see that if \( x = 0 \) then \( \sigma_i = 0 \) for \( i \neq 0, y \), which proves Theorem 1 (see below). By Eq. (D6) we see that if \( y = \lambda \) then \( \sigma_i = 0 \) for \( i \neq x, \lambda \), which answers the dual question for these designs.

3.2. The \( x = 0 \) case

In the earlier version [6] of [7], the authors state that when \( x = 0 \), the dual design of a quasi-3 design is quasi-3. We shall include two proofs of this result here because we shall use the result later, and we do not know of any published proof. The first proof is in the previous paragraph, using Eq. (D7). We also give a shorter, direct proof of this theorem, kindly supplied by a referee.

**Theorem 1.** The dual design of a quasi-3 design with \( x = 0 \) is a quasi-3 design.

**Proof.** Suppose there exists a square \((v,k,y)\) design \( D \) that is quasi-3 for points with triple containment sizes \( x = 0 \) and \( y > 1 \). Choose an incident point-block pair \((a,A)\), and let \( D' \) be the incidence structure whose points are the points of \( A \) other than \( a \), and whose blocks are the blocks of \( D \) containing \( a \), except for \( A \). Then \( D' \) has \( k - 1 \) points and \( k - 1 \) blocks. Each block of \( D' \) has \( \lambda - 1 \) points, and any two points of \( D' \) are in \( y - 1 \) blocks. Thus \( D' \) is a square \( 2 - (k - 1, \lambda - 1, y - 1) \) design, and so any two blocks of \( D' \) meet in \( y - 1 \) points. It follows that any three blocks of \( D \) which have a point in common meet in \( y \) points.

3.3. Known examples and the dual question

We consider each of the examples in Section 2 in light of the question about the dual design.

For designs in class (E1), the dual design is also quasi-3 since the dual design has the same parameters and therefore has \( \lambda \leq 2 \).

The dual design of the point-hyperplane design \( \text{PG}(n,q) \) (example (E2)) is quasi-3 because such designs always have a collineation. Alternatively, \( \text{PG}(n,q) \) has \( y = \lambda \) and the result follows from Eq. (D6) as mentioned above.

For designs in class (E3) we have the following theorem. This result is proved in [7] for the case of the “classical” SDP designs constructed from quadratic forms.

**Theorem 2.** If \( D \) is any SDP design, then the dual design \( D^T \) is also an SDP design.

**Proof.** Suppose \( D \) is an SDP design with \( v = 2^{2m} \), so that \( D^T \) is quasi-3. The 2-rank of \( D \) is \( 2m + 2 \), see [10]. Therefore the 2-rank of \( D^T \) is also \( 2m + 2 \), since row-rank equals column-rank. Since \( D^T \) has the same parameters and 2-rank as \( D \), by Corollary 11 of [10], \( D^T \) is an SDP design.

**Corollary 3.** An SDP design is quasi-3.
For designs in class (E4) we observe that they are produced by the Kronecker product of quasi-3 regular Hadamard matrices as in [3]. Since \((H \otimes K)^T = H^T \otimes K^T\) it follows from Lemma 4 of [3] that if the duals of the quasi-3 designs corresponding to \(H\) and \(K\) are also quasi-3, then the dual of the quasi-3 design corresponding to \(H \otimes K\) will be quasi-3. Since the designs we start with in (E4) have quasi-3 duals, the same is true of all designs in this class.

For designs in class (E5), suppose that \(A\) is the incidence matrix of a quasi-3 design whose dual design is also quasi-3. Then \(J - A\) is the incidence matrix of the complementary design. Since \((J - A)^T = J - A^T\), the dual design of the complementary design is quasi-3.

4. Existence and non-existence

We assume again that \(D\) is a square \((v, k, \lambda)\) design that is quasi-3 for points with triple containment sizes \(x\) and \(y\), where \(x < y\).

Fix two points \(P\) and \(Q\). Let \(\alpha = \alpha_{P,Q}\) be the number of points \(R \neq P, Q\) such that \(\{P, Q, R\}\) is contained in exactly \(x\) blocks. Let \(\beta = \beta_{P,Q}\) be the number of points \(R \neq P, Q\) such that \(\{P, Q, R\}\) is contained in exactly \(y\) blocks. Then

\[
\alpha + \beta = v - 2. \tag{1}
\]

Count in two ways the number of ordered pairs \((R, B)\) such that \(\{P, Q, R\} \subseteq B\) (where \(B\) is a block and \(R\) is a point not equal to \(P\) or \(Q\)) to get

\[
\alpha x + \beta y = \lambda(k - 2). \tag{2}
\]

**Lemma 4.** Let \(D\) be a square \((v, k, \lambda)\) design that is quasi-3 for points with triple containment sizes \(x\) and \(y\), where \(x < y\). Then

\[
x < \frac{\lambda(k - 2)}{(v - 2)} < y. \tag{3}
\]

**Proof.** By Eqs. (1) and (2) we get

\[
\lambda(k - 2) = \alpha x + \beta y > \alpha x + \beta x = x(v - 2)
\]

and

\[
\lambda(k - 2) = \alpha x + \beta y < \alpha y + \beta y = y(v - 2). \quad \square
\]

We recall the following lemma, which can be found in [3] or [12].

**Lemma 5.** Suppose there exists a quasi-symmetric 2-(\(v, k, \lambda\)) design with block intersection sizes \(x\) and \(y\). Then

\[
k(r - 1)(x + y - 1) - xy(b - 1) = k(k - 1)(\lambda - 1). \tag{4}
\]
Eq. (3) can be solved for $y$ in terms of $v, k, \lambda, x$ and combined with Lemma 4 to give the following.

**Algorithm 6.** Given $v, k, \lambda$, for each $x$ between 0 and $\lceil \lambda(k - 2)/(v - 2) \rceil - 1$ check if

$$y = \frac{\lambda(\lambda - 1)(\lambda - 2) - \lambda(k - 2)(x - 1)}{\lambda(k - 2) - x(v - 2)}$$

is an integer. If none of these $y$ values is an integer then there is no quasi-3 design with the given parameters $(v, k, \lambda)$.

The following three theorems are also of use for classification.

**Theorem 7.** If $y - x$ does not divide $k - x$ and $\lambda - x$, then there is no quasi-3 design with the parameters $(v, k, \lambda, x, y)$.

Theorem 7 (due to Goethals and Seidel) follows from Corollary 5.4 in [7], applied to the derived design of the dual.

In [5] Cameron considers quasi-3 designs with $x = 0$. He proves the following classification theorem. (The $x = 0$ case has also been considered by Hughes [8,9].)

**Theorem 8 ([5]).** Let $D$ be a square $(v, k, \lambda)$ quasi-3 design with $\lambda \geq 2$ and triple containment sizes $x$ and $y$, where $x = 0$. Then one of the following holds:

(a) $v = 2^{n+1} - 1$, $k = 2^n$, $\lambda = 2^{n-1}$ (where $n \geq 2$) and $(D^T)_R$ is a Hadamard 3-design.
(b) $v = (m + 1)(m^3 + 8m^2 + 19m + 11)$, $k = (m + 1)(m^2 + 5m + 5)$, $\lambda = (m + 1)(m + 2)$, where $m \geq 0$.

In [5] Cameron also proves the following theorem.

**Theorem 9 ([5]).** If $D$ is a Hadamard 2-design that is quasi-3 for points, then $D$ is either a projective geometry over $GF(2)$ or the unique 2-(11, 5, 2) design.

In the following table we list all parameter sets $(v, k, \lambda, x, y)$ which satisfy all of the following:

- $v \leq 144$.
- $\lambda(v - 1) = k(k - 1)$.
- $k \leq v/2$.
- they have not been eliminated by Algorithm 6.
- $y < \lambda$ (so in particular, projective geometries are omitted).
- $\lambda > 2$ (so in particular, biplanes are omitted).

The existence column refers to the existence of a quasi-3 design with those parameters, not just a square design. Here $n = k - \lambda$, the order of the design. In the comments
column we give the theorem that disproves existence immediately, or the class of the example in the case of existence. Some of these designs can also be ruled out by the Bruck–Ryser–Chowla theorem. We refer to the $v$ even case of the Bruck–Ryser–Chowla theorem as Schutzenberger’s theorem, which states that if $v$ is even and a square $(v,k,\lambda)$ design exists, then $n$ must be a square.

<table>
<thead>
<tr>
<th>$(v,k,\lambda,x,y)$</th>
<th>$n$</th>
<th>Existence</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(34,12,4,1,3)$</td>
<td>8</td>
<td>N</td>
<td>Schutzenberger</td>
</tr>
<tr>
<td>$(35,17,8,2,4)$</td>
<td>9</td>
<td>N</td>
<td>Theorem 7 or Theorem 9</td>
</tr>
<tr>
<td>$(59,29,14,4,7)$</td>
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<td>N</td>
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</tr>
<tr>
<td>$(64,28,12,4,6)$</td>
<td>16</td>
<td>Y</td>
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<td>N</td>
<td>Theorem 7 or Theorem 8</td>
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<tr>
<td>$(66,26,10,3,5)$</td>
<td>16</td>
<td>N</td>
<td>Theorem 7</td>
</tr>
<tr>
<td>$(70,24,8,2,4)$</td>
<td>16</td>
<td>N</td>
<td>Theorem 10 (below)</td>
</tr>
<tr>
<td>$(77,20,5,1,4)$</td>
<td>15</td>
<td>N</td>
<td>Theorem 7</td>
</tr>
<tr>
<td>$(78,22,6,0,2)$</td>
<td>16</td>
<td>N</td>
<td>Theorem 11 (below)</td>
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<tr>
<td>$(144,66,30,12,15)$</td>
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<td>?</td>
<td></td>
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</tbody>
</table>

**Theorem 10** (Calderbank [4]). *There does not exist a $(70,24,8,2,4)$ quasi-3 design.*

**Proof.** If such a design did exist, the derived design of the dual design would be a $(24,8,7)$ quasi-symmetric design with block intersection sizes 2 and 4. Calderbank [4] proves that such a quasi-symmetric design does not exist. □

We remark that a square $(70,24,8)$ design does exist.

**Theorem 11.** *There does not exist a $(78,22,6,0,2)$ quasi-3 design.*

**Proof.** By Theorem 1, since $x = 0$, if such a design does exist, the dual design must also be quasi-3. In [9] Hughes proves that such a design does not exist. □

We remark that a square $(78,22,6)$ design does exist. We also point out that the parameters of Theorem 11 are the $m = 1$ case of Theorem 8 part (b).

We have been unable to settle the existence of a quasi-3 $(144,66,30,12,15)$ design.
In conclusion, we have settled the existence question for all $v < 144$ (where $\lambda > 2$). The only designs which exist are the known examples. This keeps open the possibility of a classification of quasi-3 designs. In particular, it is possible that any quasi-3 design with $y < \lambda$ and $\lambda > 2$ has parameters $(4u^2, 2u^2 - u, u^2 - u)$ for some $u$.

References