On some improvements of square root iteration for polynomial complex zeros

M.S. PETKOVIĆ and L.V. STEFANOVIĆ
Department of Mathematics, Faculty of Electronic Engineering, 18000 Niš, Yugoslavia

Received 10 April 1984

Dedicated to Professor A.M. Ostrowski on his 90th Birthday

Abstract: Using Newton’s and Halley’s corrections, some modifications of the simultaneous method for finding polynomial complex zeros, based on square root iteration, are obtained. The convergence order of the proposed methods is five and six respectively. Further improvements of these methods are performed by applying the Gauss–Seidel approach. The lower bounds of the R-order of convergence and the convergence conditions for the accelerated (single-step) methods are given. Faster convergence is attained without additional calculations. The considered iterative procedures are illustrated numerically in the example of an algebraic equation.

Keywords: Determination of polynomial zeros, simultaneous iterative methods, accelerated convergence, R-order of convergence.

1. Some modifications of square root iteration

Consider a monic polynomial of degree \( n \geq 3 \)

\[
P(x) = \prod_{j=1}^{n} (x - r_j)
\]

with simple real zeros \( r_1, \ldots, r_n \). Let

\[
G(x) = -\frac{d^2}{dx^2} \ln P(x) = \frac{P'(x)^2 - P(x)P''(x)}{P(x)^2}
\]

and

\[
K(x) = \frac{\text{sgn}(P(x)P'(x))}{\sqrt{G(x)}}.
\]

The iterative formula

\[
x^{(m+1)} = x^{(m)} - K(x^{(m)}), \quad m = 0, 1, \ldots,
\]

which starts from an initial approximation \( x = x^{(0)} \) of the zero \( r \), defines the well-known square root iteration for the determination of a simple zero \( r \) of polynomial \( P \). The convergence of the
sequence \( \{ x^{(m)} \} \), defined by (1.2), is cubic if \( r \) is a simple zero. Square root iteration was analysed in detail by Ostrowski [7] and it is often called Ostrowski’s method.

The iterative formula (1.2) can be applied also for finding a simple zero of a three times differentiable function \( f \) which need not be a polynomial. Moreover, it can be proved, that even in the complex case when we start in a sufficiently close neighborhood of a (real or complex) zero of \( f \), the iterative process (1.2) converges to this zero [7, p. 123].

Consider now a polynomial

\[
P(z) = \prod_{j=1}^{n} (z - r_j) = z^n + a_{n-1}z^{n-1} + \cdots + a_1z + a_0, \quad a_i \in \mathbb{C}
\]  

(1.3)
of degree \( n \geq 3 \) with simple real or complex zeros \( r_1, \ldots, r_n \). By the function \( u(z) = \frac{P'(z)}{P(z)} \) and \( s(z) = \frac{P''(z)}{P'(z)} \) we shall define

\[
G(z) = u(z)[u(z) - s(z)] \quad \text{(Ostrowski’s function),}
\]

(1.4)
\[
N(z) = -\frac{1}{u(z)} \quad \text{(Newton’s correction),}
\]

(1.5)
\[
H(z) = \frac{1}{2s(z) - u(z)} \quad \text{(Halley’s correction).}
\]

(1.6)

We recall that the correction terms (1.5) and (1.6) appear in the iterative formulas

\[
z^{(m+1)} = z^{(m)} + N(z^{(m)}) \quad \text{(Newton’s method),}
\]

\[
z^{(m+1)} = z^{(m)} + H(z^{(m)}) \quad \text{(Halley’s method [4]),}
\]

which have quadrate and cubic convergence respectively.

Using (1.1) and (1.3) it is easy to prove that

\[
G(z) = \frac{P'(z)^2 - P(z)P''(z)}{P(z)^2} = \sum_{j=1}^{n} (z - r_j)^{-2}
\]

(1.7)

From (1.7) for \( z = z_i \) we find

\[
(z_i - r_j)^{-2} = G(z_i) - \sum_{j=1}^{n} (z_i - r_j)^{-2},
\]

wherefrom

\[
r_i = z_i - \left[ G(z_i) - \sum_{\substack{j=1 \atop j \neq i}}^{n} (z_i - r_j)^{-2} \right]^{-1/2} \quad , \quad i = 1, \ldots, n,
\]

(1.8)

where the symbol \( * \) denotes that one of the two values of the square root is chosen. Assume that reasonably good approximations \( z_1, \ldots, z_n \) of the zeros \( r_1, \ldots, r_n \) are found. Putting \( r_i := \tilde{z}_i \) in (1.8), where \( \tilde{z}_i \) is the new approximation of the zero \( r_i \), and taking certain approximations of \( r_j \) on the right-hand side of the identity (1.8), some modified iterative processes for simultaneous finding all zeros of the polynomial \( P \) follow from (1.8). Similar procedures have been considered in [5] and [8] for the simultaneous methods of the second and third order.
(TS) For \( r_j := z_j \ (j \neq i) \) we obtain the total-step square root iteration (TS)

\[
\hat{z}_i = z_i - \left[ G(z_i) - \sum_{j=1}^{n} \frac{(z_i - z_j)^{-1}}{(z_i - z_j)^{-2}} \right]^{-1/2}, \quad i = 1, \ldots, n. \tag{1.9}
\]

This method has been discussed in [9] as a special case of the generalised root iteration. It has been proved that the convergence order of TS-method (1.9) is \textit{four}. Thus, the correction term in the form of a sum enables (i) the increase of the convergence order and (ii) the determination of all zeros of a polynomial. Note that an iterative method of the form (1.9) in terms of circular regions has been analysed by Gargantini [3]. This author has also established a criterion for the choice of the appropriate value of a square root. In the sequel we shall use the abbreviation CCR for this criterion. If all zeros of a polynomial are real, then CCR reduces to the choice of sign which coincides to the sign of (real value) \( P(z_i)P'(z_i) \).

(SS) Let \( r_j := \hat{z}_j \ (j < i) \) and \( r_j := z_j \ (j > i) \), then the single-step square root iteration (SS) follows from (1.8):

\[
\hat{z}_i = z_i - \left[ G(z_i) - \sum_{j=1}^{i-1} \frac{(z_i - \hat{z}_j)^{-1}}{(z_i - \hat{z}_j)^{-2}} - \sum_{j=i+1}^{n} \frac{(z_i - z_j)^{-1}}{(z_i - z_j)^{-2}} \right]^{-1/2}, \quad i = 1, \ldots, n. \tag{1.10}
\]

It has been proved in [10] that the R-order of convergence of the SS-method is at least \( 3 + \mu_n \in (4, 5) \), where \( \mu_n \in (1, 2) \) is the unique positive zero of the equation \( \mu^n - \mu - 3 = 0 \).

(TSN) Taking \( r_j := z_j + N(z_j) \ (j \neq i) \) in (1.8), where \( N(z_j) \) is Newton’s correction given by (1.5), we obtain the total-step square root method with Newton’s correction (TSN):

\[
\hat{z}_i = z_i - \left[ G(z_i) - \sum_{j=1}^{n} \frac{(z_i - z_j - N(z_j))^{-1}}{(z_i - z_j - N(z_j))^{-2}} \right]^{-1/2}, \quad i = 1, \ldots, n. \tag{1.11}
\]

(TSN) TSN process (1.11) can be accelerated using the Gauss–Seidel approach: substituting \( r_j := \hat{z}_j \ (j < i) \), \( r_j := z_j + N(z_j) \ (j > i) \) in (1.8), we obtain the single-step method with Newton’s correction (SSN):

\[
\hat{z}_i = z_i - \left[ G(z_i) - \sum_{j=1}^{i-1} \frac{(z_i - \hat{z}_j)^{-1}}{(z_i - \hat{z}_j)^{-2}} - \sum_{j=i+1}^{n} \frac{(z_i - z_j - N(z_j))^{-1}}{(z_i - z_j - N(z_j))^{-2}} \right]^{-1/2}, \quad i = 1, \ldots, n. \tag{1.12}
\]

(TSH) Similar as for TSN-method, we can use Halley’s correction (1.6). Taking \( r_j := z_j + H(z_j) \ (j \neq i) \) in (1.8), we get the total-step method with Halley’s correction (TSH):

\[
\hat{z}_i = z_i - \left[ G(z_i) - \sum_{j=1}^{n} \frac{(z_i - z_j - H(z_j))^{-1}}{(z_i - z_j - H(z_j))^{-2}} \right]^{-1/2}, \quad i = 1, \ldots, n. \tag{1.13}
\]

(SSH) Finally, setting \( r_j := \hat{z}_j \ (j < i) \), \( r_j := z_j + H(z_j) \ (j > i) \) in (1.8), we obtain the single-step
method with Halley’s correction (SSH):

\[
\hat{z}_i = z_i - \left[ G(z_i) - \sum_{j=1}^{i-1} \left( z_i - \hat{z}_j \right)^{-2} - \sum_{j=i+1}^{n} \left( z_i - z_j - H(z_j) \right)^{-2} \right]_{*}^{-1/2}, \quad i = 1, \ldots, n.
\]

(1.14)

The four last iterative formulas will be considered in this paper.

2. Convergence analysis

In this section we shall give a convergence analysis of SSH-method (1.14). The convergence order of the TSH-method is directly obtained from this analysis. The results, relative to SSN and TSN methods, will be briefly presented because the convergence analysis of these methods is very similar to that of SSH and TSH methods.

Let \( z_1^{(0)}, \ldots, z_n^{(0)} \) be distinct reasonably close approximations to the zeros \( r_1, \ldots, r_n \) of polynomial \( P \), and let \( m = 0, 1, \ldots \) be the iteration index. Starting from (1.14) we obtain the following iterative process for the simultaneous determination of polynomial complex zeros:

\[
z_i^{(m+1)} = z_i^{(m)} - \left[ G\left( z_i^{(m)} \right) - \sum_{j=1}^{i-1} \left( z_i^{(m)} - z_j^{(m+1)} \right)^{-2} - \sum_{j=i+1}^{n} \left( z_i^{(m)} - z_j^{(m)} - H(z_j^{(m)}) \right)^{-2} \right]_{*}^{-1/2},
\]

\[
i = 1, \ldots, n, \quad m = 0, 1, \ldots.
\]

The symbol * points at the choice of the appropriate value of the square root according to CCR given in [3].

Introduce the notations

\[
d = \min_{i \neq j} |r_i - r_j|, \quad \nu_i^{(m)} = z_i^{(m)} - r_i,
\]

\[
q = \frac{2n - 1}{d}, \quad \nu^{(m)} = \max_i |\nu_i^{(m)}|,
\]

\[
w_i^{(m)} = z_i^{(m)} + H\left( z_i^{(m)} \right) \quad \text{(Halley’s approximation)}.
\]

Since

\[
u(z) = \frac{P'(z)}{P(z)} = \sum_{j=1}^{n} (z - r_j)^{-1}
\]

and

\[
G(z) = \sum_{j=1}^{n} (z - r_j)^{-2} = \nu(z)\left[ \nu(z) - s(z) \right],
\]

we have

\[
s(z) = \nu(z) - \frac{G(z)}{\nu(z)} = \sum_{j=1}^{n} (z - r_j)^{-1} - \sum_{j=1}^{n} (z - r_j)^{-2} \left/ \sum_{j=1}^{n} (z - r_j)^{-1} \right.
\]
According to this and (1.6), we obtain

\[ H(z) = \frac{-2 \sum_{j=1}^{n} (z - r_j)^{-1}}{\left( \sum_{j=1}^{n} (z - r_j)^{-1} \right)^2 + \sum_{j=1}^{n} (z - r_j)^{-2}}. \]

We have now for Halley’s approximation

\[ w_i^{(m)} - r_i = \frac{g_i^{(m)} v_i^{(m)}}{1 + \hat{g}_i^{(m)} + \sum_{j \neq i}^{n} \frac{z_i^{(m)} - r_i}{z_i^{(m)} - r_j}}, \quad i = 1, \ldots, n, \quad m = 0, 1, \ldots \quad (2.3) \]

where

\[ \hat{g}_i^{(m)} = \sum_{j \neq i}^{n} \left( \frac{z_i^{(m)} - r_i}{z_i^{(m)} - r_j} \right)^2 + \sum_{j=1}^{n} \sum_{k \neq i}^{n} \frac{(z_i^{(m)} - r_i)^2}{(z_i^{(m)} - r_j)(z_i^{(m)} - r_k)}. \quad (2.4) \]

In the similar way, the following relation can be derived for the iterative process (2.1)

\[ z_i^{(m+1)} - r_i = v_i^{(m+1)} = \frac{g_i^{(m)} v_i^{(m)}}{\left[ g_i^{(m)} + 1 \right]^{1/2} \left[ g_i^{(m)} + 1 \right]^{1/2} + 1}, \quad i = 1, \ldots, n, \quad m = 0, 1, \ldots \quad (2.5) \]

where

\[ g_i^{(m)} = \sum_{j=1}^{i-1} \frac{\left( z_i^{(m)} - r_i \right)^2 \left( r_j - z_j^{(m+1)} \right)(2z_i^{(m)} - z_i^{(m+1)} - r_j)}{\left( z_i^{(m)} - r_j \right)^2 (z_i^{(m)} - z_j^{(m+1)})^2} \]

\[ + \sum_{j=i+1}^{n} \frac{\left( z_i^{(m)} - r_i \right)^2 \left( r_j - w_j^{(m)} \right)(2z_i^{(m)} - w_i^{(m)} - r_j)}{\left( z_i^{(m)} - r_j \right)^2 (z_i^{(m)} - w_j^{(m)})^2}. \quad (2.6) \]

Suppose that the initial conditions

\[ |v_i^{(0)}| < \frac{d}{2n - 1} = \frac{1}{q}, \quad i = 1, \ldots, n \quad (2.7) \]

are satisfied. Then

\[ |z_i^{(0)} - r_i| > |r_i - r_j| - |z_i^{(0)} - r_i| > d - \frac{d}{2n - 1} > 2(n - 1) \]

\[ \frac{q}{2n - 1} > \frac{2n - 3}{q} > \frac{1}{q}. \quad (2.8) \]

\[ |z_i^{(0)} - z_j^{(0)}| > |z_i^{(0)} - r_j| - |z_j^{(0)} - r_j| > \left( d - \frac{d}{2n - 1} \right) - \frac{d}{2n - 1} = \frac{2n - 3}{q} > \frac{1}{q}. \quad (2.9) \]
Using the two last inequalities, we find from (2.4)

\[
|\hat{g}_i^{(0)}| \leq |v_i^{(0)}|^2 \sum_{j=1}^{n} |z_i^{(0)} - r_j|^2 + |v_i^{(0)}|^2 \sum_{j=1}^{n} \left( \sum_{k=1}^{n} \left( \frac{|z_i^{(0)} - r_j|}{|z_i^{(0)} - r_k|} \right)^{-1} \right)
\]

\[
< |v_i^{(0)}|^2 \left[ \frac{(n-1)q^2 + (n-1)(n-2)}{4(n-1)^2} \frac{q^2}{2} \right]
\]

\[
= \frac{1}{8(n-1)} |v_i^{(0)}|^2 < \frac{1}{4} q^2 |v_i^{(0)}|^2 ,
\]

wherefrom

\[
|\hat{g}_i^{(0)}| < \frac{1}{4} .
\]

By (2.7) and (2.8) we obtain

\[
\sum_{j=1}^{n} \left| \frac{z_i^{(0)} - r_j}{z_i^{(0)} - r_j} \right| < |v_i^{(0)}| \frac{(n-1)q}{2(n-1)} = \frac{1}{2} q |v_i^{(0)}| < \frac{1}{2} .
\]

Using (2.10) and the last inequality, we find from (2.3) for \( m = 0 \)

\[
|r_i - w_i^{(0)}| \leq \frac{|\hat{g}_i^{(0)}| |v_i^{(0)}|}{1 - |\hat{g}_i^{(0)}| - \sum_{j=1}^{n} \left| \frac{z_i^{(0)} - r_j}{z_i^{(0)} - r_j} \right|} < \frac{1}{4} q^2 |v_i^{(0)}|^3 = q^2 |v_i^{(0)}|^3 ,
\]

wherefrom, with regard to (2.7),

\[
|r_i - w_i^{(0)}| < \frac{1}{q} .
\]

Further, we have for \( i \neq j \)

\[
|z_i^{(0)} - w_j^{(0)}| \geq |z_i^{(0)} - r_j| - |r_j - w_j^{(0)}| > (2n - 3)/q > 1/q ,
\]

which, together with (2.8) and (2.11), gives the following estimates

\[
\frac{|r_j - w_i^{(0)}|}{|z_i^{(0)} - r_j| |z_i^{(0)} - w_j^{(0)}|} \leq q \frac{2(n-1)}{2(n-1)} ,
\]

\[
\frac{|2z_i^{(0)} - w_j^{(0)} - r_j|}{|z_i^{(0)} - r_j| |z_i^{(0)} - w_j^{(0)}|} \leq \frac{1}{|z_i^{(0)} - w_j^{(0)}|} + \frac{1}{|z_i^{(0)} - r_j|} < \frac{(2n-1)q}{2(n-1)} .
\]

Using (2.7), (2.12) and (2.13), we get from (2.6)

\[
|\hat{g}_1^{(0)}| = \sum_{j=2}^{n} \frac{(z_i^{(0)} - r_j)^3 (r_j - w_j^{(0)}) (2z_i^{(0)} - w_j^{(0)} - r_j)}{(z_i^{(0)} - r_j)^2 (z_i^{(0)} - w_j^{(0)})^2} < \frac{2n-1}{4(n-1)} q^2 |v_i^{(0)}|^3 ,
\]

wherefrom, for \( n \geq 2 \)

\[
|\hat{g}_1^{(0)}| < \frac{1}{4} .
\]
According to CCR and (2.14), we estimate
\[ \left| \left[ g_i^{(0)} + 1 \right]^{1/2} \right| > \frac{1}{q}, \quad \left| \left[ g_i^{(0)} + 1 \right]^{1/2} + 1 \right| > \frac{3}{q}, \]
so that
\[ \left| \left[ g_i^{(0)} + 1 \right]^{1/2} \left( \left[ g_i^{(0)} + 1 \right]^{1/2} + 1 \right) \right| > \frac{3}{4}. \] (2.15)

In view of (2.14) and (2.15), we obtain from (2.5)
\[ |v^{(1)}_1| < |v^{(0)}_1| < 1/q. \]

Using (2.8) and the inequality
\[ |z_2^{(0)} - z_1^{(1)}| \geq |z_2^{(0)} - r_1| - |v^{(1)}_1| \geq \frac{2(n-1)}{q} - \frac{1}{q} = \frac{2n-3}{q} \geq \frac{1}{q}, \]
we find
\[ \frac{|2z_2^{(0)} - z_1^{(1)} - r_1|}{|z_2^{(0)} - r_1| \left| z_2^{(0)} - z_1^{(1)} \right|} \leq \frac{1}{|z_2^{(0)} - r_1|} + \frac{1}{\left| z_2^{(0)} - z_1^{(1)} \right|} < \frac{(2n-1)q}{2(n-1)}. \]

On use of the above reasoning, by a successively estimation procedure we obtain
\[ |g_i^{(0)}| < |v^{(0)}_i|^3 \left( \frac{2n-1}{4(n-1)^2} \sum_{j=1}^{i-1} |v^{(1)}_{j+1}| + \frac{2n-1}{4(n-1)^2} q^2 \sum_{j=i+1}^{n} |v^{(0)}_{j+1}| \right) < \frac{3}{4}, \]
\[ i = 2, \ldots, n. \] (2.16)

On the basis of (2.15) and (2.16) it follows
\[ |v^{(1)}_i| < \frac{3}{4} |g_i^{(0)}| |v^{(0)}_i| < \frac{2n-1}{3(n-1)^2} |v^{(0)}_i|^3 \left( \sum_{j=1}^{i-1} |v^{(1)}_{j+1}| + q^2 \sum_{j=i+1}^{n} |v^{(0)}_{j+1}| \right), \]
that is
\[ |v^{(1)}_i| < \frac{q^3}{n-1} |v^{(0)}_i|^3 \left( \sum_{j=1}^{i-1} |v^{(1)}_{j+1}| + q^2 \sum_{j=i+1}^{n} |v^{(0)}_{j+1}| \right) < \frac{1}{q}. \] (2.17)

Applying the same argumentations as for \( m = 0 \) and the mathematical induction, we can prove that the inequality
\[ |v^{(m+1)}_i| < \frac{q^3}{n-1} |v^{(m)}_i|^3 \left( \sum_{j=1}^{i-1} |v^{(m+1)}_{j+1}| + q^2 \sum_{j=i+1}^{n} |v^{(m)}_{j+1}| \right) < \frac{1}{q} \] (2.18)
holds for each \( m = 0, 1, \ldots \) and \( i = 2, \ldots, n. \)

In the sequel, following Ortega and Rheinboldt [6], the R-order of convergence of an iterative process \( IP \) with the limit point \( r \) will be denoted by \( O_R((IP), r). \)

**Theorem 1.** Under the conditions (2.7) SSH-method (2.1) is convergent with the R order of convergence
\[ O_R((2.1), r) \geq 3(1 + \nu_n) \in (6, 7), \]
where $\sigma_n$ is the unique positive root of the equation $\sigma^n - 1 = 0$ and $r = [r_1, r_2, \ldots r_n]^T$ is the limit point (the vector of the exact zeros).

Proof. Substituting $|v_i^{(m)}| = h_i^{(m)}/q$ in (2.18) we obtain

$$h_i^{(m+1)} < \frac{1}{n-1} \left( h_i^{(m)} \right)^3 \left( \sum_{j=1}^{i-1} h_j^{(m+1)} + \sum_{j=i+1}^n \left( h_j^{(m)} \right)^3 \right).$$

Let $h_i^{(m)} = \max_i h_i^{(m)}$ and $h^{(0)} = h$. By virtue of (2.7) it follows $h_i^{(0)} \leq h < 1$ ($i = 1, \ldots, n$). According to the last inequalities and (2.19) we conclude that the sequences $\{h_i^{(m)}\}$ ($i = 1, \ldots, n$) converge to 0. Consequently, the sequences $\{z_i^{(m)}\}$ converge to the zeros $r_i$ ($i = 1, \ldots, n$).

In view of (2.19) we can write

$$h_i^{(m+1)} \leq h_i^{x_i^{(m+1)}}$$

for $i = 1, \ldots, n$, $m = 0, 1, \ldots$.

Defining the matrix $B$ by $B = 3A$, where

$$A = \begin{bmatrix}
1 & 1 & 1 & 0 \\
1 & 1 & \ddots & \ddots \\
0 & \ddots & \ddots & 1 \\
1 & 1 & 0 & \cdots & 0 & 1
\end{bmatrix},$$

the vectors $x^{(m)} = [x_1^{(m)} \ldots x_n^{(m)}]^T$ can be successively evaluated by

$$x^{(m+1)} = Bx^{(m)},$$

starting with $x^{(0)} = [1 \ldots 1]^T$.

The matrix $A$ is irreducible and primitive so that it has the unique positive eigenvalue equal to its spectral radius $\rho(A)$. Using the same consideration as in [2, Chapter 8] (see, also [1]) and definition of the R-order of convergence (see [6]), it can be concluded that

$$O_R((2.1), r) \geq \rho(B) = 3\rho(A),$$

where $\rho(A)$ and $\rho(B)$ are the spectral radii of the matrices $A$ and $B$.

The characteristic polynomial $\phi_n$ of the matrix $A$ is

$$\phi_n(\lambda) = (\lambda - 1)^n - (\lambda - 1) - 1.$$ 

Substituting $\sigma = \lambda - 1$, we obtain

$$\phi_n(\sigma) = \phi_n(1 + \lambda) = \sigma^n - \sigma - 1.$$ 

Since $\phi_n(\sigma) < 0$ if $\sigma \in (0, 1)$, $\phi_n(1/2) > 0$ and $d\phi_n(\sigma)/d\sigma > 0$ if $\sigma > 1$, there exists a zero $\sigma_n \in (1, 1)$ of $\phi_n$ and there can be no other positive zero of $\phi_n$. Hence, we deduce that $\rho(A) = 1 + \sigma_n$. Finally, from (2.21) we find the lower bound for the R-order of convergence of the iterative method (2.1)

$$O_R((2.1), r) \geq 3(1 + \sigma_n) \in (6, 7).$$

Using Halley's correction in the total-step procedure (TSH-method, formula (1.13)), we obtain
the following iterative process:

\[
z_i^{(m+1)} = z_i^{(m)} - \left[ G \left( z_i^{(m)} \right) - \sum_{j=1}^{n} \left( z_i^{(m)} - z_j^{(m)} - H \left( z_j^{(m)} \right) \right)^{-2} \right]^{-1/2},
\]

\[i = 1, \ldots, n, \quad m = 0, 1, \ldots.\]

(2.22)

**Theorem 2.** Under the conditions (2.7) TSH-method (2.22) is convergent with the convergence order equal to six.

**Proof.** Using the relations (2.7)–(2.18), in the same way as in the derivation of (2.18), we find

\[
|v_i^{(m+1)}| < \frac{q^3}{n-1} |v_i^{(m)}|^3 \left( q^2 \sum_{j \neq i}^{n} |v_j^{(m)}|^3 \right) < \frac{1}{q}.
\]

(2.23)

The relation (2.23) can be obtained directly from (2.18) by omitting the first sum on the right-hand side. Putting \(|v_i^{(m)}| = h_i^{(m)}/q\), the inequality (2.23) becomes

\[
h_i^{(m+1)} < \frac{(h_i^{(m)})^3}{n-1} \sum_{j \neq i}^{n} (h_j^{(m)})^3 < (h_i^{(m)})^3 < (h_i^{(m)})^6.
\]

(2.24)

The conditions (2.7) imply \(q |v_i^{(0)}| = h < 1\) \((i = 1, \ldots, n)\). According to this it follows from (2.24) that the sequences \(\{h_i^{(m)}\}\) \((i = 1, \ldots, n)\) converge to 0 and, consequently, \(z_i^{(m)} \to r_i\) \((i = 1, \ldots, n)\).

Further, we find from (2.23)

\[
|v_i^{(m+1)}| < q^5 |v_i^{(m)}|^3 (v_i^{(m)})^3 < q^5 (v_i^{(m)})^6
\]

or

\[
|v_i^{(m+1)}| \leq v^{(m+1)} < q^5 (v_i^{(m)})^6.
\]

From the last relation we may infer that the convergence order of the iterative method (2.22) is equal to six. □

Starting from the formulas (1.11) and (1.12), where Newton’s correction (1.5) is applied, we establish the following iterative methods:

(TSN)

\[
z_i^{(m+1)} = z_i^{(m)} - \left[ G \left( z_i^{(m)} \right) - \sum_{j=1}^{n} \left( z_i^{(m)} - z_j^{(m)} - N \left( z_j^{(m)} \right) \right)^{-2} \right]^{-1/2},
\]

\[i = 1, \ldots, n, \quad m = 0, 1, \ldots.
\]

(2.25)
Theorem 3. Under the conditions (2.7) SSN-method (2.26) is convergent with the R-order of convergence

$$O_R((2.26), r) \geq 3 + \tau_n \in (5, 6),$$

where $\tau_n$ is the unique positive root of the equation $\tau^n - 2^{n-1} \tau - 3 \cdot 2^{n-1} = 0$.

Proof. The convergence analysis of the iterative process (2.26) is essentially equal to that of the iterative method (2.1). Therefore, only the final part of the analysis will be presented.

Under the conditions (2.7) the following relation, similar to (2.18), can be derived:

$$|v_i^{(m+1)}| < \frac{q}{n-1} |v_i^{(m)}| \left( \sum_{j=1}^{i-1} |v_j^{(m+1)}| + q \sum_{j=i+1}^{n} |v_j^{(m)}|^2 \right) < \frac{1}{q},$$

(2.27)

where the previous notations are used. Substituting $|v_i^{(m)}| = h_i^{(m)}/q$ in (2.27), we obtain

$$h_i^{(m+1)} < \frac{1}{n-1} \left( h_i^{(m)} \right)^3 \left( \sum_{j=1}^{i-1} h_j^{(m+1)} + \sum_{j=i+1}^{n} (h_j^{(m)})^2 \right)$$

(2.28)

$$i = 1, \ldots, n, \quad m = 0, 1, \ldots.$$

The conditions (2.7) imply $h_i^{(0)} \leq \max h_i^{(0)} = h < 1$ $(i = 1, \ldots, n)$. According to this, we conclude from (2.28) that the sequences $\{ h_i^{(m)} \}$ $(i = 1, \ldots, n)$ converge to 0, which means that the iterative process (2.26) is convergent.

The components of vector $x_i^{(m)} = [x_i^{(m)} \ldots x_n^{(m)}]^T$ from the relation of the form (2.20), corresponding to (2.28), can be determined from the recurrent formula

$$x_i^{(m+1)} = Cx_i^{(m)}, \quad x_i^{(0)} = [1 \ldots 1]^T,$$

where $C$ is the matrix given by

$$C = \begin{bmatrix} 3 & 2 & \hdots & 0 \\ 3 & 2 & \hdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 3 & 2 & \hdots & 0 \end{bmatrix}.$$

The matrix $C$ is irreducible and primitive so that it has the unique positive eigenvalue equal to its spectral radius $\rho(C)$. The lower bound of the R-order of convergence of the simultaneous method (2.26) is given by

$$O_R((2.26), r) \geq \rho(C).$$

(2.29)
The characteristic polynomial \( \psi_n \) of the matrix \( C \) is
\[
\psi_n(\lambda) = (\lambda - 3)^n - (\lambda - 3)2^{n-1} - 3 \cdot 2^{n-1}.
\]
Replacing \( \tau = \lambda - 3 \), we obtain
\[
\tilde{\psi}_n(\tau) = \tau^n - 2^{n-1}\tau - 3 \cdot 2^{n-1}.
\]
Since \( \tilde{\psi}_n(\tau) < 0 \) if \( \tau \in (0, 2) \), \( \tilde{\psi}_n(3) = 3^n - 3 \cdot 2^n > 0 \) and \( d\tilde{\psi}_n(\tau)/d\tau = n\tau^{n-1} - 2^{n-1} > 0 \) if \( \tau > 2 \), we infer that the polynomial \( \tilde{\psi}_n \) has a zero \( \tau_n \) on the interval \((2, 3)\) and it is only positive zero. Hence, we have \( \rho(C) = 3 + \tau_n \in (5, 6) \). In regard to (2.29) the lower bound of the R-order of convergence of the iterative method (2.26) is
\[
O_R((2.26), r) \geq 3 + \tau_n \in (5, 6).
\]

Starting from the relation (2.27), in the similar way as in the proof of Theorem 2, the following inequalities can be derived
\[
\begin{align*}
&h^{(m+1)} < (h^{(m)})^5, \quad h^{(0)} \leq h < 1 \quad \text{under the conditions (2.7)} \\
&q^{(m+1)} < (q^{(m)})^5.
\end{align*}
\]

Hence, using the conclusions deduced in proving Theorem 2, we have the following statement:

**Theorem 4.** Under the conditions (2.7) TSN-method (2.25) is convergent with the convergence order equal to five.

The lower bounds for \( O_R((1.10), r) \), \( O_R((2.26), r) \) and \( O_R((2.1), r) \) are tabulated for \( n = 3(1)10 \) in Table 1.

The increase of the convergence speed of the single-step methods (1.10), (2.26) and (2.1) (relative to the corresponding total-step methods (1.9), (2.25) and (2.22) respectively) is larger if the degree of the polynomial is lower. The acceleration of convergence is attained without additional calculations. Besides, single-step methods occupy less storage space in digital computer (because the new approximations take positions of the former ones).

In practical realization of the considered iterative methods (1.11)–(1.14) with Newton’s and Halley’s corrections, it is desirable, before calculation of new approximations, to find the values of functions \( u(z) \) and \( s(z) \) and then, by the formulas (1.4), (1.5) and (1.6), to calculate \( G(z) \) and the wanted corrections \( N(z) \) or \( H(z) \). In such a way, the method with the correction terms require slightly more numerical operations in comparison with the basic method (1.9) of the fourth order. Thus, significant increase of the convergence speed, compared with the basic method (1.9), is attained by inconsiderable increase of arithmetic operations, which points at the efficiency of the proposed modifications of square root method.

**Table 1**

<table>
<thead>
<tr>
<th>( n )</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>SSN: (2.26)</td>
<td>5.862</td>
<td>5.586</td>
<td>5.443</td>
<td>5.357</td>
<td>5.299</td>
<td>5.257</td>
<td>5.225</td>
<td>5.201</td>
</tr>
</tbody>
</table>
3. Numerical results

The considered iterative methods of square root type, based on the formulas (1.9)–(1.14), was tested in the example of the polynomial

\[ P(z) = z^5 - (4 + 5i)z^4 + (6 + 20i)z^3 - (4 + 30i)z^2 + (-15 + 20i)z + 75i, \]  

(3.1)

whose the exact zeros are \( r_1, r_2 = 1 + 2i \), \( r_3 = -1 \), \( r_4 = 3 \) and \( r_5 = 5i \).

The routine on FORTRAN was realized on a HONEYWELL 66 system in double precision arithmetic (about 18 significant digits). Before calculating new approximations, the values \( u(z^{(m)}) \) and \( s(z^{(m)}) \) \((i = 1, \ldots, n)\), necessary for evaluation of Ostrowski’s function (1.4), were calculated. The same values were used for evaluation of Newton’s and Halley’s corrections in the formulas (2.25), (2.26), (2.22) and (2.1).

Table 2

<table>
<thead>
<tr>
<th>( i )</th>
<th>( \text{Re}(z_i^{(m)}) )</th>
<th>( \text{Im}(z_i^{(m)}) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>TS</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(1.9)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0.99999938019776821</td>
<td>2.0000001707170553462</td>
</tr>
<tr>
<td>2</td>
<td>1.00000027930352643</td>
<td>-2.000000176464657521</td>
</tr>
<tr>
<td>3</td>
<td>-0.999999790801744628</td>
<td>1.47\times10^{-7}</td>
</tr>
<tr>
<td>4</td>
<td>3.000000008454234552</td>
<td>-2.06\times10^{-7}</td>
</tr>
<tr>
<td>5</td>
<td>5.12\times10^{-7}</td>
<td>5.000000353285864895</td>
</tr>
<tr>
<td>SS</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(1.10)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1.00000016008831563</td>
<td>1.999999846637151023</td>
</tr>
<tr>
<td>2</td>
<td>1.000000232361937907</td>
<td>-1.999999875334209145</td>
</tr>
<tr>
<td>3</td>
<td>-0.99999999974857274</td>
<td>8.58\times10^{-11}</td>
</tr>
<tr>
<td>4</td>
<td>3.00000000004794713</td>
<td>-1.53\times10^{-11}</td>
</tr>
<tr>
<td>5</td>
<td>-1.25\times10^{-15}</td>
<td>5.0000000000000117</td>
</tr>
<tr>
<td>TSN</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(1.11)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0.9999996666718872</td>
<td>2.000000554250890694</td>
</tr>
<tr>
<td>2</td>
<td>1.00000013100207197</td>
<td>-1.99999987734416132</td>
</tr>
<tr>
<td>3</td>
<td>-1.00000022569099023</td>
<td>4.27\times10^{-7}</td>
</tr>
<tr>
<td>4</td>
<td>3.00000003609140354</td>
<td>4.78\times10^{-9}</td>
</tr>
<tr>
<td>5</td>
<td>-4.04\times10^{-9}</td>
<td>4.999999989567260054</td>
</tr>
<tr>
<td>SSN</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(1.12)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0.999999944040282847</td>
<td>1.999999846677047645</td>
</tr>
<tr>
<td>2</td>
<td>0.99999998785395964</td>
<td>-2.00000000513604734</td>
</tr>
<tr>
<td>3</td>
<td>-1.0000000002193334</td>
<td>-3.77\times10^{-12}</td>
</tr>
<tr>
<td>4</td>
<td>2.99999999888187</td>
<td>6.01\times10^{-14}</td>
</tr>
<tr>
<td>5</td>
<td>-1.61\times10^{-15}</td>
<td>5.00000000000000783</td>
</tr>
<tr>
<td>TSH</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(1.13)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0.99999999931345461</td>
<td>1.99999999885598444</td>
</tr>
<tr>
<td>2</td>
<td>0.99999999988968412</td>
<td>-1.9999999991093962</td>
</tr>
<tr>
<td>3</td>
<td>-1.000000000053598353</td>
<td>-6.91\times10^{-11}</td>
</tr>
<tr>
<td>4</td>
<td>3.00000000031266106</td>
<td>5.52\times10^{-11}</td>
</tr>
<tr>
<td>5</td>
<td>-4.84\times10^{-11}</td>
<td>5.000000000045326267</td>
</tr>
<tr>
<td>SSH</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(1.14)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1.000000000028365003</td>
<td>1.99999999977318455</td>
</tr>
<tr>
<td>2</td>
<td>1.000000000004753</td>
<td>-2.000000000354773</td>
</tr>
<tr>
<td>3</td>
<td>-1.00000000001134</td>
<td>6.88\times10^{-16}</td>
</tr>
<tr>
<td>4</td>
<td>2.99999999999874</td>
<td>-2.41\times10^{-16}</td>
</tr>
<tr>
<td>5</td>
<td>4.85\times10^{-19}</td>
<td>5.0000000000000000</td>
</tr>
</tbody>
</table>
As the initial approximations to the zeros of polynomial (3.1) the following complex numbers were taken:

\[ z^{(0)}_1 = 1.8 + 1.3i, \quad z^{(0)}_2 = 1.8 - 1.3i, \quad z^{(0)}_3 = -1.8 - 0.7i, \]
\[ z^{(0)}_4 = 3.7 + 0.7i, \quad z^{(0)}_5 = 0.7 + 4.3i. \]

These starting values were chosen under weaker conditions than (2.7); namely, the condition (2.7) requires \( |z^{(0)}_i - r_i| < 1/q \approx 0.314 \), while in our example we have \( |z^{(0)}_i - r_i| \approx 1 \).

In spite of crude initial approximations, the presented iterative methods demonstrate good behaviour and very fast convergence. Numerical results, obtained after the second iteration, are displayed in Table 2.

Let \( z^{(m)} = [z^{(m)}_1 \ldots z^{(m)}_n]^T \) be the vector of approximations to the zeros in the \( m \)th iteration. Take Euclid’s norm,

\[ e^{(m)} := \| z^{(m)} - r \|_E = \left( \sum_{i=1}^{n} |z^{(m)}_i - r_i|^2 \right)^{1/2}, \]

as a measure of closeness of approximations with regard to the exact zeros.

In the presented example for the initial approximations we have even \( e^{(0)} \approx 5.35 \) ! For this reason, the results of the first iteration are not so good: the values of \( e^{(1)} \) for the considered methods belong to the range \((0.069, 0.184)\). But, the second iteration produces significantly better approximations to the zeros. The values \( e^{(2)} \) for every applied process are given in Table 3.

From Table 2 and Table 3 we observe very fast convergence of the square root methods with Halley’s correction.

References


