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ULTRAFILTERS, ULTRAPOWERS AND FINITENESS IN A TOPOS

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Introduction

This paper started from the following question raised by A. Day: What is the correct definition of an ultrafilter and an ultrapower in an elementary topos in the sense of Lawvere and Tierney (cf. [14], [19])? To be more precise, we are looking for a generalization of the set-theoretic ultrapower construction which is internal, i.e. which can be described within the topos.

Thus an internal filter on an object X in a topos E should be a subobject of Ω^X with appropriate closure properties, which can be given as preservation properties of the characteristic function $u : \Omega^X \to \Omega$. In particular, an ultrafilter will be a Heyting algebra morphism from Ω^X to Ω with a certain section. On the other hand, an external filter on X would be a filter on $E(1, \Omega^X)$, the set of subobjects of X.

Therefore the construction of the ultrapower A^X/U should use the internal power A^X rather than the external power $A^{E(1,X)}$ which might not even exist without further assumptions on external limits. Usually the ultrapower A^X/U of a set A with respect to an ultrafilter U on the set X is defined as the quotient of A^X, obtained by identifying two functions if they agree on a subset in the filter (cf. [3]). Okhuma observed in [17] that the ultrapower may be viewed as the filtered colimit of the partial powers A^Y with Y in U. Rephrasing this idea we arrive at the following definition. The ultrapower is defined as the quotient of $\tilde{A}^X | U$, the set of partial functions with domain in U, obtained by identifying two functions if they agree on a subset in the filter. This is the approach we will use.

In the first section we will study various types of filters. In particular, every topology $j : \Omega \to \Omega$ is a filter on 1. The main result of the second section can be stated as follows. The filterpower functor is left exact and it preserves a propositional operation iff the filter preserves it. The filterpower functor preserves Ω iff the filter is an ultrafilter. As a corollary we obtain the left exactness of the associated sheaf functor, which is obtained by applying twice the filterpower functor. Another application yields the set of quotients with respect to a topology in the category of

M-sets, where M is a monoid. Moreover, in a topos with the internal axiom of choice the ultrapower functor is a first order functor i.e. it is left exact and preserves the propositional operations and the quantification. This is a generalization of the basic result on ultrapowers of sets which states that the diagonal morphism from A to A^X/U is an elementary embedding.

A property which characterizes finite sets in the category of sets can be used to define a concept of finiteness in an arbitrary topos. In the third section we study the following two variants which depend on ultrafilters. An object will be called ultrafinite iff it is isomorphic to all its ultrapowers. It will be called principally finite iff every ultrafilter on it is principal. Ultrafinite objects in categories with external ultrapowers have been studied by Day and Higgs [4]. It will be shown that the class of ultrafinite resp. principally finite objects in a topos contains Ω and is closed under finite limits. The class of principally finite objects is in general not closed under the power set operation $\Omega^{(-)}$. However, we do not know whether the class of ultrafinite objects is closed under this operation.

In .he following we will work in a fixed elementary topos E unless stated otherwise. The category of sets S is a fixed 2-valued topos with natural number object and the axiom of choice. Any model of ZFC will determine such a category. With regard to the basic results on elementary topoi, the rea'.er is referred to Kock, Wraith [12] and Freyd [7]. A list of notations is given at the end of the paper.

Last not least I w nt to acknowledge the valuable help I received from A. Day, D. Higgs and R. Diaconescu during a visit to Lakehead University and to McGill University in fall 1973.

1. Filters in a topos

Let X be an object in a topos E. An internal filter on X is a subobject U of Ω^X with appropriate closure properties which are presented as preservation properties of the characteristic function $u : \Omega^X \to \Omega$ in the following definitions.

U resp. u is called a *filter* on X iff U is closed under finite intersections i.e. $u \wedge^X = \wedge(u \times u)$ and $ut^X = t$. A filter u is called *proper* iff $u \leqslant \exists ! X$. A proper filter u is called *prime* iff $u \vee^X = \vee(u \times u)$. A proper filter u is called *ultrafilter* iff $u \Rightarrow^X = \Rightarrow(u \times u)$. A proper filter is called *maximal* iff for every proper filter v $u \leqslant v$ implies $u = v$. A filter u is called the *principal* filter generated by the subobject $k : K \to X$ iff for every filter v $u \leqslant v$ iff $v|\chi(k)| = t$. Clearly, every filter u on X determines an external filter on $E(1, \Omega^X)$, the set of subobjects of X, given by $E(1, u)$. Every topology $j : \Omega \to \Omega$ in E is a filter on 1, which is proper iff $j = \Omega$.

The following proposition shows that internal filters have nearly all the usual properties.

1.1. *Let* $u : \Omega^X \to \Omega$ *be a filter on* X.

(1) *u is proper iff* $u\Omega^{!X} = \Omega$ *iff* $u\Omega^{!X} \leqslant \Omega$.
(2) *If u is proper then u and* $!X$ *are epic.*
(3) *If u is an ultrafilter then u is prime and maximal.*
(4) *If u is a filter (proper, prime, ultra-) then* $u\Omega^f$ *is a filter (proper, prime, ultra-) for every* $f: X \to Y$.
(5) *If u is a filter (proper, prime, ultra-) then* $u \exists k$ *is a filter (proper, prime, ultra-) for every subobject* $k: K \to X$ *with* $u|\chi(k)| = t$.
(6) $u \Rightarrow^X (|\chi(k)|\,!\Omega^X, \Omega^X)$ *is the filter generated by u and the subobject* $k: K \to X$.
(7) $\forall\, !K\Omega^k$ *is the principal filter generated by the subobject* $k: K \to X$. *Moreover, u has a left adjoint* $g: \Omega \to \Omega^X$ *with* $gt = |\chi(k)|$ iff $u = \forall\, !K\Omega^k$. $\forall\, !K\Omega^k$ *is an ultrafilter iff* $!K$ *is an isomorphism.*

1) We have always $\Omega \leqslant u\Omega^{!X}$ because of $u\Omega^{!X} t = t$. Moreover, the adjointness of $\exists\, !X$ and $\Omega^{!X}$ yields $u \leqslant \exists\, !X$ iff $u\Omega^{!X} \leqslant \Omega$.

2) $u\Omega^{!X} = \Omega$ implies $\exists\, !X\Omega^{!X} = \Omega$. Hence $!X$ must be epic, too.

3) If u preserves $\Rightarrow$ then $u(\phi_1 \vee \phi_2) \leqslant u\mu$ iff $u\phi_1 \vee u\phi_2 \leqslant u\mu$ because of $(\phi_1 \vee \phi_2) \Rightarrow \mu = (\phi_1 \Rightarrow u) \wedge (\phi_2 \Rightarrow \mu)$. Using $\lambda = u\Omega^{!X}\lambda$ we obtain $u\phi_1 \vee u\phi_2 = u(\phi_1 \vee \phi_2)$ as required. Let v be a proper filter with $u \leqslant v$. Since u preserves $\Leftrightarrow$ we have $u \Leftrightarrow^X (h_1, h_2) = \Leftrightarrow (uh_1, uh_2) = t_Z$, where $(h_1, h_2) = \mathrm{kp}(u)$. Then $u \leqslant v$ and $v \Leftrightarrow^X \leqslant \Leftrightarrow (v \times v)$ imply $t_Z = v \Leftrightarrow^X (h_1, h_2) \leqslant \Leftrightarrow (vh_1, vh_2)$ and hence $vh_1 = vh_2$. Since u is epic there exists q with $v = qu$. However, $q = qu\Omega^{!X} = v\Omega^{!X} = \Omega$ implies $u = v$ as required.

4) Here we use that Ω^f is a Heyting algebra morphism and that $\Omega^f\Omega^{!Y} = \Omega^{!X}$.

5) $\exists k$ preserves $\wedge$ and $\vee$ and we have $\exists k\Omega^{!K} \leqslant \Omega^{!X}$, $\exists k[t_X] = \chi(k)$ and $\exists k[\phi_1 \Rightarrow \phi_2] = \chi(k) \wedge (\exists k[\phi_1] \Rightarrow \exists k[\phi_2])$.

6) Let u_k abbreviate $u \Rightarrow^X (|\chi(k)|\,!\Omega^X, \Omega^X)$. Obviously u_k is a filter such that $u_k|\chi(k)| = t$ and $u \leqslant u_k$. Conversely, $v|\chi(k)| = t$ and $u \leqslant v$ imply $u_k = u \Rightarrow^X (|\chi(k)|\,!\Omega^X, \Omega^X) \leqslant v \Rightarrow^X (|\chi(k)|\,!\Omega^X, \Omega^X) = v$.

7) Since $\forall\, !X$ is the smallest filter on X, $\forall\, !X \Rightarrow^X (|\chi(k)|\,!\Omega^X, \Omega^X)$ is the filter generated by $k: K \to X$. The result follows now from the identity $\forall\, !X \Rightarrow^X (|\chi(k)|\,!\Omega^X, \Omega^X) = \forall\, !K\Omega^k$. $\exists\, k\Omega^{!K}$ is left adjoint to $\forall\, !K\Omega^k$ and $\exists\, k\Omega^{!K} t = |\chi(k)|$. On the other hand, g is left adjoint to u iff $\forall\, !X \Rightarrow^X (g \times \Omega^X) = \Rightarrow (\Omega \times u)$. This implies $\forall\, !X \Rightarrow^X (|\chi(k)|\,!\Omega^X, \Omega^X) = \Rightarrow (t\,!\Omega^X, u) = u$ as required. It follows from (4), (5) and $\Omega^k \exists k = \Omega$ that $\forall\, !K$ is a proper filter resp. ultrafilter iff $\forall\, !K\Omega^k$ is proper filter resp. ultrafilter. However, $\forall\, !K\Omega^{!K} = \Omega$ iff $!K$ is epic, and $\forall\, !K$ preserves $\Rightarrow$ iff $!K$ is monic.

The following result generalizes the wellknown characterization of an ultrafilter as a filter such that either a set or its complement belongs to the filter. It owes its final form to an observation of C.J. Mikkelsen.

1.2. *A filter u is an ultrafilter iff* $u \Leftrightarrow^X (\Omega^X, \Omega^{!X}u) = t\,!\Omega^X$ *and u is epic.*

If u is an ultrafilter then $u \Leftrightarrow^X (\Omega^X, \Omega^{!X}u) = \Leftrightarrow (u, u\Omega^{!X}u) = \Leftrightarrow (u,u) = t\,!\Omega^X$ and u is epic since u is proper. Conversely, by transitivity $u \Leftrightarrow^X (\Omega^X, \Omega^{!X}u) = t\,!\Omega^X$ implies $\mathrm{kp}(u) \leqslant \chi^{-1}(u \Leftrightarrow^X)$. This yields $u \Leftrightarrow^X = \Leftrightarrow (u \times u)$ since the other inclusion is true in general. Expressing implication by means of biimplication and intersection we see that that u preserves implication. Since u is epic, $\Leftrightarrow (u, u\Omega^{!X}u) = t\,!\Omega^X$ implies $u = u\Omega^{!X}u$ and hence $u\Omega^{!X} = \Omega$.

The following lemma of D. Higgs will be needed later.

1.3. (1) *If* $j : \Omega \to \Omega$ *is monic then* $j^2 = \Omega$.
(2) *If* $j^2 = \Omega$ *and* $\Omega \leqslant j$ *then* $j = \Omega$.

1) The morphism j is monic iff $\Leftrightarrow (j \times j) = \Leftrightarrow$. It suffices to prove the analogous result for a Heyting algebra H, i.e. if a map $g: H \to H$ satisfies $g(a) \Leftrightarrow g(b) = a \Leftrightarrow b$ for all a, b in H then g is the identity. The equation $g^3 = g$ and hence $g^2 = H$ is obtained by applying several times the following equivalence: $x \wedge g(a) = x \wedge g(b)$ iff $x \wedge a = x \wedge b$.

2) $\Omega = \Omega \wedge j$ and $j^2 = \Omega$ imply $j = j \wedge j^2 = j \wedge \Omega$.

In the following we will study filters in particular topoi. Let G be a group. A filter u on an object X in S^G corresponds to a filter U on the underlying set which is closed under the action of G. Analogous statements hold for the various types of filters. As a consequence we have that there is no ultrafilter on G itself if G has a nontrivial subgroup of finite index.

Let M be a monoid. A filter u on an object X in S^M corresponds to a filter U on the underlying set of $M \times X$ which is closed under the action of M in the following sense. If Y is in U then $m^{-1}Y$ is in U for every m in M, where $m^{-1}Y = \{(n,x) : (nm, x) \in Y\}$. In particular every topology in S^M determines such a filter U on M. These are the topologies described by Stenstroem [18].

Let Top(Y) be the category of sheaves over the topological space Y. Let $p: X \to Y$ be an etale space over Y and let op(X) resp. op(Y) be the open subsets of X resp. Y. A filter on $p: X \to Y$ corresponds to a map $v : \mathrm{op}(X) \to \mathrm{op}(Y)$ which preserves intersection and satisfies $\mathrm{op}(Y) \leqslant vp^{-1}$. The filter is prime resp. an ultrafilter iff $\mathrm{op}(Y) = vp^{-1}$ and v preserves union resp. implication.

In S^2 a filter u on an object $\alpha : A_0 \to A_1$ determines a pair of filters v_0, v_1 on A_0 resp. A_1 which are defined as follows: $v_0(Y) = 1$ iff $u_0((Y, A_1)) = 1$, $v_1(Y) = 1$ iff $u_1(Y) = 1$.

The various properties of the filter u are inherited by the filters v_0 and v_1. Moreover, there is a bijective correspondence between ultrafilters on $\alpha : A_0 \to A_1$ and pairs of ultrafilters v_0, v_1 satisfying $v_1 = v_0\Omega^\alpha$.

In S^N, where N is the ordered set of natural numbers, we have an analogous situation. A filter u on an object A with transition maps $\alpha_n : A_n \to A_{n+1}$ determines a sequence of filters v_n on A_n for n in N, which are defined as follows:

$$v_n(Y) = 1 \quad \text{iff} \quad u_n(Y, A_{n+1}, \ldots) = 1 \;.$$

As above the various filterproperties are inherited by the filters v_n for n in N. If A is increasing i.e. the transition maps are injective then there is a bijective correspondence between ultrafilters on A and sequences of countably complete ultrafilters v_n on A_n satisfying $v_{n+1} = v_n \Omega^{\alpha_n}$ for n in N.

2. Filterpowers in a topos

For an object A and a filter u on X in a topos E the filterpower A^X/u will be defined as the quotient of $\tilde{A}^X|u$, the object of partial morphisms from X to A with domain in u, obtained by identifying two partial morphisms if they agree on a subobject which is in the filter u. As we will see in the appendix, A^X/u can be viewed as the filtered colimit of the partial powers A^K with K in u.

Let u be a filter on X. The subobject $j_{u,A} : \tilde{A}^X|u \to \tilde{A}^X$ is determined by the characteristic function $u(\mathrm{dom}_A)^X$. The *filterpower* A^X/u with its projection $q_{u,A} : \tilde{A}^X|u \to A^X/u$ is defined by $q_{u,A} = \mathrm{coeq}(k_{u,A})$, where $k_{u,A} : K_u(A) \to \tilde{A}^X|u \times \tilde{A}^X|u$ is determined by its characteristic function $u(\mathrm{dom}_A \sqcap_A)^X(j_{u,A} \times j_{u,A})$. Here we have used the intersection operation $\sqcap_A : \tilde{A} \times \tilde{A} \to \tilde{A}$ on A.

$$\begin{array}{ccccccccc}
K_u(A) & \longrightarrow & & \longrightarrow & \tilde{A}^X|u & \longrightarrow & U & \longrightarrow & 1 \\
\downarrow{\scriptstyle k_{u,A}} & & & & \downarrow{\scriptstyle j_{u,A}} & & \downarrow & & \downarrow{\scriptstyle t} \\
\tilde{A}^X|u \times \tilde{A}^X|u & \xrightarrow[j_{u,A}\times j_{u,A}]{} & \tilde{A}^X \times \tilde{A}^X & \xrightarrow[(\sqcap_A)^X]{} & \tilde{A}^X & \xrightarrow[(\mathrm{dom}_A)^X]{} & \Omega^X & \xrightarrow[u]{} & \Omega \\
 & \searrow\!\searrow & & & & & & & \\
 & & \tilde{A}^X|u & \xrightarrow[q_{u,A}]{} & A^X/u & & & &
\end{array}$$

Since there exists a factorization $j_{u,A}\eta_{u,A}$ of $(\eta_A)^X$, we can define the *diagonal morphism* $d_{u,A} : A \to A^X/u$ by $q_{u,A}\eta_{u,A}A^{!X}$. The above constructions are functorial. Thus we obtain a functor $(-)^X/u : E \to E$ and a natural transformation $d_u : \mathrm{id}_E \to (-)^X/u$.

The following proposition gives basic properties of the filterpower A^X/u.

2.1. (1) *If u is proper then $d_{u,A}$ is monic for every A;*

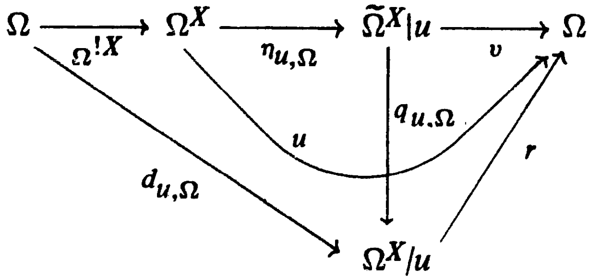
(2) *u is a proper filter resp. ultrafilter iff $d_{u,\Omega}$ is monic resp. an isomorphism;*

(3) *If $k : K \to X$ is a subobject of X then $A^K \simeq A^X/\forall\, !K\Omega^k$. In particular, $A \simeq A^X/\Omega^i$ for $i : 1 \to X$.*

1) Since u is a filter we can verify that $k_{u,A}$ is an equivalence relation. Thus we

obtain $\chi(\mathrm{kp}(d_{u,A})) = \chi(\mathrm{kp}(q_{u,A}))(\eta_{u,A}A^{!X} \times \eta_{u,A}A^{!X}) = u(\mathrm{dom}_A\sqcap_A)^X(\eta_A A^{!X} \times \eta_A A^{!X}) = u(\equiv_A)^X(A^{!X} \times A^{!X}) = u\Omega^{!X} \equiv_A$ because of $\mathrm{dom}_A\sqcap_A(\eta_A \times \eta_A) = \equiv_A$. Thus $d_{u,A}$ has to be monic if $u\Omega^{!X} = \Omega$.

2) Let $v: \tilde{\Omega}^X|u \to \Omega$ be $u(d_0)^X j_{u,\Omega}$ and let $s: \tilde{\Omega}^X|u \to \tilde{\Omega}^X|u$ be $\eta_{u,\Omega}(d_0)^X j_{u,\Omega}$. Making use of $u \Leftrightarrow^X \leqslant \Leftrightarrow (u \times u)$, $\Leftrightarrow = \mathrm{dom}_\Omega\sqcap_\Omega(\eta_\Omega \times \eta_\Omega)$, $\mathrm{dom}_\Omega = d_1$ and $\eta_\Omega d_0 \geqslant \Omega$ we obtain $\chi(\mathrm{kp}(v)) = \Leftrightarrow (u(d_0)^X j_{u,\Omega} \times u(d_0)^X j_{u,\Omega}) \geqslant u \Leftrightarrow^X ((d_0)^X j_{u,\Omega} \times (d_0)^X j_{u,\Omega}) = u(\mathrm{dom}_\Omega\sqcap_\Omega)^X((\eta_\Omega d_0)^X j_{u,\Omega} \times (\eta_\Omega d_0)^X j_{u,\Omega}) = \chi(k_{u,\Omega})(s \times s) \geqslant \chi(k_{u,\Omega})$ and hence $k_{u,\Omega} \leqslant \mathrm{kp}(v)$. Moreover, $\tilde{\Omega} = \sqcap_\Omega(\tilde{\Omega}, \eta_\Omega d_0)$ yields $t\,!\tilde{\Omega}^X|u = u(\mathrm{dom}_\Omega)^X j_{u,\Omega} = u(\mathrm{dom}_\Omega\sqcap_\Omega)^X(j_{u,\Omega} \times j_{u,\Omega})(\tilde{\Omega}^X|u, \eta_{u,\Omega}(d_0)^X j_{u,\Omega}) = \chi(k_{u,\Omega})(\tilde{\Omega}^X|u, s)$. Therefore $(\tilde{\Omega}^X|u, s)$ factors through $k_{u,\Omega}$ and by transitivity $\chi(k_{u,\Omega}) \geqslant \chi(k_{u,\Omega})(s \times s)$. Hence we can conclude that $k_{u,\Omega} = \mathrm{kp}(v)$ iff $u \Leftrightarrow^X = \Leftrightarrow (u \times u)$, since $(d_0)^X j_{u,\Omega}$ is right invertible.



Therefore there exists $r: \Omega^X/u \to \Omega$ such that $v = rq_{u,\Omega}$. This implies $rd_{u,\Omega} = rq_{u,\Omega}\eta_{u,\Omega}\Omega^{!X} = u(d_0)^X j_{u,\Omega}\eta_{u,\Omega}\Omega^{!X} = u\Omega^{!X}$. If u is proper then $d_{u,\Omega}$ is monic. Conversely, if $d_{u,\Omega}$ is monic then $\Leftrightarrow = \chi(\mathrm{kp}(d_{u,\Omega})) = \chi(\mathrm{kp}(q_{u,\Omega}))(\eta_{u,\Omega}\Omega^{!X} \times \eta_{u,\Omega}\Omega^{!X}) = u(\mathrm{dom}_\Omega\sqcap_\Omega)^X(\eta_\Omega\Omega^{!X} \times \eta_\Omega\Omega^{!X}) = u \Leftrightarrow^X (\Omega^{!X} \times \Omega^{!X}) = u\Omega^{!X} \Leftrightarrow$. Hence $u\Omega^{!X}$ must be monic and thus $u\Omega^{!X} = \Omega$ by 1.3. Here we have used again $\Leftrightarrow = \mathrm{dom}_\Omega\sqcap_\Omega(\eta_\Omega \times \eta_\Omega)$. Combining the above results we obtain: $d_{u,\Omega}$ is an isomorphism iff r is the inverse of $d_{u,\Omega}$ iff $k_{u,\Omega} = \mathrm{kp}(v)$ and $u\Omega^{!X} = \Omega$ iff $u \Leftrightarrow^X = \Leftrightarrow (u \times u)$ and u is proper iff u is an ultrafilter.

3) Let u_k abbreviate $\forall\, !K\Omega^k$. Because of $\Omega^k(\mathrm{dom}_A)^X = (\mathrm{dom}_A)^K A^k$ there exists $\xi_k: \tilde{A}^X|u_k \to \tilde{A}^K|\Omega^k$ with $(\xi_k, j_{u_k,A}) = \mathrm{pb}((\eta_A)^K, \tilde{A}^k)$. Since $\tilde{A}^k$ is epic ξ_k has to be epic, too. The equation $\Omega^k(\mathrm{dom}_A\sqcap_A)^X(j_{u_k,A} \times j_{u_k,A}) = (\mathrm{dom}_A\sqcap_A(\eta_A \times \eta_A))^K(\xi_k \times \xi_k)$ yields $\mathrm{kp}(\xi_k) = k_{u_k,A}$ and hence $A^K \simeq A^X/u_k$.

The functor $(-)^X/u$ has the following preservation properties:

2.2. (1) *$(-)^X/u$ is left exact;*

(2) *$(-)^X/u$ preserves $\alpha: \Omega \times \Omega \to \Omega$ iff u preserves α;*

(3) *If u is prime then $(-)^X/u$ preserves finite coproducts;*

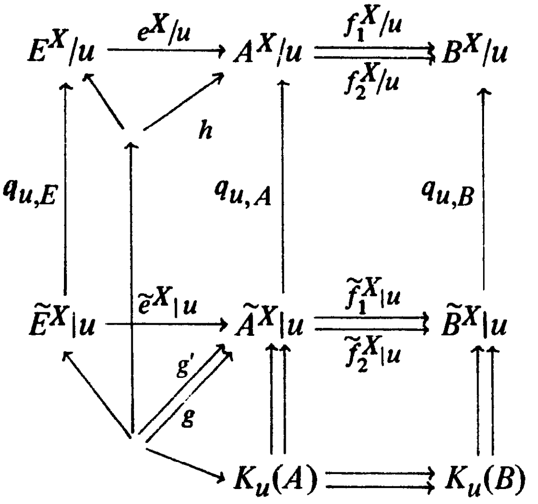
(4) *$(-)^X/u$ preserves epics if $(\tilde{-})^X|u$ preserves epics;*

(5) *$(-)^X/u$ preserves Ω iff u is an ultrafilter.*

1) First we want to show that $(-)^X/u$ preserves binary products. The morphism

$(\tilde{p}_A, \tilde{p}_B):(A \times B)^{\sim} \rightarrow \tilde{A} \times \tilde{B}$ has a left inverse c determined by the partial morphism $(\eta_A \times \eta_B; A \times B)$. Specializing to $A = 1$ and $B = 1$ yields $\Delta_\Omega: \Omega \rightarrow \Omega \times \Omega$ and its left inverse $\wedge: \Omega \times \Omega \rightarrow \Omega$. Since $(\tilde{p}_A, \tilde{p}_B)$ and c commute with the intersection and domain operation, one obtains a morphism $r_u:(A \times B)^X/u \rightarrow (A^X/u) \times (B^X/u)$ with a left inverse c_u from $(\tilde{p}_A, \tilde{p}_B)$ and c. After rewriting $r_u c_u$ with the help of the restriction operations $\rho_A: \tilde{A} \times \Omega \rightarrow \tilde{A}$ and $\rho_B: \tilde{B} \times \Omega \rightarrow \tilde{B}$, it can be verified that $r_u c_u$ is the identity. $1^X/u \simeq 1$ can be shown as follows. $\tilde{1} \simeq \Omega$ and $\tilde{1}^X|u \simeq U$ imply $\chi(k_{u,1}) = u(\mathrm{dom}_1 \sqcap_1)^X (j_{u,1} \times j_{u,1}) = u \wedge^X (\chi^{-1}(u) \times \chi^{-1}(u)) = \wedge(u\chi^{-1}(u) \times u\chi^{-1}(u))$ $= t!(U \times U)$ and hence $1^X/u \simeq 1$ since $U \rightarrow 1$ is epic.

Let $m: A \rightarrow B$ be monic. Since $\tilde{m}^X|u$ is again monic and commutes with the intersection operation, we obtain $\mathrm{kp}((\tilde{m}^X|u)q_{u,B}) = \mathrm{kp}(q_{u,B})$. This together with $q_{u,B}$ being epic implies that m^X/u is monic by a lemma for regular categories (cf. [2, p. 156]). Let e be the equalizer of $f_1, f_2: A \rightarrow B$. Let h be the equalizer of f_1^X/u and f_2^X/u. Pulling h back along the epic $q_{u,A}$ we obtain g such that $((\tilde{f}_1^X|u)g, (\tilde{f}_2^X|u)g)$ factors through $k_{u,B}$.



Making use of $\rho_A(p_1, \mathrm{dom}_A \sqcap_A) = \rho_A(p_2, \mathrm{dom}_A \sqcap_A)$ we obtain g' such that $(\tilde{f}_1^X|u)g' = (\tilde{f}_2^X|u)g'$ and $q_{u,A}g = q_{u,A}g'$. Since $(\tilde{-})^X|u$ preserves equalizers, g' factors through $\tilde{e}^X|u$ and h must factor through e^X/u as required.

2) As we have seen in 2.1.2 $(\tilde{\Omega}^X|u, \eta_{u,\Omega}(d_0)^X j_{u,\Omega})$ factors through $k_{u,\Omega}$. Thus $q_{u,\Omega} = q_{u,\Omega}\eta_{u,\Omega}(d_0)^X j_{u,\Omega}$ implies that $q_{u,\Omega}\eta_{u,\Omega}$ is epic. Moreover, by 2.1.2 there exists $r: \tilde{\Omega}^X|u \rightarrow \Omega$ such that $rq_{u,\Omega} = u(d_0)^X j_{u,\Omega}$ and hence $rq_{u,\Omega}\eta_{u,\Omega} = u$. Pulling back along the epic $q_{u,\Omega}\eta_{u,\Omega}: \Omega^X \rightarrow \Omega^X/u$ we are able to verify that r is the characteristic function of t^X/u. Since $(-)^X/u$ is left exact, $(-)^X/u$ preserves $\alpha: \Omega \times \Omega \rightarrow \Omega$ iff $r(\alpha^X/u) = \alpha(r \times r)$, where $r = \chi(t^X/u)$. The required equivalence follows from $r(\alpha^X/u)(q_{u,\Omega}\eta_{u,\Omega} \times q_{u,\Omega}\eta_{u,\Omega}) = rq_{u,\Omega}\eta_{u,\Omega}\alpha^X = u\alpha^X$ and $\alpha(u \times u) =$ $\alpha(r \times r)(q_{u,\Omega}\eta_{u,\Omega} \times q_{u,\Omega}\eta_{u,\Omega})$ since $q_{u,\Omega}\eta_{u,\Omega}$ is epic.

3) This follows from (2) since every left exact functor which preserves $\vee$ has to preserve binary coproducts. Since u preserves false we have $\tilde{O}^X|u \simeq O$ and hence $O^X/u \simeq O$.

4) If $\tilde{g}^X|u$ is epic then g^X/u is epic, since every $q_{u,A}$ is epic.

5) This is a restatement of 2.1.2.

Diaconescu suggested to prove the left exactness of the functor $(-)^X/u$ using the description of A^X/u as the internal colimit of the powers A^K for K in u. This will be done in the appendix.

As we have seen earlier, every topology is a filter on 1. This leads to the following corollary.

2.3. (1) *If $j: \Omega \to \Omega$ is a topology in a topos E then $((-)^1/j)^2$ is the associated sheaf functor and it is left exact.*

(2) *If $j: \Omega \to \Omega$ is a topology in S^M, where M is a monoid, then $(-)^1/j$ is the functor which associates with an M-set its set of quotients.*

1) It can be verified that $(-)^1/j$ coincides with the functor $(-)^+$ described by Johnstone [10]. Hence the associated sheaf functor is obtained by applying twice $(-)^1/j$. Moreover, $((-)^1/j)^2$ is left exact by 2.2.1.

2) Let J be the filter of right ideals corresponding to j. Then A^1/j is the filtered colimit of the $S^M(D, A)$ for D in J. However, this is the set of quotients of A described by Stenstroem [18]. In particular, M^1/j is the monoid of quotients.

If the internal axiom of choice holds in E then the functor $(\tilde{-})^X$ preserves epics. Using results of Diaconescu and Freyd, we know that E is boolean in this case. Thus the above results give rise to the following corollary.

2.4. *Let E be a topos with the internal axiom of choice and let u be an ultrafilter on X. Then $(-)^X/u$ is a first order functor, i.e. it is left exact and preserves the propositional operations as well the existential and universal quantification.*

Thus the theorem of Kock–Mikkelsen [11] on the factorization of first order functors can be applied in this case. The above corollary generalizes the wellknown fact that in the category of sets the diagonal morphism $d_{u,A}: A \to A^X/u$ is an elementary embedding.

3. Ultrafinite and principally finite objects

Using the axiom of choice and in particular the existence of nonprincipal ultrafilters on the natural numbers, one can characterize finite sets in the category of sets as follows.

(1) A is finite iff A is isomorphic to all its ultrapowers.

(2) A is finite iff every ultrafilter on A is principal.

This motivates the following definitions.

An object A in a topos E is called *principally finite* iff for every ultrafilter u on A

there exists $a: 1 \to A$ such that $u = \Omega^a$. The object A is called *ultrafinite* iff for every X and every ultrafilter u on X the diagonal morphism $d_{u,A}: A \to A^X/u$ is an isomorphism.

The two classes of finite objects have the following closure properties.

3.1. (1) ***The full subcategory of ultrafinite objects contains Ω and is closed under finite limits and finite coproducts. It is closed under quotients if the functor $(\tilde{-})^X$ preserves epics for every X.***

(2) ***If A is ultrafinite then A is principally finite, but not conversely.***

(3) ***The full subcategory of principally finite objects contains Ω and is closed under finite limits. It is closed under quotients if every proper filter can be extended to an ultrafilter.***

(4) ***The class of principally finite objects is not closed under the power object operation $\Omega^{(-)}$, in general.***

It remains an open question whether the class of ultrafinite objects is closed under the operation $\Omega^{(-)}$.

1) The closure properties of the ultrafinite objects follow from the preservation properties of the ultrapower functor given in 2.2.

2) Let u be an ultrafilter on A. Since $d_{u,A}: A \to A^A/u$ is an isomorphism we have $q_{u,A}\eta_{u,A}|A| = q_{u,A}\eta_{u,A}A^{!A}a$, where $a = (d_{u,A})^{-1}q_{u,A}\eta_{u,A}|A|$. This implies $t = \chi(k_{u,A})(|A|, A^{!A}a) = u^X(|A|, A^{!A}a) = u\{a\}$ and hence $u = \Omega^a$ by 1.1.7. – The counterexample can be found in S^2. In S^2 an ultrafilter on an object $\alpha: A_0 \to A_1$ corresponds to a pair of ultrafilters v_0, v_1 on A_0 resp. A_1 with $v_1 = v_0\Omega^\alpha$. Hence $\alpha: A_0 \to A_1$ is principally finite iff A_0 is finite. If $\alpha: A_0 \to A_1$ is ultrafinite then not only A_0 but also A_1 must be finite. Otherwise we could take a nonprincipal ultra filter on $\mathrm{id}_X: X \to X$ with X infinite, which yields $(A^X/v)_1 \simeq A_1^X/v_1 \neq A_1$. Conversely, if A_0 and A_1 are finite then $\alpha: A_0 \to A_1$ is ultrafinite. Thus any object $\alpha: A_0 \to A_1$ with A_0 finite and A_1 infinite provides a counterexample.

3) 1 and Ω are ultrafinite and hence principally finite. Let A_1, A_2 be principally finite. If u is an ultrafilter on $A_1 \times A_2$ then $u\Omega^{p_1}$ and $u\Omega^{p_2}$ are again ultrafilters by 1.1.4. Hence there exist a_1 and a_2 with $u\Omega^{p_1} = \Omega^{a_1}$ and $u\Omega^{p_2} = \Omega^{a_2}$. This yields $u|A_1 \times a_2| = t$ and $u|a_1 \times A_2| = t$ and hence $u\{(a_1, a_2)\} = t$. By 1.1.7 we can conclude $u = \Omega^{(a_1, a_2)}$ as required. Let $m: A \to B$ be monic and B principally finite. If u is an ultrafilter on A then $u\Omega^m$ is an ultrafilter on B by 1.1.4 and there exists $b: 1 \to B$ such that $u\Omega^m = \Omega^b$. Since u is proper, $t = u\Omega^m\{b\} = u|\chi(m^{-1}(b))|$ implies $!A|\chi(m^{-1}(b))| = t$ and hence $K \simeq 1$. This implies $u = \Omega^a$ with $a = m^{-1}(b)$ as required. Let A be principally finite and let $q: A \to B$ be epic. If u is an ultrafilter on B then $u \forall q$ is a proper filter on A because of $u \forall q\Omega^{!A} = u \forall q\Omega^q\Omega^{!B} = u\Omega^{!B} = \Omega$. Here we have used that q is epic. By assumption there exists an ultrafilter v on A with $v \geqslant u \forall q$. However, then $v\Omega^q$ is a proper filter with $v\Omega^q \geqslant u \forall q\Omega^q = u$ and hence $v\Omega^q = u$ by maximality of u. Since A is principally finite there exists a such that $v = \Omega^a$, which yields $u = \Omega^{qa}$ as required.

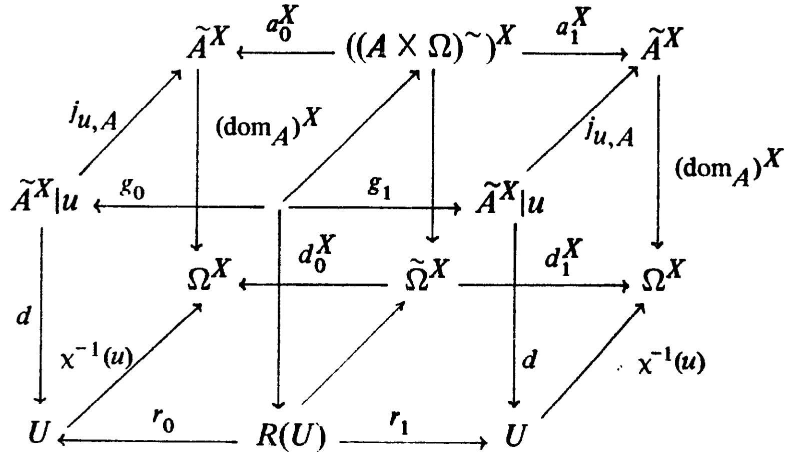
4) As we have seen in (2) an object $\alpha: A_0 \to A_1$ is principally finite iff A_0 is finite. In particular, $o: 1 \to N$ is principally finite. However, its power set object is not principally finite since its o-th component is infinite.

Moreover, it follows from the above that the natural number object in S^2 is not principally finite and hence not ultrafinite. However, the natural number object in S^N, where N is the ordered set of natural numbers, is principally finite. This follows from the description of ultrafilters on increasing objects in S^N given earlier.

Appendix

Following a suggestion of Diaconescu we will show that A^X/u is the colimit of the internal contravariant functor $A^{(-)}$ from U to E. With regard to the concepts of category objects, internal functors and their colimits the reader is referred to Kock, Wraith [12] and Diaconescu [5].

The order relation on Ω^X is given by $(d_0^X, d_1^X): \tilde{\Omega}^X \to \Omega^X \times \Omega^X$, where d_0 and d_1 are determined by the partial morphisms $(\eta_\Omega t; 1)$ and $(\eta_\Omega; !\Omega)$. In this way Ω^X becomes a category object. The contravariant functor $A^{(-)}$ from Ω^X to E is given by the triple $((\mathrm{dom}_{\tilde{A}})^X, a_0^X, a_1^X)$, where $(a_0^X, a_1^X): ((A \times \Omega)^{\sim})^X \to \tilde{A}^X \times \tilde{A}^X$ is the order relation on $\tilde{A}^X$. Here a_0 and a_1 are determined by the partial morphisms $(\eta_{A\times\Omega}(A \times t); A)$ and $(\eta_{A\times\Omega}; p_A)$.



The induced order relation $(r_0, r_1): R(U) \to U \times U$ is obtained by pulling (d_0^X, d_1^X) back along $\chi^{-1}(u) \times \chi^{-1}(u)$. The restriction of the functor $A^{(-)}$ to U is given by the triple (d, g_1, g_0) where d is obtained by pulling $(\mathrm{dom}_A)^X$ back along $\chi^{-1}(u)$ and g_1 resp. g_0 is obtained by pulling r_1 resp. a_0^X back along d resp. $j_{u,A}$. Finally, the colimit is obtained as the coequalizer of g_0 and g_1.

Using the intersection operation on $\tilde{A}^X$ one verifies that $(g_0, g_1) * (g_0, g_1)^{-1}$ factors through $k_{u,A}$, where $*$ and $^{-1}$ are relational composition and inverse formation. This proves that $k_{u,A}$ is the equivalence relation generated by (g_0, g_1). This implies $q_{u,A} = \mathrm{coeq}(g_0, g_1)$ as required.

Let $\mathrm{colim}_U: E(U^{\mathrm{op}}) \to E$ be the colimit functor with respect to U, where $E(U^{\mathrm{op}})$ is the category of contravariant E-valued functors from U. Let $D_U: E \to E(U^{\mathrm{op}})$ be the functor which associates with A the internal functor $A^{(-)}$ defined above. We have shown that $(-)^X/u$ is obtained by composing D_U with colim_U. By Theorem I.2.5. in Diaconescu [5], the functor colim_U is left exact iff U^{op} is filtered. However, U^{op} is filtered in the sense of Diaconescu since U is a filter. Therefore $(-)^X/u$ is left exact, since D_U is left exact. Here we have used that $(\tilde{-})$ preserves pullbacks.

Notations

1	terminal object
0	initial object
$!X: X \to 1$	unique morphism from X to 1
Ω	subobject classifier
$\chi(m)$	char. function of the subobject m
$\chi^{-1}(\psi)$	subobject classified by $\psi: X \to \Omega$
$t: 1 \to \Omega$	true
$\wedge: \Omega \times \Omega \to \Omega$	intersection
$\vee: \Omega \times \Omega \to \Omega$	union
$\Rightarrow: \Omega \times \Omega \to \Omega$	implication
$\Leftrightarrow: \Omega \times \Omega \to \Omega$	biimplication
$\equiv_X: X \times X \to \Omega$	equality on X
$\exists f: \Omega^X \to \Omega^Y$	existential quantification along $f: X \to Y$
$\forall f: \Omega^X \to \Omega^Y$	universal quantification along $f: X \to Y$
$\lvert\psi\rvert: Z \to \Omega^X$	exponential adjoint of $\psi: X \times Z \to \Omega$
$\mathrm{dom}_X: \tilde{X} \to \Omega$	domain of X, $\mathrm{dom}_X = (!X)^\sim$
$\eta_X: X \to \tilde{X}$	$\eta_X = \chi^{-1}(\mathrm{dom}_X)$
$\sqcap_X: \tilde{X} \times \tilde{X} \to \tilde{X}$	intersection operation on X, determined by the partial morphism $((\eta_X, \eta_X); X)$
$\mathrm{pb}(f, g)$	pullback of f and g
$\mathrm{kp}(g)$	kernel pair of g, $\mathrm{kp}(g) = \mathrm{pb}(g, g)$
$\mathrm{eq}(f, g)$	equalizer of f and g
$\mathrm{coeq}(f, g)$	coequalizer of f and g

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