A Wedderburn–Artin–Jacobson Structure Theorem*

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The Wedderburn–Artin structure theorem characterizes the rings which
are semisimple (in the classical sense) as the cartesian products of finitely
many rings, each of which is the ring of linear transformations of a finite
dimensional vector space over a division ring. There are two obvious direc-
tions in which this class of rings may be enlarged: one may drop the require-
ment that the product have only finitely many factors, and one may drop
the finite-dimensionality of the vector spaces. The resulting class of rings
has been the subject of intensive investigation, principally by Jacobson [4].

In this note we give a simple characterization of the rings of this enlarged
class; a necessary and sufficient condition is that the ring be complete in a
certain quite natural topology. This topology, which we call the intrinsic
topology, is defined in Section 1, which contains some of its more immediate
properties. Among other things, we relate this notion to one introduced
by Rubin [5] of semisimplicity relative to a kernel functor, this in turn being
a variant of a notion of topological semisimplicity (see for example [2]).

In Section 2 we show that an arbitrary product of full linear rings is
complete in its intrinsic topology, while the third section contains the proof
of the converse. The main result can also be obtained as a consequence
of some of the ideas of Chapter VI of [4]; the proof given here is essentially
self-contained.

1. THE INTRINSIC TOPOLOGY

If \( A \) is a ring, then the class of those (left) \( A \)-modules which are at the
same time projective and semisimple is closed under the formation of sub-
module, factor module and arbitrary direct sums. Furthermore, if a module

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is a sum (not necessarily direct) of submodules, each of which is in this class, then the module is also in this special class. (In the terminology of Gabriel [1], this special class of modules is a localizing subcategory of the category of all left modules.) Because of these closure properties one may topologize the ring \( A \) by taking as a base for the neighborhoods of 0 those left ideals \( a \) such that \( A/a \) is a projective semisimple module. This topology will be called the intrinsic topology of \( A \), and, relative to it, \( A \) is a topological ring (not necessarily Hausdorff).

It is immediate from the definition that \( A \) is discrete in its intrinsic topology if, and only if, \( A \) is a semisimple ring.

The condition on a left ideal \( a \) of \( A \) that \( A/a \) be projective is, of course, equivalent to the condition that \( a \) be generated by an idempotent. Thus, a left ideal \( a \) is open in the intrinsic topology if, and only if, \( a = Ae \) with \( e^2 = e \) and \( A(1 - e) \) semisimple. This description of the open ideals will be used frequently in what follows. There are other useful characterizations of the open ideals which will shortly be described.

Recall that the socle, \( S \) of \( A \), is defined to be the sum of all minimal left ideals; \( S \) is the largest left ideal of \( A \) which is semisimple. Furthermore, \( S \) is a two-sided ideal.

**Proposition 1.1.** A left ideal \( a \) of \( A \) is open in the intrinsic topology if, and only if, \( S + a = A \).

**Proof.** If \( a \) is open, then \( a = Ae \) with \( A(1 - e) \) semisimple. This means that \( A(1 - e) \subseteq S \), so that \( a + S = A \). On the other hand, suppose that \( a + S = A \). Then, \( a \cap S \) is a direct summand of \( S \), so that \( S = (a \cap S) \oplus S' \), and hence \( a \oplus S' = A \).  

**Corollary 1.2.** \( S \) is dense in the intrinsic topology. If \( b \) is any left ideal of \( A \), then there is a left ideal \( c \) such that \( b \cap c = 0 \) and \( b + c \) is dense.

**Proof.** The previous proposition shows that \( A \) is the only neighborhood of \( S \), hence that \( S \) is dense. Given the ideal \( b \), we have a direct sum decomposition \( S = (b \cap S) \oplus c \) for some left ideal \( c \subseteq S \). Then, \( b \cap c = 0 \), while \( b + c \supset S \), so that \( b + c \) is dense.

Another, quite useful, description of the intrinsic topology is contained in the following proposition.

**Proposition 1.3.** A left ideal \( a \) is open in the intrinsic topology if, and only if, \( a \) is the intersection of a finite number of maximal left ideals each of which is generated by an idempotent.

**Proof.** If \( m \) is such a maximal ideal, then \( A/m \) is simple and projective, so that \( m \) is open. Certainly a finite intersection of such maximal ideals is
still open. Suppose that \( a \) is an open left ideal. Then, \( A/a \) is the direct sum of a finite number of simple modules, say \( A/a = S_1 \oplus \cdots \oplus S_n \). If \( a_i \) is the image of 1 in \( S_i \) and \( m_i \) is its annihilator, then \( m_i \) is a maximal left ideal, \( a = m_1 \cap \cdots \cap m_n \) and \( m_i \) is open. The last condition implies that \( m_i \) is generated by an idempotent. 

There are two important corollaries of Proposition 1.3.

**Corollary 1.4.** If \( b \) is a proper closed two-sided ideal of \( A \), then the topology of \( A/b \) arising from the intrinsic topology of \( A \) is the intrinsic topology of the ring \( A/b \).

**Proof.** Suppose that \( m \) is an open maximal ideal of \( A \) which contains \( b \). Then its image \( m' \) in \( A/b \) is a maximal left ideal generated by an idempotent because \( m \) is so generated. This means that \( m' \) is open in the intrinsic topology of \( A/b \). The same is the case, therefore, for the image of any open left ideal of \( A \) containing \( b \).

Now suppose that \( m' \) is a maximal left ideal of \( A/b \) which is open in the intrinsic topology of that ring. Then \( m' \) is generated by an idempotent \( e' \). Let \( e \) be a preimage of \( e' \) in \( A \) and let \( m \) be the preimage of \( m' \). Then, \( m \) is a maximal left ideal of \( A \) with \( m = b + Ae \). (Also \( e - e^2 \in b \).) We must show that \( m \) is open. Assume on the contrary that \( m \) is not open, then it is dense. (Any neighborhood of \( m \) is a left ideal properly larger than \( m \).) Now, \( m(1 - e) \subseteq b \), while \( \{ x \in A \mid x(1 - e) \in b \} \) is closed because \( b \) is closed. But this implies that \( 1 - e \in b \) which is not so. Thus, \( m \) is not dense and it is, in fact, open. 

**Corollary 1.5.** Let \( \{ A_\alpha \} \) be a collection of rings each topologized by its own intrinsic topology. Then the product topology of \( \Gamma = \prod A_\alpha \) coincides with the intrinsic topology of \( \Gamma \).

**Proof.** The assertion follows immediately from Proposition 1.3; the details are left to the reader. 

**Remark.** It is well-known that an infinite product of classical simple rings is not semisimple. A reason for this lies in Corollary 1.5, for an infinite product of discrete rings is never discrete in its product topology.

It had been remarked earlier that \( A \) is not necessarily Hausdorff in its intrinsic topology. The fact that a sub-basis at 0 consists of certain maximal ideals makes it very easy to determine when \( A \) is Hausdorff; indeed there is a simple description of the closure of 0. Denote by \( J \) the Jacobson radical of \( A \), i.e., the intersection of all maximal left ideals, and by \( J_0 \) the closure of 0, i.e., the intersection of those maximal left ideals which are direct summands of \( A \). Clearly \( J_0 \subseteq J \), and also \( J_0 \) is a two-sided ideal.
PROPOSITION 1.6. \( J_0 \) is the annihilator of the left module \( S/S \cap J \).

Proof. Let us first see that \( J_0 S \subset J \). Suppose that \( m \) is a maximal left ideal. If \( m \) is open, then \( J_0 \subset m \), so that \( J_0 S \subset m \). If \( m \) is not open, then \( S + m \neq A \), so that \( S \subset m \), and hence also \( J_0 S \subset m \). Thus, \( J_0 S \subset J \). On the other hand, suppose that \( x S \subset J \). If \( m \) is an open maximal left ideal, then \( S + m = A \), so that \( x \in x S + x m \), and hence \( x \in J + m \subset m \). This means that \( x \) is in every neighborhood of 0, and consequently \( x \in J_0 \).

COROLLARY 1.7. The intrinsic topology is Hausdorff if, and only if, \( S \) is a faithful left \( A \)-module.

Proof. Since the annihilator of \( S \) is contained in the annihilator of \( S/S \cap J \), it follows that \( J_0 \) contains the annihilator of \( S \). Thus, if \( A \) is Hausdorff, then \( S \) is faithful. On the other hand, if \( S \) is a faithful module, then \( J = 0 \) because \( J S = 0 \). This means that \( J_0 \) is the annihilator of \( S \) itself, or that \( J_0 = 0 \).

COROLLARY 1.8. The intrinsic topology is "anti-Hausdorff" (the only neighborhood of 0 is \( A \) itself) if, and only if, \( S \subset J \).

Proof. This is immediate from Proposition 1.6.

Remarks. Should \( S \) be a faithful module (i.e., \( A \) is Hausdorff) then, in addition to being semisimple, \( S \) is also projective. For, the condition on \( S \) implies that \( J = 0 \), and hence that the minimal left ideals of \( A \) are each generated by an idempotent. Thus, \( S \) is the sum of projective simple modules, which implies that \( S \) itself is projective. One should not suppose that the converse is the case. It can certainly happen that \( S \) is projective without \( S \) being faithful, e.g., \( S = 0 \). We have noted that \( J \subset J_0 \); it is possible to have \( J = 0 \) with \( J_0 \neq 0 \). For example, if \( A \) is a primitive ring without minimal ideals, then \( J = 0 \) while \( J_0 = A \). This suggests that one should consider the condition that \( A \) be Hausdorff in its intrinsic topology as a more stringent condition than the vanishing of its Jacobson radical. If \( A \) is Hausdorff, then \( A \) has many projective simple modules. However, one should not conclude that in this case every simple \( A \)-module is projective. For, unless \( A \) is a semisimple ring in the classical sense, \( S \) is a proper ideal and no maximal left ideal which contains \( S \) can be a direct summand of \( A \). This may also be interpreted in another form: if \( A \) is not a semisimple ring, then \( A \) has dense maximal left ideals.

The class of projective semisimple \( A \)-modules constitutes the class of torsion modules relative to a kernel functor, which we denote by \( \iota \) (see [3] for the definition and properties of kernel functors). In [5], Rubin introduces the notion of semisimplicity relative to a kernel functor. By Theorem 1.3
of [5], \( A \) is \( \sigma \)-semisimple (\( \sigma \) some kernel functor) if, and only if, \( S \) is dense in the \( \sigma \)-topology of \( A \). Proposition 1.1 may now be interpreted as follows: \( A \) is \( \iota \)-semisimple and \( A \) is \( \sigma \)-semisimple if, and only if, \( \sigma \leq \iota \). Thus, \( \iota \) is the largest kernel functor relative to which \( A \) is semisimple.

2. ENDOMORPHISM RINGS

Let \( V \) be a left \( A \)-module. The annihilator topology of \( A \) defined by \( V \) has for a basis of neighborhoods of 0 the left ideals \( a_F \) as follows: if \( F \) is a finite subset of \( V \) then \( a_F = \{ x \in A \mid xF = 0 \} \). (This topology is called the finite topology in [4].) It is routine that \( A \) is a topological ring in this topology, and is Hausdorff if, and only if, \( V \) is faithful. We shall consider some of the relations between the intrinsic topology and various annihilator topologies of \( A \).

**Proposition 2.1.** If \( V \) is a faithful \( A \)-module then the intrinsic topology of \( A \) is weaker (i.e., fewer open sets) than the annihilator topology defined by \( V \).

**Proof.** Suppose that \( m \) is a maximal left ideal of \( A \) which is open in the intrinsic topology, so that \( m = Ae \) with \( e = e^2 \). Since \( V \) is faithful, there is an element \( y \in V \) with \( x = (1 - e)y + 0 \). Then, \( ex = 0 \) so that the annihilator of \( x \) is exactly \( m \). This means that \( m \) is open in the annihilator topology defined by \( V \). It is clear that the same conclusion is valid for any left ideal of \( A \) which is open in the intrinsic topology.

**Proposition 2.2.** Let \( V \) be a faithful \( A \)-module. Then the following two conditions are equivalent:

1. \( V \) is projective and semisimple.
2. The intrinsic topology of \( A \) coincides with the annihilator topology defined by \( V \).

**Proof.** (1) \( \Rightarrow \) (2). Let \( z \) be an element of \( V \) and let \( a \) be its annihilator. Then, \( A/a \cong Az \), while \( Az \) is a projective semisimple module. Hence \( a \) is open in the intrinsic topology. This means that the annihilator topology defined by \( V \) is weaker than the intrinsic topology, and therefore, using Proposition 2.1, the two topologies coincide.

(2) \( \Rightarrow \) (1). Let \( z \) and \( a \) have the same meaning as above. Because \( a \) is open in the annihilator topology, it is open in the intrinsic topology and hence \( Az \) is a projective semisimple module. This implies that \( V \) is a sum of projective semisimple modules so that \( V \) is also a module of this type.
Remark. If there is to exist any module $V$ as described above, then $A$ must be Hausdorff in its intrinsic topology, in which case the socle $S$ is such a module. Thus, whenever the intrinsic topology of $A$ is Hausdorff, that topology coincides with the annihilator topology defined by any faithful projective semisimple module ($S$ being one such).

Suppose now that a ring $\mathcal{Q}$ has a faithful simple module $V$. Assume first that the socle $S$ of $\mathcal{Q}$ is not zero. Then, there is some element $x \in V$ with $Sx \neq 0$, so that $Sx = V$. This means (remembering that $S$ is semisimple) that $V$ is isomorphic to a submodule of $S$. But then, $S$ is also faithful, and we may conclude that the intrinsic topology of $\mathcal{Q}$ is Hausdorff, that $S$ is projective and therefore that $V$ is also projective, and finally that the annihilator topology defined by $V$ coincides with the intrinsic topology of $\mathcal{Q}$. Conversely, if $V$ is a projective $\mathcal{Q}$-module, then the annihilator of any of its nonzero elements would be an open maximal left ideal. Consequently, the intrinsic topology of $\mathcal{Q}$ is not anti-Hausdorff, hence $S \neq 0$. Thus, we have proved the following:

**Proposition 2.3.** Assume $V$ is a faithful simple $\mathcal{Q}$-module. Then the following assertions are equivalent:

1. $V$ is projective.
2. $S \neq 0$.
3. The intrinsic topology of $\mathcal{Q}$ is Hausdorff.
4. The annihilator topology defined by $V$ is the intrinsic topology of $\mathcal{Q}$.

As a special case, we have the following:

**Corollary 2.4.** Let $D$ be a division ring, $V$ a vector space over $D$ and $\mathcal{Q}$ the ring of all $D$-linear endomorphisms of $V$. Then the intrinsic topology of $\mathcal{Q}$ coincides with the annihilator topology defined by $V$.

*Proof.* One need merely note that $V$ is a projective simple $\mathcal{Q}$-module.

To complete the consideration of annihilator topologies we include the following very useful fact.

**Proposition 2.5.** Let $A$ be a ring, $V$ an $A$-module and $\mathcal{Q}$ the $A$-endomorphism ring of $V$. Then $\mathcal{Q}$ is complete in the annihilator topology defined by $V$.

*Proof.* By its very definition, $V$ is a faithful $\mathcal{Q}$-module, so that $\mathcal{Q}$ is Hausdorff in the annihilator topology. Assign the discrete topology to $V$; then $V$ is a topological $\mathcal{Q}$-module in the sense that the composition mapping $\mathcal{Q} \times V \to V$ is continuous. (This is the essence of the annihilator topology.) As a result, if $\tilde{\mathcal{Q}}$ denotes the completion of $\mathcal{Q}$, the map $\mathcal{Q} \times V \to V$ extends to a map $\tilde{\mathcal{Q}} \times V \to V$. This gives $V$ also the structure of an $\tilde{\mathcal{Q}}$-module.
Furthermore, because $\Omega$ is dense in $\mathcal{D}$ and the elements of $\Omega$ commute with those of $\mathcal{A}$ on $V$, it follows that the elements of $\Omega$ also commute with those of $\mathcal{A}$ on $V$. Thus, the only way that $\Omega$ may be properly larger than $\Omega$ is the possible existence of elements of $\mathcal{D}$ which act like $0$ on $V$. Denote by $\mathfrak{d}$ the annihilator, in $\mathcal{Q}$, of $V$; we shall show that $\mathfrak{d} = 0$. Let $N$ be a closed neighborhood of $0$ in $\Omega$, so that $N$ is the closure of $N \cap \Omega$. The latter is a neighborhood of $0$ in $\Omega$, so that there is a finite subset $F$ of $V$ with $N \cap \mathfrak{a}_F = \{x \in \Omega \mid xF = 0\}$. The closure of $\mathfrak{a}_F$ in $\mathcal{D}$ is a subneighborhood of $N$, and we may use it in place of $N$. Since $\Omega$ is dense in $\Omega$, we have $\Omega + \mathfrak{a}_F = \Omega$. Given any $d \in \mathfrak{d}$, we have $d = x + \alpha$ with $x \in \Omega$ and $\alpha \in \mathfrak{a}_F$. The elements of $\mathfrak{a}_F$ annihilate $F$; the extension of the module structure of $V$ to $\mathcal{D}$ implies that the elements of $\mathfrak{a}_F$ also annihilate $F$. Hence, $\alpha F = 0$, while, of course, $dF = 0$. Hence also $xF = 0$, or $x \in \mathfrak{a}_F$. Consequently, $d \in \mathfrak{a}_F$, i.e., $\mathfrak{d}$ is contained in every closed neighborhood of $0$ in $\mathcal{D}$, or $\mathfrak{d} = 0$.

Suppose once more that $V$ is a vector space over a division ring $D$, and that $\Omega$ is the $D$-endomorphism ring of $V$. Then, by the result just proved, $\Omega$ is complete in the annihilator topology defined by $V$, and hence by Corollary 2.4, is complete in its intrinsic topology. Applying Corollary 1.5, we see that we have also proved the following:

**Theorem 2.6.** Let $\{D_\alpha\}$ be a set of division rings, and $V_\alpha$ a vector space over $D_\alpha$. Denote by $\mathcal{Q}_\alpha$ the $D_\alpha$-endomorphism ring of $V_\alpha$. Then the ring $\prod \mathcal{Q}_\alpha$ is complete in its intrinsic topology.

3. **The Structure Theorem**

Throughout this section $\mathcal{A}$ stands for a ring which is complete in its intrinsic topology. In particular, $\mathcal{A}$ is Hausdorff, so that, among others, $J = 0$ and hence $\mathcal{A}$ has no nonzero nilpotent ideals. The essence of the proof of the structure theorem is the following:

**Proposition 3.1.** If $\mathfrak{n}$ is a closed two-sided ideal of $\mathcal{A}$, then

1. $\mathfrak{n} = \mathcal{A}e$ with $e$ a central idempotent,
2. $\mathcal{A}/\mathfrak{n}$ is complete in the topology induced by that of $\mathcal{A}$,
3. this topology of $\mathcal{A}/\mathfrak{n}$ is the intrinsic topology of the ring $\mathcal{A}/\mathfrak{n}$.

**Proof.** First note that (3) follows from Corollary 1.4. Also observe that (2) follows from (1), for $\mathcal{A}/\mathfrak{n}$ is isomorphic as a topological group with $\mathcal{A}(1 - e)$ and the latter is a closed subgroup of $\mathcal{A}$. ($\mathcal{A}(1 - e)$ is the annihilator of $e$ in $\mathcal{A}$.) Thus, we need only prove (1).

By Corollary 1.2 there is a left ideal $\mathfrak{a}$ of $\mathcal{A}$ such that $\mathfrak{n} \cap \mathfrak{a} = 0$ and
$n + a$ is dense. Denote by $d$ the set of all $x \in A$ such that $nx = 0$; $d$ is a closed two-sided ideal of $A$. Since $na \subseteq n \cap a = 0$, we have $a \subseteq d$ and hence $n + d$ is dense. Also, $(n \cap d)^2 = 0$, from which it follows that $n \cap d = 0$.

Both $n$ and $d$ are closed ideals of $A$; the completeness of $A$ implies that $n$ and $d$ are complete in their induced topologies, and hence also that $n \times d$ is complete. Consider the map $\alpha: n \times d \to A$ given by $\alpha(x, y) = x + y$. This is clearly continuous, and is one-to-one because $n \cap d = 0$. We shall show that $\alpha$ is open. To do so we must prove the following: Given neighborhoods $U$ and $U'$ of 0 in $n$ and $d$, there is a neighborhood $W$ of 0 in $A$ such that $W \cap (n + d) \subseteq \alpha(U, U')$. The neighborhood $U$ contains one of the form $n = b$ with $b$ an open left ideal of $A$. Such an ideal $b$ has the form $Af$ with $f^2 = f$, and (because $n$ is a two-sided ideal), $n \cap b = nf$. Similarly, $U' \supseteq bg$ with $g^2 = g$ and $Ag$ an open ideal of $A$. Then, $\alpha(U, U') \supseteq nf + bg \supseteq (n + b) \cap Af \cap Ag$ as required. The completeness of $n \times d$ implies that the image of $\alpha$, i.e., $n + d$, is closed in $A$; we know that $n + d$ is dense and hence $n + d = A$.

Let $m$ be an open maximal left ideal of $A$, and let $n$ be the annihilator of the simple module $V = A/m$. It is clear that $n$ is a closed two-sided ideal of $A$, so that Proposition 3.1 may be applied. In particular, $A/n$ is Hausdorff in its intrinsic topology, so that the intrinsic topology of $A/n$ is the same as the annihilator topology of $A/n$ defined by the faithful simple module $V$. If $D$ is the $A$-endomorphism ring of $V$, then the Jacobson density theorem tells us that $A/n$ is dense in $\text{End}_D(V)$ in the annihilator topology defined by $V$. The completeness of $A/n$ therefore yields the equality $A/n = \text{End}_D(V)$.

Keeping the same notation, suppose that $m'$ is an open maximal left ideal of $A$ which contains $n$. Then, $m' = Af$ with $f = f^2$, and the inclusion $n \subseteq m'$ implies that $1 - f$ does not annihilate $V$. Hence, there is a nonzero element $x = (1 - f)y$, for which $fx = 0$, i.e., $m'$ is the annihilator of $x$. This implies that $A/m'$ is isomorphic to $V$. Summarizing the results, we have the following.

**Proposition 3.2.** If $m$ is an open maximal left ideal of $A$ and $n$ is the annihilator of $V = A/m$, then $A/n$ is isomorphic (as a topological ring) with $\text{End}_D(V)$, $D$ a division ring. If $m'$ is any open maximal left ideal of $A$ which contains $n$, then $A/m'$ is isomorphic to $V$. ■

Choose a set $\{m_\alpha\}$ of open maximal left ideals of $A$ with the properties:

1. If $\alpha \neq \beta$, then $A/m_\alpha$ is not isomorphic to $A/m_\beta$.
2. If $m$ is an open maximal left ideal, then $A/m$ is isomorphic to $A/m_\alpha$, some $\alpha$. 
Denote by $n_\alpha$ the annihilator of $A/m_\alpha$. The set of all open maximal left ideals is partitioned as follows: To each $\alpha$ is associated the set of those $m$ which contain $n_\alpha$; it is the same set as those $m$ with $A/m \cong A/m_\alpha$.

Since each $n_\alpha$ is the intersection of those open maximal left ideals which contain it, we have $\bigcap \n_\alpha = 0$. Form $\Gamma = \prod A/n_\alpha$ where each $A/n_\alpha$ is given its intrinsic topology and $\Gamma$ is given the product topology. (This gives $\Gamma$ its own intrinsic topology.) Define $\varphi: A \to \Gamma$ in the obvious way so that $\varphi$ is a continuous ring monomorphism.

**Proposition 3.3.** If $\alpha \neq \beta$, then $n_\alpha + n_\beta = A$. If $\alpha_1, \alpha_2, \ldots, \alpha_n$ are distinct indices, then $\bigcap n_\alpha + \bigcap n_\alpha = A$. The image of $\varphi$ is dense in $\Gamma$.

**Proof.** Because the $n_\alpha$'s are two-sided ideals, the second assertion follows from the first. The last assertion is a Chinese-Remainder type of conclusion from the second assertion. Thus, we need only prove the first statement. According to the partitioning of the open maximal left ideals, there is none which contains both $n_\alpha$ and $n_\beta$. This implies that $n_\alpha + n_\beta$ is dense. However, each $n_\alpha$ is generated by a central idempotent, so that the same is true of $n_\alpha + n_\beta$, i.e., $n_\alpha + n_\beta$ is closed. Thus $n_\alpha + n_\beta = A$ as desired. □

**Proposition 3.4.** The topology induced in $A$ by the inclusion map $\varphi$ coincides with the intrinsic topology of $A$.

**Proof.** Because $\varphi$ is continuous we need only prove that it is an open map. Let $a$ be an open left ideal of $A$, so that $a = m_1 \cap \cdots \cap m_n$ with $m_i$ an open maximal left ideal. Each $m_i$ belongs to one of the sets of the previously discussed partition; let $n_{\alpha_1}, \ldots, n_{\alpha_r}$ be the different annihilators of the various simple modules $A/m_i$. Then, we have $a = b_1 \cap \cdots \cap b_r$ with $b_j$ an open left ideal and $b_j \supseteq n_{\alpha_j}$. Define a neighborhood $T$ of 0 in $\Gamma$ as follows: an element of $\Gamma$ belongs to $T$ if its $a_j$-component is in $b_j/\cap n_{\alpha_j}$, $1 \leq j \leq r$, and no condition imposed on the other components. It is clear that $\varphi(A) \cap T = \varphi(a)$.

The completeness of $A$ now implies that $\varphi(A)$ is closed in $\Gamma$. We have already noted that the image is dense, and thus $\varphi(A) = \Gamma$. □

Thus, we have proved the structure theorem:

**Theorem 3.5.** A ring is the product of full linear rings if, and only if, it is complete in its intrinsic topology. □

The reader might wish to note that the product decomposition just arrived at will have only finitely many factors if, and only if, the center of $A$ is discrete. Independently, each of the vector spaces involved will be finite-dimensional (over the corresponding division ring) if, and only if, the intrinsic
topology as defined (and with respect to which \( A \) is complete) coincides with the intrinsic topology of \( A \) using right instead of left ideals.

We conclude with a remark concerning the situation in which the given ring is not necessarily complete in its intrinsic topology. Given a ring \( \Omega \), let \( J_0 \) be the closure of 0. If \( J_0 = \Omega \), then the intrinsic topology contributes nothing to the study of \( \Omega \). Assume \( J_0 \neq \Omega \); then Corollary 1.4 permits us to pass to \( \Omega/J_0 \), whose induced topology is the same as the intrinsic topology of \( \Omega/J_0 \). One may now take the step of completing the ring. The possibility of doing so, and still keeping track of the topology, is covered by the following, whose proof is left to the reader.

**Proposition 3.6.** Suppose that \( \Omega \) is Hausdorff in its intrinsic topology. Then, the topology of the completion \( \tilde{\Omega} \), as completion, is the same as the intrinsic topology of \( \tilde{\Omega} \) as a ring. ■

**References**