Journal of Mathematical Analysis and Applications **250,** 204–221 (2000)
doi:10.1006/jmaa.2000.7078, available online at http://www.idealibrary.com on IDEAL®

Analytical and Numerical Solutions of Nonlinear Differential Equations Arising in Non-Newtonian Fluid Flows

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Received May 15, 2000

Solutions for a class of nonlinear second order differential equations, arising in a viscoelastic fluid flow at a rotating cylinder, are obtained. Furthermore, using the Shauder theory and the perturbation technique existence, uniqueness and analyticity results are established. Moreover, the exact analytical solutions (in integral form) are compared with the corresponding numerical ones.

1. INTRODUCTION

During the past three decades there have been several studies of boundary layer flows of non-Newtonian fluids. These investigations have been for non-Newtonian fluids of the differential type. In the case of fluids of differential type, the equations of motion are an order higher than the Navier–Stokes equations, and thus the adherence boundary condition is insufficient to determine the solution completely. The same is also true for the approximate boundary layer approximations of the equations of motion. In the absence of a clear means of obtaining additional boundary conditions, Beard and Walters [1], in their study of an incompressible fluid of second grade, suggested a method for overcoming this difficulty. They suggested a perturbation approach in which the velocity and the pressure field were expanded in a series in terms of a small parameter ϵ. This parameter ϵ in question multiplied the highest order spatial derivatives in their equation. Though this approximation reduces the order of the equation, it treats a singular perturbation problem as a regular perturbation problem.

0022-247X/00 $35.00

In 1991, Garg and Rajagopal suggested that it would be preferable to overcome the difficulty associated with the paucity of boundary conditions by augmenting them on the basis of physically reasonable assumptions. They thought that it is possible to do this in the case of flows which take place in unbounded domains by using the fact that either the solution is bounded or the solution has certain smoothness at infinity. To demonstrate this, Garg and Rajagopal [2] studied the stagnation flow of fluid of second grade by augmenting the boundary conditions. Their results agreed well with the results of Rajeswari and Rathna [3] who studied the problem based on the perturbation approach, for a small value of the perturbation parameter. The advantage of augmenting the boundary conditions over the perturbation approach is that the analysis is valid even for large values of the parameter ϵ, and significant deviations from the Newtonian behavior are possible for even moderately large values of ϵ.

The Cauchy stress $\mathbf{T}$ in an incompressible homogeneous fluid of second grade has the form

$$\mathbf{T} = -p\mathbf{I} + \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2, \tag{1.1}$$

where

$$\begin{aligned} \mathbf{A}_1 &= (\operatorname{grad}\mathbf{v}) + (\operatorname{grad}\mathbf{v})^T \qquad \text{and} \\ \mathbf{A}_2 &= d\mathbf{A}_1/dt + \mathbf{A}_1(\operatorname{grad}\mathbf{v}) + (\operatorname{grad}\mathbf{v})^T\mathbf{A}_1. \end{aligned} \tag{1.2}$$

In the above equations, the spherical stress $-p\mathbf{I}$ is due to the constraint of incompressibility, μ is the viscosity, α_1 and α_2 are material moduli and usually referred to as the normal stress moduli, d/dt denotes the material time derivative, $\mathbf{v}$ denotes the velocity field, and $\mathbf{A}_1$ and $\mathbf{A}_2$ are the first two Rivlin–Ericksen tensors. This model is applicable to some dilute polymer solutions [such as (i) the 5.4% solution of polyisobutylene in cetane (see Markovitz and Coleman [4]) and (ii) the 0.83% solution of ammonium alginate in water (see Acrivos [5])] at low rates of shear.

The above model has been studied in great detail. The sign of the coefficient α_1 has been a subject of much controversy, and a thorough discussion of the issues involved can be found in the recent critical review by Dunn and Rajagopal [6]. We shall not get into a discussion of these issues here. In this study we shall assume that Eq. (1.1) models the fluid exactly. If the fluid modeled by (1.1) is to be compatible with thermodynamics, in the sense that all motions of the fluid meet the Clausius–Duhem inequality and the assumption that the specific Helmholtz free energy of the fluid is a minimum when the fluid is locally at rest, then

$$\mu \ge 0, \qquad \alpha_1 \ge 0, \qquad \alpha_1 + \alpha_2 = 0. \tag{1.3}$$

In 1991, Vajravelu and Rollins [7] studied the flow of an incompressible second order fluid due to stretching of a plane elastic surface. They examined the effects of viscous dissipation and internal heat generation or absorption in a viscoelastic boundary layer flow. Recently, Sarma and Rao [8] analyzed the effects of work due to deformation in the energy equation. In the above references [7, 8] (also see Troy et al. [9], Chang [10], Lawrence and Rao [11], and Chang et al. [12]), the sign for the material constant α_1 (in Eq. (3)) was taken as negative; however, this is not compatible with the stability criteria (see Dunn and Rajagopal [6]).

Hence, Vajravelu and Roper [13] recently studied the flow and heat transfer in a viscoelastic fluid over a stretching sheet with power law surface temperature, including the effects of viscous dissipation, internal heat generation or absorption, and work due to deformation in the energy equation. Furthermore, they augment the boundary conditions, use the proper sign for the material constant ($\alpha_1 \geq 0$), and analyze the salient features of the flow and heat transfer characteristics.

In this paper, we study the existence, uniqueness, and behavior of exact solutions of second-order nonlinear differential equations arising in viscoelastic fluid flows at a rotating cylinder. In Section 2, we shall consider the mathematical model; in Section 3 we shall prove the existence, uniqueness results; in Section 4, we shall present the perturbation analysis and exact solution; and in Section 5, we shall present the numerical results, through graphs.

2. FORMULATION OF THE PROBLEM

Consider the flow of a second-order fluid, obeying (1.1), maintained at a cylinder by its angular velocity. Let the cylinder have radius R and angular velocity Ω. The steady equation for this fluid (for details see Beard and Walters [1]) in usual notation is

$$\mu\left[\frac{d^2 v}{dr^2} + \frac{1}{r}\frac{dv}{dr} - \frac{v}{r^2}\right] + \beta\left(\frac{dv}{dr} - \frac{v}{r}\right)^2\left[6\frac{d^2 v}{dr^2} - \frac{2}{r}\frac{dv}{dr} + \frac{2v}{r^2}\right] = 0, \tag{2.1}$$

with boundary conditions

$$\begin{aligned} v &= R\Omega \qquad \text{at } r = R, \\ v &\to 0 \qquad \text{as } r \to \infty, \end{aligned} \tag{2.2}$$

where $v = v_\theta$ is the nonzero velocity in polar coordinates, and μ and β are material constants due to viscosity and viscoelasticity, respectively.

Defining nondimensional variables

$$\bar{r} = r/R \qquad \text{and} \qquad \bar{v} = v/R\Omega \tag{2.3}$$

and substituting (2.3) in (2.1) and (2.2), we get (after dropping the bar)

$$\frac{d^2 v}{dr^2} + \frac{1}{r}\frac{dv}{dr} - \frac{v}{r^2} = \epsilon\left(\frac{dv}{dr} - \frac{v}{r}\right)^2\left(6\frac{dv^2}{dr^2} - 2\frac{1}{r}\frac{dv}{dr} + 2\frac{v}{r^2}\right) = 0, \tag{2.4}$$

with boundary conditions

$$\begin{aligned} v &= 1 \qquad \text{at } r = 1, \\ v &\to 0 \qquad \text{as } r \to \infty, \end{aligned} \tag{2.5}$$

where $\epsilon = \Omega^2\beta/\mu$, the nondimensional number related to the material constants and the rotation of the cylinder. For practical purposes this number ϵ can be very small or very large or in between. Hence we study the differential equation (2.4) along with conditions (2.5) for all values of the parameter ϵ.

3. EXISTENCE AND UNIQUENESS RESULTS

Problem. Find $v = v(r)$ which satisfies

$$\begin{gathered} v'' + \frac{1}{r}v' - \frac{1}{r^2}v + \epsilon\left(v' - \frac{v}{r}\right)^2\left(6v'' - \frac{2}{r}v' + \frac{2v}{r^2}\right) = 0, \qquad 1 < r < \infty, \\ v(1) = 1, \\ \lim_{r\to\infty} v(r) = 0. \end{gathered} \tag{3.1}$$

Preliminaries

Set $A = v' - \frac{v}{r}$. Then (3.1) can be rewritten as

$$\begin{gathered} (1 + 6\epsilon A^2)v'' + \frac{1}{r}(1 - 2\epsilon A^2)v' - \frac{1}{r^2}(1 - 2\epsilon A^2)v = 0, \qquad 1 < r < \infty, \\ v(1) = 1, \\ \lim_{r\to\infty} v(r) = 0. \end{gathered} \tag{3.2}$$

As $(1 + 6\epsilon A^2) > 0$ we can divide through to obtain

$$v'' + \frac{1}{r}\frac{(1 - 2\epsilon A^2)}{(1 + 6\epsilon A^2)}A = 0. \tag{3.3}$$

Thus a knowledge of A would be useful in finding v.

Consider

$$A' = \left(v' - \frac{v}{r}\right)' = v'' - \frac{v'}{r} + \frac{v}{r^2}, \tag{3.4}$$

which can be written as

$$v'' = A' + \frac{1}{r}A. \tag{3.5}$$

Using (3.5), to eliminate v'' in (3.3), we obtain

$$\begin{gathered} A' + \frac{1}{r}\left(1 + \frac{(1 - 2\epsilon A^2)}{(1 + 6\epsilon A^2)}\right)A = 0, \qquad 1 < r < \infty, \\ A(1) = \lambda, \end{gathered} \tag{3.6}$$

where λ is an unknown parameter which must be determined later in order to specify v.

Next, consider the function

$$f(w) = \frac{(1 - 2w)}{(1 + 6w)}, \qquad w > 0 \tag{3.7}$$

as $w = \epsilon A^2 > 0$, subject to

$$\lim_{w \to 0} f(w) = 1 \tag{3.8}$$

and

$$\lim_{w \to \infty} f(w) = -\frac{1}{3}. \tag{3.9}$$

Also, consider

$$f'(w) = \frac{-8}{(1 + 6w)^2} < 0, \tag{3.10}$$

which implies that

$$-\frac{1}{3} < f(w) < 1, \qquad w > 0, \tag{3.11}$$

and

$$|f'(w)| < 8, \qquad w > 0. \tag{3.12}$$

Applying the usual exponential integrating factor to (3.6) we see that

$$A(r) = \lambda \exp\left\{ -\int_1^r \frac{1}{\eta}\left(1 + f(\epsilon A^2)\right) d\eta \right\}. \tag{3.13}$$

A Priori Bounds

We observe that

$$\text{sign } A(r) = \text{sign } \lambda \tag{3.14}$$

and that

$$|A(r)| \leq |\lambda|. \tag{3.15}$$

Next, we employ the properties of $f(w)$ above to obtain sharper estimates of $A(r)$ via the strictly decreasing nature of e^{-x}. First, we see from (3.13) that

$$\begin{aligned} |A(r)| &= |\lambda| \exp\left\{ -\int_1^r \frac{1}{\eta}\left(1 + f(\epsilon A^2)\right) d\eta \right\} \\ &\leq |\lambda| \exp\left\{ -\int_1^r \frac{1}{\eta}\left(1 - \frac{1}{3}\right) d\eta \right\} \\ &\leq |\lambda| \exp\left\{ -\frac{2}{3}(\ln r - \ln 1) \right\} \\ &= |\lambda| r^{-2/3}, \qquad 1 \leq r < \infty. \end{aligned} \tag{3.16}$$

Likewise, we see that

$$|A(r)| \geq |\lambda| \exp\{-2(\ln r - \ln 1)\} = |\lambda| r^{-2}, \qquad 1 \leq r < \infty. \tag{3.17}$$

Recalling Eq. (3.6) and using property (3.11) of $f(w)$ along with estimates (3.16) and (3.17), we obtain

$$\begin{aligned} |A'(r)| &= \frac{1}{r}|1 + f(\epsilon A^2)|\,|A| \\ &\leq \frac{2}{r}|\lambda| r^{-2/3} \\ &= 2|\lambda| r^{-5/3}, \qquad 1 \leq r < \infty \end{aligned} \tag{3.18}$$

and

$$
\begin{aligned}
|A'(r)| &= \frac{1}{r}|1 + f(\epsilon A^2)|\,|A| \\
&\geq \frac{2}{3}|\lambda| r^{-3}, \qquad 1 \leq r < \infty.
\end{aligned}
\tag{3.19}
$$

From estimate (3.18) we see that

$$
|A'(r)| \leq 2|\lambda|, \qquad 1 \leq r < \infty, \tag{3.20}
$$

which implies a uniform continuity on $1 \leq r < \infty$.

A Nonlinear Mapping

Let B denote the Banach space of bounded continuous functions on $1 \leq r < \infty$ with norm

$$
\|g\| = \sup_{1 \leq r < \infty} |g(r)|, \tag{3.21}
$$

for all $g \in B$. We define the mapping T: $B \to B$ via the formula

$$
(Tg)(r) = \lambda \exp\left\{ -\int_1^r \frac{1}{\eta}\left(1 + f\left(\epsilon g^2(\eta)\right)\right) d\eta \right\}, \qquad 1 \leq r < \infty. \tag{3.22}
$$

Clearly, $(Tg)(r)$ is a continuously differentiable function on $1 \leq r < \infty$ with norm $\|Tg\| = |\lambda| < \infty$. Also from the analysis for (3.16) we see that

$$
|(Tg)(r)| \leq |\lambda|^{-2/3}, \qquad 1 \leq r < \infty. \tag{3.23}
$$

Differentiating Tg with respect to r we obtain

$$
(Tg)'(r) = -r^{-1}\left(1 + f\left(\epsilon g^2(r)\right)\right)(Tg)(r). \tag{3.24}
$$

Consequently, employing the analysis for (3.18) via (3.21) and the property (3.11) of $f(w)$, we see that

$$
|(Tg)'(r)| \leq 2|\lambda| r^{-5/3}, \qquad 1 \leq r < \infty, \tag{3.25}
$$

which implies that the image TB consists of functions bounded by (3.23) and equi-continuous by (3.25).

At this point it is convenient to define a subspace S_λ of B via

$$
\begin{aligned}
S_\lambda = \big\{ g \in B : |g(r)| \leq |\lambda| r^{-2/3},\ 1 \leq r < \infty \quad \text{and} \\
|g(r_1) - g(r_2)| \leq 2|\lambda|\,|r_1 - r_2|,\ 1 \leq r_1 \leq r_2 < \infty \big\}.
\end{aligned}
\tag{3.26}
$$

Via the Ascoli–Arzela theorem on finite intervals coupled to the Cantor diagonalization process we see that S_λ is compact and as the inequalities in the definition admit convex combinations S_λ is also convex. Thus, we have shown that via (3.21) and (3.23) that

$$TB \subset S_\lambda. \tag{3.27}$$

Continuity of the Map

We must now estimate $\|Tg_1 - Tg_2\|$ in terms of $\|g_1 - g_2\|$. Using the mean value theorem twice we see that

$$\begin{aligned}
(Tg_1)(r) - Tg_2(r) &= \lambda \exp\left\{ -\int_1^r \frac{1}{\eta}\left(1 + f\left(\epsilon g_1^2\right)\right) d\eta \right\} \\
&\quad - \lambda \exp\left\{ -\int_1^r \frac{1}{\eta}\left(1 + f\left(\epsilon g_2^2\right)\right) d\eta \right\} \\
&= -\lambda e^{-\xi} \left(\int_1^r \frac{1}{\eta}\left(f\left(\epsilon g_1^2\right) - f\left(\epsilon g_2^2\right)\right) d\eta \right) \\
&= -\lambda e^{-\xi} \int_1^r \frac{1}{\eta} f'(\mu_\eta)\, \epsilon \left(g_1^2 - g_2^2\right) d\eta. \qquad (3.28)
\end{aligned}$$

Consequently, for $1 \le r < \infty$,

$$\begin{aligned}
|(Tg_1)(r) - (Tg_2)(r)| &\le |\lambda| \left(\int_1^r \frac{1}{\eta} |f'(\mu_\eta)| \epsilon |g_1 + g_2| \, d\eta \right) \|g_1 - g_2\| \\
&\le 8|\lambda|\epsilon \left(\int_1^r \frac{1}{\eta} |\lambda| \eta^{-2/3} \, d\eta \right) \|g_1 - g_2\| \\
&\le 16|\lambda|^2 \epsilon \frac{3}{2}\left(1 - r^{-2/3}\right) \|g_1 - g_2\| \\
&\le 24|\lambda|^2 \epsilon \|g_1 - g_2\|. \qquad (3.29)
\end{aligned}$$

Thus, we see that

$$\|Tg_1 - Tg_2\| \le 24|\lambda|^2 \epsilon \|g_1 - g_2\|, \tag{3.30}$$

which implies the continuity of the map $T\colon B \to S_\lambda$.

Existence Theorem

From the estimates above, we have shown that T is a continuous map of S_λ into S_λ. The Shauder theorem asserts that there exists a fixed point of T. We state this as follows.

THEOREM 1. *For each λ, $-\infty < \lambda < \infty$, there exists a function $A_\lambda \in S_\lambda$ so that*

$$A_\lambda(r) = \lambda \exp\left\{-\int_1^r \frac{1}{\eta}\left(1 + f\left(\epsilon A_\lambda^2(\eta)\right)\right) d\eta\right\}, \tag{3.31}$$

where

$$|A'_\lambda(r)| \le 2|\lambda| r^{-5/3}. \tag{3.32}$$

Proof. See the analysis preceding the statement of the theorem.

We shall now consider the continuity of A with respect to λ and as a corollary the unicity of A. Let $A_{\lambda_1}(r)$ and $A_{\lambda_2}(r)$ be solutions which satisfy (3.29). We form the difference

$$\begin{aligned}
&A_{\lambda_1}(r) - A_{\lambda_2}(r)\\
&\quad = \lambda_1 \exp\left\{-\int_1^r \frac{1}{\eta}\left(1 + f\left(\epsilon A_{\lambda_1}^2(\eta)\right)\right) d\eta\right\}\\
&\qquad - \lambda_2 \exp\left\{-\int_1^r \frac{1}{\eta}\left(1 + f\left(\epsilon A_{\lambda_2}^2(\eta)\right)\right)\right\}\\
&\quad = (\lambda_1 - \lambda_2) \exp\left\{-\int_1^r \frac{1}{\eta}\left(1 + f\left(\epsilon A_{\lambda_1}^2(y)\right)\right) d\eta\right\}\\
&\qquad + \lambda_2\left[\exp\left\{-\int_1^r \frac{1}{\eta}\left(1 + f\left(\epsilon A_{\lambda_1}^2(\eta)\right)\right) d\eta\right\}\right.\\
&\qquad \left. - \exp\left\{-\int_1^r \frac{1}{\eta}\left(1 + f\left(\epsilon A_{\lambda_2}^2(\eta)\right)\right) d\eta\right\}\right]\\
&\quad = (\lambda_1 - \lambda_2) \exp\left\{-\int_1^r \frac{1}{\eta}\left(1 + f\left(\epsilon A_{\lambda_1}^2(\eta)\right)\right) d\eta\right\}\\
&\qquad - \lambda_2 e^{-\xi}\left[\int_1^r \frac{1}{\eta}\left\{f\left(\epsilon A_{\lambda_1}^2(\eta)\right) - f\left(\epsilon A_{\lambda_2}^2(\eta)\right)\right\} d\eta\right]\\
&\quad = (\lambda_1 - \lambda_2) \exp\left\{-\int_1^r \frac{1}{\eta}\left(1 + f\left(\epsilon A_{\lambda_1}^2(\eta)\right)\right) d\eta\right\}\\
&\qquad - \lambda_2 e^{-\xi} \epsilon \int_1^r \frac{1}{\eta} f'(\mu_\eta)\left(A_{\lambda_1}(\mu) + A_{\lambda_2}(\eta)\right)\left(A_{\lambda_1}(\eta) - A_{\lambda_2}(\eta)\right) d\eta.
\end{aligned} \tag{3.33}$$

Taking absolute values of both sides and using the triangle inequalities, it follows from estimates (3.12) and (3.13) that

$$|A_{\lambda_1}(r) - A_{\lambda_2}(r)| \leq |\lambda_1 - \lambda_2| + 8\epsilon(|\lambda_1| + |\lambda_2|)|\lambda_2| \int_1^r |A_{\lambda_1}(\eta) - A_{\lambda_2}(\eta)|\, d\eta. \tag{3.34}$$

Applying Gronwall's inequality we see that

$$|A_{\lambda_1}(r) - A_{\lambda_2}(r)| \leq \left[\exp\{c(r-1)\} + \frac{1}{c}(\exp\{c(r-1)\} - 1)\right] |\lambda_1 - \lambda_2|, \tag{3.35}$$

where

$$c = 8\epsilon(|\lambda_1| + |\lambda_2|)|\lambda_2|.$$

Hence, we see that for each r, $A_\lambda(r)$ is continuous with respect to λ. Moreover, if $B_1(r)$ and $B_2(r)$ are two solutions of (3.29) for λ_1, then (3.33) implies that $B_1(r) \equiv B_2(r)$. We summarize these results in the following statement.

THEOREM 2. *The solution $A_\lambda(r)$ of Eq.* (3.29) *is unique and depends continuously upon the initial condition λ.*

Proof. See the estimates prior to the statement of the theorem.

Representation for v

We now recall by definition that

$$v' - \frac{1}{r}v = A, \qquad 1 < r < \infty \tag{3.36}$$

$$v(1) = 1, \tag{3.37}$$

and

$$\lim_{r \to \infty} v(r) = 0. \tag{3.38}$$

Multiplying Eq. (3.36) by $\frac{1}{r}$, we see that

$$\left(\frac{1}{r}v\right)' = \frac{A_\lambda(r)}{r}$$

and integrating with respect to r we obtain

$$\frac{1}{r}v(r) = 1 + \int_1^r \frac{A_\lambda(\eta)}{\eta}\, d\eta$$

or

$$v(r) = r + r\int_1^r \frac{A_\lambda(\eta)}{\eta}\, d\eta. \tag{3.39}$$

From (3.31) we can rewrite (3.39) as

$$v(r) = r + \lambda r\int_1^r \frac{1}{\zeta}\exp\left\{-\int_1^\zeta \frac{1}{\eta}\left(1 + f\left(\epsilon A_\lambda^2(\eta)\right)\right) d\eta\right\} d\zeta, \qquad 1 < r < \infty. \tag{3.40}$$

First, it is clear that λ needs to be negative for $v \to 0$ as $r \to \infty$. Let $\lambda = -\alpha$, $\alpha > 0$. Then,

$$v(r) = r - \alpha r\int_1^r \frac{1}{\zeta}\exp\left\{-\int_1^\zeta \frac{1}{\eta}\left(1 + f\left(\epsilon A_{-\alpha}^2(\eta)\right)\right) d\eta\right\} d\zeta.$$

Considering the case $v(R) = 0$, we see that α must satisfy the nonlinear equation

$$\alpha = \frac{1}{\int_1^R \frac{1}{\zeta}\exp\left\{-\int_1^\zeta \frac{1}{\eta}\left(1 + f\left(\epsilon A_{-\alpha}^2(\eta)\right)\right) d\eta\right\} d\zeta}. \tag{3.41}$$

Using the estimates for $f(w)$ above we can estimate the integral I in the denominator of (3.41) to obtain

$$\frac{1}{2}(1 - R^{-2}) \le I \le \frac{3}{2}(1 - R^{-2/3}). \tag{3.42}$$

From whence follows

$$\frac{2}{3(1 - R^{-2/3})} \le \alpha \le \frac{2}{(1 - R^{-2})} \tag{3.43}$$

which must be satisfied for each $R > 1$ in order that $v(R) = 0$, allowing $R \to \infty$ in (3.43), we see that

$$\frac{2}{3} \le \alpha \le 2, \tag{3.44}$$

for α so that $v \to 0$ as $r \to \infty$.

4. PERTURBATION ANALYSIS AND SOLUTION

The problem of our interest is

$$v'' + \frac{1}{r}v' - \frac{1}{r^2}v + \epsilon\left(v' - \frac{v}{r}\right)^2\left(6v'' - \frac{2}{r}v' + \frac{2v}{r^2}\right) = 0, \tag{4.1}$$
$$v(1) = 1 \quad \text{and} \quad v(\infty) = 0, \qquad 1 < r < \infty.$$

It will be shown that there is an exact implicit solution to (4.1) for any ϵ, but it still instructive to give an approximate solution for small ϵ. When $\epsilon \ll 1$, we let $v = v_0 + \epsilon v_1 + O(\epsilon^2)$ in (4.1) and obtain the $O(1)$ problem

$$v_0'' + \frac{1}{r}v_0' - \frac{v_0}{r^2} = 0, \qquad v_0(1) = 1, \qquad v_0(\infty) = 0 \tag{4.2}$$

and the $O(\epsilon)$ problem

$$v_1'' + \frac{1}{r}v_1' - \frac{v_1}{r^2} = -\left(v_0' - \frac{v_0}{r}\right)^2\left(6v_0'' - \frac{2}{r}v_0' + \frac{2v_0}{r^2}\right), \qquad v_1(1) = 0, \qquad v_1(\infty) = 0. \tag{4.3}$$

Solving (4.2) and (4.3), we get

$$v = \frac{1}{r} + \frac{8}{3}\epsilon\left(\frac{1}{r} - \frac{1}{r^5}\right) + O(\epsilon^2). \tag{4.4}$$

A similar approach can be taken for large ϵ. Let $\eta = \frac{1}{\epsilon}$ in (4.1) to obtain

$$\left(v' - \frac{v}{r}\right)^2\left(6v'' - \frac{2}{r}v' + \frac{2v}{r^2}\right) + \eta\left(v'' + \frac{1}{r}v' - \frac{v}{r^2}\right) = 0, \qquad v(1) = 1, \qquad v(\infty) = 0. \tag{4.5}$$

Let $v = v_0 + \eta v_1 + O(\eta^2)$; then either $v_0' = v_0/r$ which implies $v_0 = cr$, or

$$6v_0'' - \frac{2}{r}v_0' + \frac{2v_0}{r^2} = 0,$$

which has solution $v_0 = c_1 r + c_2 r^{1/3}$. In either case it is not possible to satisfy the infinity condition. This suggests that for large ϵ, the terms neglected at the lowest order need to be retained. This observation is the motivation toward solving (4.1) exactly.

Let $v(r) = \frac{rw(r)}{\sqrt{\epsilon}}$ in (4.1) to get

$$\begin{gathered} rw'' + 3w' + r2w'^2(6rw'' + 10w') = 0 \\ w(1) = \sqrt{\epsilon} \qquad w(\infty) = 0, \end{gathered} \tag{4.6}$$

which shows that all terms in the equation need to be retained in a perturbative expansion. Noting that the equation in (4.6) is invariant under scaling of the r variable, let $\tau = \ln r$ be a new variable. This reduces (4.6) to the constant coefficient equation

$$\begin{gathered} \ddot{w} + 2\dot{w} + \dot{w}^2(6\ddot{w} + 4\dot{w}) = 0 \\ w(0) = \sqrt{\epsilon} \qquad w(\infty) = 0, \end{gathered} \tag{4.7}$$

where $\cdot$ is differentiation with respect to τ. Integration of (4.7) gives

$$\dot{w}^3 + \frac{1}{2}\dot{w} - ce^{-2\tau} = 0 \tag{4.8}$$

which is a cubic in $\dot{w}(\tau)$. When the discriminant of this cubic is examined, it is found to be positive which implies there is one real and two complex conjugate solutions. Using Maple to solve, one finds the real solution as

$$\dot{w}(\tau) = \frac{1}{6}(f(\tau))^{1/3} - (f(\tau))^{-1/3} \equiv F(\tau; c), \tag{4.9}$$

where

$$f(\tau) = 108ce^{-2\tau} + 6\sqrt{6 + 324c^2e^{-4t}}. \tag{4.10}$$

Integrating (4.9) from τ to ∞ and using the boundary condition at infinity in (4.7) gives

$$w(\tau) = -\int_\tau^\infty F(p; c)\, dp. \tag{4.11}$$

When (4.11) is evaluated using the boundary condition at $\tau = 0$, an implicit equation for the constant c is found to be

$$\sqrt{\epsilon} = -\int_0^\infty F(p; c)\, dp. \tag{4.12}$$

With c determined, the exact solution to (4.1) is found to be

$$v(r) = \frac{-r}{\sqrt{\epsilon}} \int_{\ln r}^{\infty} F(p;c)\, dp. \tag{4.13}$$

Using the asymptotic behavior of F for large τ, the behavior of $v(r)$ for large r can be found. Since $F(\tau; c) \sim 2ce^{-2\tau}$ for $\tau \to \infty$

$$v(r) \sim \frac{-c(\epsilon)}{\sqrt{\epsilon}} r^{-1} \qquad \text{as } r \to \infty, \tag{4.14}$$

where the ϵ dependence of c has been noted. Comparing (4.14) with the perturbative result for small ϵ, it must be true that $c(\epsilon) = O(\sqrt{\epsilon})$ for $\epsilon \to 0^+$. Also, if $c(\epsilon) = o(\sqrt{\epsilon})$ for $\epsilon \to \infty$, this would explain the failure of the standard large ϵ approach in solving (4.1).

Using Eq. (4.13), the velocity $v(r)$ is evaluated numerically for several values of the parameter ϵ. Some of the interesting results are presented in Figs. A and B. Comparing these figures with the results in Figs. 1 and 2, obtained numerically, we see that the analytical and numerical results are in good agreement.

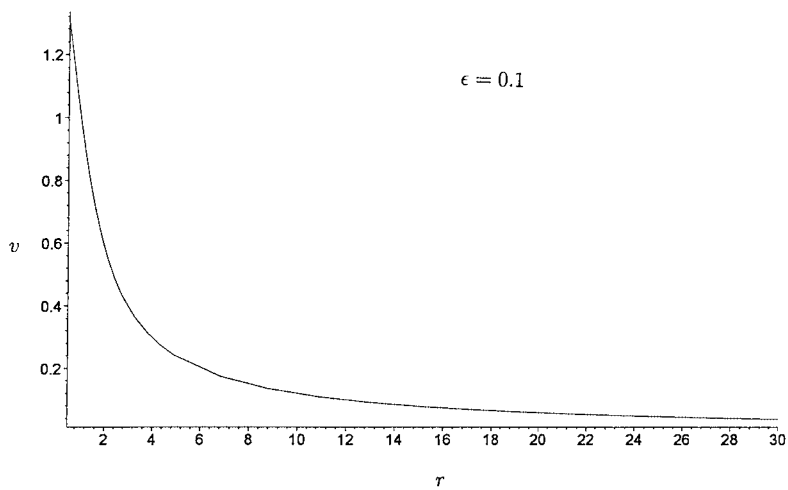

FIG. A. Velocity profile for $\epsilon = 0.1$ (from analytical results).

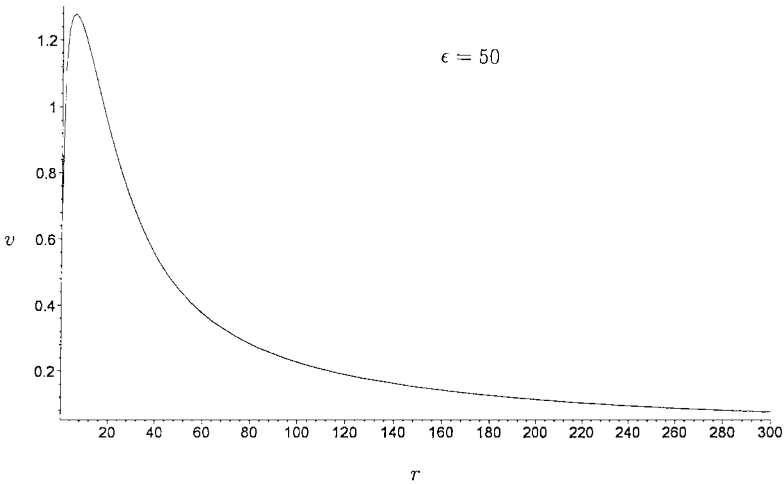

FIG. B. Velocity profile for $\epsilon = 50$ (from analytical results).

5. NUMERICAL SOLUTION AND DISCUSSION OF THE RESULTS

The nonlinear differential equation

$$v'' + \frac{1}{r}v' - \frac{v}{r^2} + \epsilon\left(v' - \frac{v}{r}\right)^2\left(6v'' - \frac{2}{r}v' + \frac{2v}{r^2}\right) = 0, \qquad (5.1)$$

subject to the boundary conditions

$$v(1) = 1 \qquad \text{and} \qquad v(\infty) = 0, \qquad (5.2)$$

is solved numerically for several values of the parameter ϵ. Some of the qualitatively interesting results are presented in Figs. 1 through 3.

In Fig. 1, we have plotted the velocity distribution $v(r)$ for several values of the nondimensional parameter ϵ. From Fig. 1 it is evident that the velocity increases with an increase in the parameter ϵ. Furthermore, from Figs. 1–3, we observe that for large values of ϵ (that is, in the case of low viscosity or highly rotating cylinder or highly viscoelastic fluid) the fluid velocity close to the cylinder increases to its maximum and then decreases (before it reaches its infinity condition). This phenomenon is not true when $\epsilon \leq 1$, where the velocity decreases from 1 at $r = 1$ to 0 as $r \to \infty$. Hence it should be mentioned that the parameter ϵ affects the fluid velocity significantly.

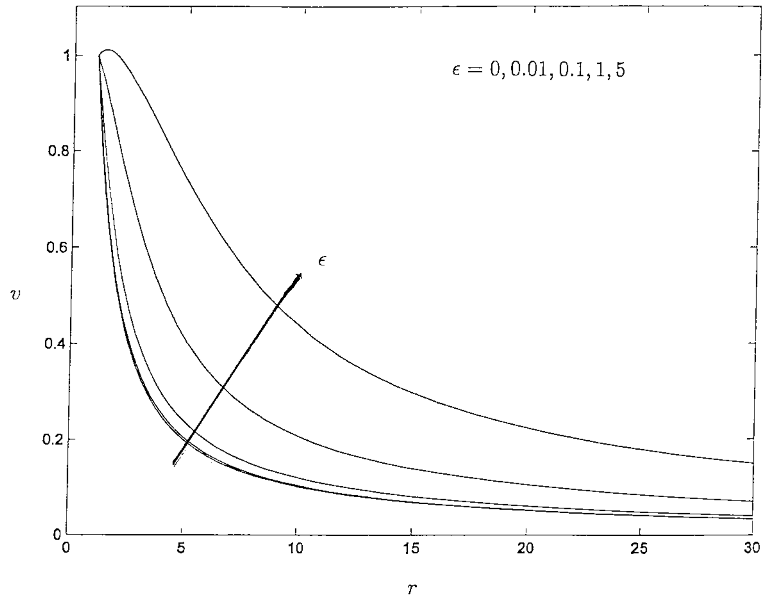

FIG. 1. Velocity profiles for various values of ϵ (from numerical calculations).

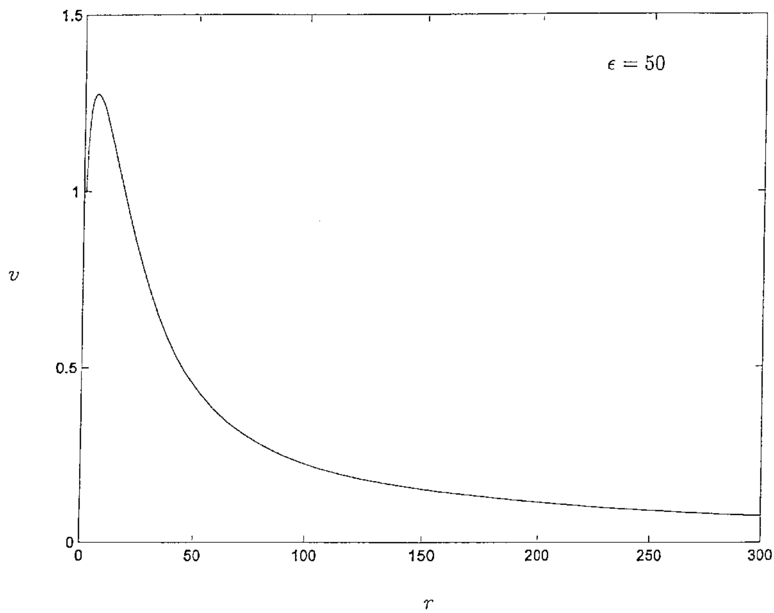

FIG. 2. Velocity profile for $\epsilon = 50$ (from numerical calculations).

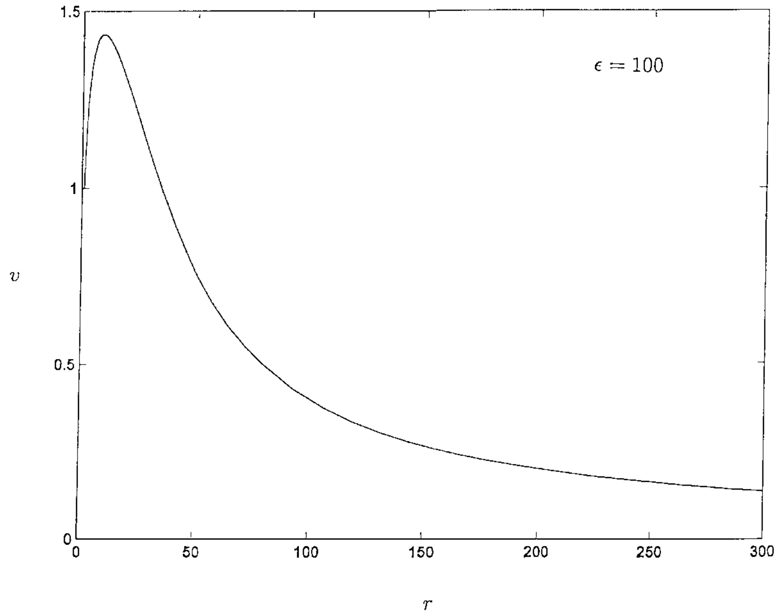

FIG. 3. Velocity profile for $\epsilon = 100$ (from numerical calculations).

ACKNOWLEDGMENTS

This problem originated as a result of a sabbatical spent at Texas A & M University by one of the authors, K. Vajravelu. Professor K. R. Rajagopal's hospitality and support is acknowledged.

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