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Singular Loci of Ladder Determinantal Varieties and Schubert Varieties

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We relate certain ladder determinantal varieties (associated to one-sided ladders) to certain Schubert varieties in $SL(n)/Q$, for a suitable n and a suitable parabolic subgroup Q, and we determine the singular loci of these varieties. We state a conjecture on the irreducible components of the singular locus of a Schubert variety in the flag variety, which is a refinement of the conjecture of Lakshmibai and Sandhya (*Proc. Indian Acad. Sci. Math. Sci.* **100** (1990), 45–52). We prove the conjecture for a certain class of Schubert varieties. © 2000 Academic Press

INTRODUCTION

Let k be the base field, which we assume to be algebraically closed of arbitrary characteristic. Let $X = (x_{ba})$, $1 \leq b, a \leq n$ be a matrix of variables, and let $L \subset X$ be an *one-sided ladder* with outside corners $(b_1, a_1), \dots, (b_h, a_h)$, i.e.,

$$L = \{x_{ba} \mid \text{ there exists } 1 \leq i \leq h \text{ such that } b_i \leq b \leq m, 1 \leq a \leq a_i\},$$

where $1 \leq b_1 < \cdots < b_h < n$, $1 < a_1 < \cdots < a_h \leq n$. We suppose that n is large enough so that $b_i > a_i$, for all i, $1 \leq i \leq h$. Let $k[L]$ denote the polynomial ring $k[x_{ba}, x_{ba} \in L]$, and let $\mathbb{A}(L) = \mathbb{A}^{|L|}$ be the associated affine space. For $1 \leq i \leq l$, let i^* denote the largest integer in $\{1, \dots, h\}$ such that $b_{i^*} \leq s_i$. Let $\mathbf{s} = (s_1, \dots, s_l) \in \mathbb{Z}_+^l$, $\mathbf{t} = (t_1, \dots, t_l) \in \mathbb{Z}_+^l$ be such that $b_1 = s_1 < \cdots < s_l \leq n$, $t_1 \geq \cdots \geq t_l$, $1 \leq t_i \leq \min\{n - s_i + 1, a_{i^*}\}$ for

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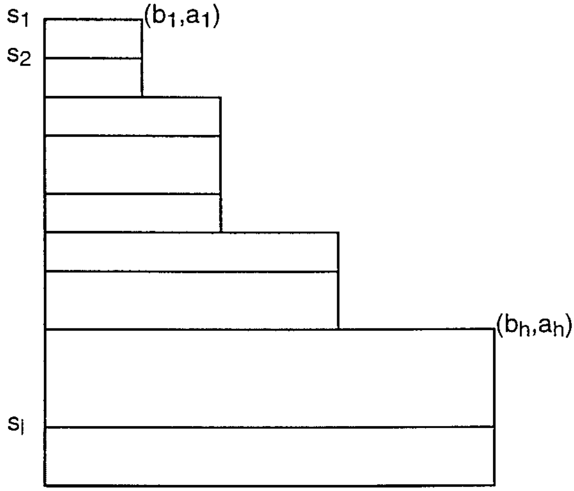

FIG. 1. The one-sided ladder L.

$1 \leq i \leq l$, and $s_i - s_{i-1} > t_{i-1} - t_i$ for $1 < i \leq l$. For each $1 \leq i \leq l$, let $L_i = \{x_{ba} \mid s_i \leq b \leq n\}$. Let $I_{\mathbf{s},\mathbf{t}}(L)$ be the ideal of $k[L]$ generated by all the t_i-minors in L_i, $1 \leq i \leq l$. Let $D_{\mathbf{s},\mathbf{t}}(L) \subset \mathbb{A}(L)$ be the variety defined by $I_{\mathbf{s},\mathbf{t}}(L)$, and we call it a *ladder determinantal variety* (the ladder being one-sided). The variety $D_{\mathbf{s},\mathbf{t}}(L)$ is isomorphic to $D_{\mathbf{s}',\mathbf{t}'}(L') \times \mathbb{A}^d$, for suitable l'-tuples $\mathbf{s}'$, $\mathbf{t}'$, a suitable one-sided ladder $L' \subset L$ in X defined by the outside corners $(b'_1, a'_1), \ldots, (b'_{h'}, a'_{h'})$ such that $\{b'_1, \ldots, b'_{h'}\} \subset \{s'_1, \ldots, s'_{l'}\}$ and $d = |L| - |L'|$ (see Section 1 for details). Thus it is enough to study the variety $D_{\mathbf{s},\mathbf{t}}(L)$ under the assumption $\{b_1, \ldots, b_h\} \subset \{s_1, \ldots, s_l\}$. Without loss of generality, we can also assume that $t_l \geq 2$, and $t_{i-1} > t_i$ if $s_i \notin \{b_1, \ldots, b_h\}$ for $1 < i \leq l$.

For each $1 \leq i \leq l$, let $L(i) = \{x_{ba} \mid s_i \leq b \leq n, 1 \leq a \leq a_{i^*}\}$. It is easy to see that the ideal $I_{\mathbf{s},\mathbf{t}}(L)$ is generated by the t_i-minors of X contained in $L(i)$, $1 \leq i \leq l$. First we relate the ladder determinantal varieties (associated to one-sided ladders) to Schubert varieties as given by the following (cf. Theorem 5.6).

THEOREM 1. *The variety $D_{\mathbf{s},\mathbf{t}}(L) \times \mathbb{A}^r$ is identified with the "opposite cell" in a certain Schubert variety $X(w)$ in $SL(n)/Q$, for a suitable parabolic subgroup Q of $SL(n)$, where $r = \dim SL(n)/Q - |L|$.*

As a consequence, we obtain (cf. Theorem 5.7)

THEOREM 2. *The variety $D_{\mathbf{s},\mathbf{t}}(L)$ is irreducible, normal, and Cohen–Macaulay and has rational singularities.*

We also determine the singular locus of $D_{\mathbf{s},\mathbf{t}}(L)$, as described below. Let V_j, $1 \leq j \leq l$, be the subvariety of $D_{\mathbf{s},\mathbf{t}}(L)$ defined by the vanishing of all $(t_j - 1)$-minors in $L(j)$. We prove (cf. Theorem 7.1)

THEOREM 3. *We have* $\operatorname{Sing} D_{\mathbf{s},\mathbf{t}}(L) = \bigcup_{j=1}^{l} V_j$.

We further prove the following (cf. Theorem 8.2).

THEOREM 4. *For* $1 \leq j \leq l$, *the subvariety* $V_j \times \mathbb{A}^r$ *of* $D_{\mathbf{s},\mathbf{t}}(L) \times \mathbb{A}^r$ (*r being as above*) *is identified with the "opposite cell" in a certain Schubert subvariety* $X(\theta_j)$ *of* $X(w)$.

As a consequence, we obtain (cf. Theorem 8.3)

THEOREM 5. *The irreducible components of* $\operatorname{Sing} D_{\mathbf{s},\mathbf{t}}(L)$ *are precisely the* V_j*'s*, $1 \leq j \leq l$.

Let $X(w^{\max})$ (resp. $X(\theta_j^{\max})$, $1 \leq j \leq l$) be the pull-back in $SL(n)/B$ of $X(w)$ (resp. $X(\theta_j)$, $1 \leq j \leq l$) under the canonical projection π: $SL(n)/B \to SL(n)/Q$ (here B is a Borel subgroup of $SL(n)$ such that $B \subset Q$). Then using Theorems 1, 3, and 4, we obtain (cf. Theorem 8.4)

THEOREM 6. *The irreducible components of* $\operatorname{Sing} X(w^{\max})$ *are precisely* $X(\theta_j^{\max})$, $1 \leq j \leq l$.

We state a conjecture on the irreducible components of the singular locus of a Schubert variety in $SL(n)/B$, which is a refinement of the conjecture in [12] (see Section 9 for the statement of the conjecture). Using Theorem 6, we prove (cf. Theorem 9.24)

THEOREM 7. *The conjecture holds for* $X(w^{\max})$.

We now briefly describe how the above theorems are proved. Let $Q = \bigcap_{i=1}^{h} P_{a_i}$, where P_{a_i} is the maximal parabolic subgroup of $SL(n)$ obtained by "omitting" the simple root α_{a_i}, the simple roots being indexed as in [2] (see Section 2 for details). Let O^- be the "opposite big cell" in G/Q (see Section 2 for details). We identify O^- ($\simeq \mathbb{A}^N$, $N = \dim G/Q$) as a subvariety of the variety of lower triangular matrices in $SL(n)$. This in turn gives rise to an embedding $\mathbb{A}(L) \subset O^-$. Let $Z_w = X(w) \cap O^-$ be the "opposite cell" in $X(w)$, and let I_w be the ideal defining Z_w in O^-. Then one knows that the Plücker coordinates vanishing on Z_w generate I_w. Let $I^*_{\mathbf{s},\mathbf{t}}(L)$ be the ideal generated by $I_{\mathbf{s},\mathbf{t}}(L)$ in $k[\mathbb{A}^N]$. We prove Theorem 1 by showing that the Plücker coordinates vanishing on Z_w belong to $I^*_{\mathbf{s},\mathbf{t}}(L)$ and, conversely, a typical t_i-minor in $L(i)$, $1 \leq i \leq l$, belongs to I_w. Theorem 2 is a consequence of Theorem 1 and the fact that Schubert varieties are irreducible, normal, and Cohen–Macaulay and have rational singularities (cf.

[10, 18–20]). Theorem 3 is proved using the Jacobian criterion for smoothness. Toward this end, we first construct a Gröbner basis for $I_{\mathbf{s},\mathbf{t}}(L)$, which then enables us to compute the codimension of $D_{\mathbf{s},\mathbf{t}}(L)$ in $\mathbb{A}(L)$. Theorem 4 is proved in the same spirit as Theorem 1. As one sees, Theorem 5 is an immediate consequence of Theorems 3 and 4, and Theorem 6 is an immediate consequence of Theorems 1, 3, and 4. Theorem 7 is proved through a relative study of $X(w^{\max})$ and $X(\theta_j^{\max})$. Thus we have used the theory of Schubert varieties to prove results on ladder determinantal varities, and vice versa. To be more precise, geometric properties such as normality, Cohen–Macaulayness, etc., for ladder determinantal varities are concluded by relating these varieties to Schubert varieties. The components of singular loci of Schubert varieties are determined by first determining them for ladder determinantal varieties and then using the above-mentioned relationship between ladder determinantal varieties and Schubert varieties.

An identification similar to that in Theorem 1 for the case $t_1 = \cdots = t_l$ has also been obtained by Mulay (see [16]). Results similar to those of Theorem 2 for certain other ladder determinantal varieties have been obtained by several authors (see [4, 5, 7, 15, 17]). To the best of our knowledge, Theorem 5 is the only result in the literature on the determination of the singular locus of a ladder determinantal variety, except for the case of the classical determinantal variety, i.e., $h = 1$ and $l = 1$ (see [13, 14, 21]).

The sections are organized as follows. In Section 1 we define ladder determinantal varieties and set up a few notations. In Section 2, we recall some generalities on G/Q. In Section 3, we recall some generalities on Schubert varieties in the flag variety. In Section 4, we prove two lemmas related to the evaluation of Plücker coordinates on the "opposite big cell." In Section 5, we bring out the relationship between ladder determinantal varieties and Schubert varieties. In Section 6, we compute the dimension of ladder determinantal varities by constructing Gröbner bases for their defining ideals. In Section 7, we determine the singular loci of ladder detrminantal varieties. In Section 8, we determine the irreducible components of the singular loci of ladder determinantal varieties. In Section 9, we state a conjecture on the irreducible components of the singular locus of a Schubert variety in $SL(n)/B$ and prove it for a certain class of Schubert varieties, namely those Schubert varieties that are related to ladder determinantal varieties as in Section 5. This conjecture is a refinement of the conjecture in [12].

1. LADDER DETERMINANTAL VARIETIES

Let $X = (x_{ba})$, $1 \leq b \leq m$, $1 \leq a \leq n$ be a $m \times n$ matrix of indeterminates.

Given $1 \le b_1 < \cdots < b_h < m$, $1 < a_1 < \cdots < a_h \le n$, we consider the subset of X, defined by

$$L = \{x_{ba} \mid \text{ there exists } 1 \le i \le h \text{ such that } b_i \le b \le m, 1 \le a \le a_i\}.$$

We call L a *one-sided ladder* in X, defined by the *outside corners* $\omega_i = x_{b_i a_i}$, $1 \le i \le h$. For simplicity of notation, we identify the variable x_{ba} with just (b, a).

For $1 \le i \le l$, let i^* be the largest integer such that $b_{i^*} \le s_i$.

Let $\mathbf{s} = (s_1, s_2 \dots, s_l) \in \mathbb{Z}_+^l$, $\mathbf{t} = (t_1, t_2 \dots, t_l) \in \mathbb{Z}_+^l$ such that

$$\begin{aligned} b_1 = s_1 < s_2 < \cdots < s_l \le m, \\ t_1 \ge t_2 \ge \cdots \ge t_l, \\ 1 \le t_i \le \min\{m - s_i + 1, a_{i^*}\} \qquad \text{for } 1 \le i \le l, \qquad \text{and} \\ s_i - s_{i-1} > t_{i-1} - t_i \qquad \text{for } 1 < i \le l. \end{aligned} \tag{L1}$$

For $1 \le i \le l$, let

$$L_i = \{x_{ba} \in L \mid s_i \le b \le m\}.$$

Let $k[L]$ denote the polynomial ring $k[x_{ba} \mid x_{ba} \in L]$, and let $\mathbb{A}(L) = \mathbb{A}^{|L|}$ be the associated affine space. Let $I_{\mathbf{s},\mathbf{t}}(L)$ be the ideal in $k[L]$ generated by all the t_i-minors contained in L_i, $1 \le i \le l$, and let $D_{\mathbf{s},\mathbf{t}}(L) \subset \mathbb{A}(L)$ be the variety defined by the ideal $I_{\mathbf{s},\mathbf{t}}(L)$. We call $D_{\mathbf{s},\mathbf{t}}(L)$ a *ladder determinantal variety* (associated to an one-sided ladder).

Let $\Omega = \{\omega_1, \dots, \omega_h\}$. For each $1 < j \le l$, let

$$\Omega_j = \left\{\omega_i \mid 1 \le i \le h \text{ such that } s_{j-1} < b_i < s_j \text{ and } s_j - b_i \le t_{j-1} - t_j\right\}.$$

Let

$$\Omega' = \left(\Omega \setminus \bigcup_{j=2}^{l} \Omega_j\right) \bigcup_{\Omega_j \ne \varnothing} \{(s_j, a_{j^*})\}.$$

Let L' be the one-sided ladder in X defined by the set of outside corners Ω'. Then it is easily seen that $D_{\mathbf{s},\mathbf{t}}(L) \simeq D_{\mathbf{s},\mathbf{t}}(L') \times \mathbb{A}^d$, where $d = |L| - |L'|$.

Let $\omega'_k = (b'_k, a'_k) \in \Omega'$, for some k, $1 \le k \le h'$, where $h' = |\Omega'|$. If $b'_k \notin \{s_1, \dots, s_l\}$, then $b'_k = b_i$ for some i, $1 \le i \le h$, and we define $s_{j^-} = b_i$, $t_{j^-} = t_{j-1}$, $s_{j^+} = s_j$, $t_{j^+} = t_j$, where j is the unique integer such that $s_j < b_i < s_{j+1}$. Let $\mathbf{s}'$ (resp. $\mathbf{t}'$) be the sequence obtained from $\mathbf{s}$ (resp. $\mathbf{t}$) by replacing s_j (resp. t_j) with s_{j^-} and s_{j^+} (resp. t_{j^-} and t_{j^+}) for all k such that $b'_k \notin \{s_1, \dots, s_l\}$, j being the unique integer such that $s_{j-1} < b_i < s_j$, and i being given by $b'_k = b_i$. Let $l' = |\mathbf{s}'|$. Then $\mathbf{s}'$ and $\mathbf{t}'$ satisfy (L1), and in addition we have $\{b'_1, \ldots, b'_{h'}\} \subset \{s'_1, \ldots, s'_{l'}\}$. It is easily seen that $D_{\mathbf{s},\mathbf{t}}(L') = D_{\mathbf{s}',\mathbf{t}'}(L')$, and hence

$$D_{\mathbf{s},\mathbf{t}}(L) \simeq D_{\mathbf{s}',\mathbf{t}'}(L') \times \mathbb{A}^d.$$

Therefore it is enough to study $D_{\mathbf{s},\mathbf{t}}(L)$ with $\mathbf{s}, \mathbf{t} \in \mathbb{Z}_+^l$ such that

$$\{s_1, \ldots, s_l\} \supset \{b_1, \ldots, b_h\}. \tag{L2}$$

Without loss of generality, we can also assume that

$$t_l \geq 2 \quad \text{and} \quad t_{i-1} > t_i \qquad \text{if } s_i \notin \{b_1, \ldots, b_h\}, \quad 1 < i \leq l. \tag{L3}$$

For $1 \leq i \leq l$, let

$$L(i) = \{x_{ba} \mid s_i \leq b \leq m, 1 \leq a \leq a_{i^*}\}.$$

Note that the ideal $I_{\mathbf{s},\mathbf{t}}(L)$ is generated by the t_i-minors of X contained in $L(i)$, $1 \leq i \leq l$.

2. GENERALITIES ON G/Q

Let G be a semisimple and simply connected algebraic group defined over an algebraically closed field of arbitrary characteristic. Let $T \subset G$ be a maximal torus, and let $B \supset T$ be a Borel subgroup. Let R be the root system of G relative to T. Let R^+ (resp. S) be the system of positive (resp. simple) roots of R with respect to B. Let R^- be the corresponding system of negative roots.

2.1. *The Chevalley–Bruhat Order*

Let $w \in W$. A minimal expression for w as a product of simple reflections is called a reduced expression for w. We denote by $l(w)$ the length of a reduced expression for w (as a product of simple refelections). We have a partial order on W, the well-known Chevalley–Bruhat order, namely $w_1 \geq w_2$, if a reduced expression for w_1 contains a subexpression that is a reduced expression for w_2.

2.2. *The Weyl Subgroup W_Q*

Let Q be a parabolic subgroup of G containing B. Associated to Q, there is a subset S_Q of S such that Q is the subgroup of G generated by B and $\{U_{-\alpha} \mid \alpha \in R_Q^+\}$, where $R_Q^+ = \{\alpha \in R^+ \mid \alpha = \sum_{\beta \in S_Q} a_\beta \beta\}$ (here, for $\beta \in R$, U_β denotes the one-dimensional unipotent subgroup of G associated to β). Let W_Q be the Weyl group of Q (note that W_Q is simply the subgroup of W generated by $\{s_\alpha \mid \alpha \in S_Q\}$; here, for $\alpha \in S$, s_α denotes the simple reflection (considered as an element of W), associated to α).

2.3. *The Set $W_Q^{\min}$ of Minimal Representatives of W/W_Q*

In each coset wW_Q, there exists a unique element of minimal length (cf. [2]). Let $W_Q^{\min}$ be this set of representatives of W/W_Q. The set $W_Q^{\min}$ is called the *set of minimal representatives of* W/W_Q. We have

$$W_Q^{\min} = \{w \in W \mid l(ww') = l(w) + l(w'), \text{ for all } w' \in Q\}.$$

The set $W_Q^{\min}$ may also be characterized as

$$W_Q^{\min} = \{w \in W \mid w(\alpha) > 0, \text{ for all } \alpha \in S_Q\}$$

(here by a root being > 0 we mean $\beta \in R^+$).

In the sequel, given $w \in W$, the minimal representative of wW_Q in W will be denoted by $w_Q^{\min}$.

2.4. *The Set $W_Q^{\max}$ of Maximal Representatives of W/W_Q*

In each coset wW_Q there exists a unique element of maximal length. Let $W_Q^{\max}$ be the set of these representatives of W/W_Q. We have

$$W_Q^{\max} = \{w \in W \mid w(\alpha) < 0 \text{ for all } \alpha \in S_Q\}.$$

Furthermore, if we denote by w_Q the element of maximal length in W_Q, then we have

$$W_Q^{\max} = \{ww_Q \mid w \in W_Q^{\min}\}.$$

In the sequel, given $w \in W$, the maximal representative of wW_Q in W will be denoted by $w_Q^{\max}$.

2.5. *Maximal Parabolic Subgroups*

The set of maximal parabolic subgroups is in one-to-one correpondence with S, namely given $\alpha \in S$, the parabolic subgroup Q, where $S_Q = S \setminus \{\alpha\}$ is a maximal parabolic subgroup, and conversely. We shall denote Q, where $S_Q = S \setminus \{\alpha\}$ by $P_{\widehat{\alpha}}$, and refer to it as the *maximal parabolic subgroup obtained by omitting* α.

2.6. *Schubert Varieties in G/Q*

For $w \in W$, let us denote the point in G/Q corresponding to the coset wQ by $e_{w,Q}$. Then the set of T-fixed points in G/Q for the action given by left multiplication is presisely $\{e_{w,Q} \mid w \in W\}$. Let $w \in W$, and let $X_Q(w)$ be the Zariski closure of $Be_{w,Q}$ in G/Q. Then $X_Q(w)$ with the canonical reduced structure is called the Schubert variety in G/Q associated to wW_Q. In particular, we have bijections between $W_Q^{\min}$ and the set of Schubert varieties in G/Q, and between $W_Q^{\max}$ and the set of Schubert varieties in G/Q. We have the well-known Bruhat decomposition

$$G/Q = \dot{\bigcup} Be_{w,Q}, \quad X_Q(\theta) = \dot{\bigcup_{w \le \theta}} Be_{w,Q}, \qquad \theta \in W.$$

As above, let $w_Q^{\min}$ (resp. $w_Q^{\max}$) denote the minimal (resp. maximal) representative of wW_Q. Let π: $G/B \to G/Q$ be the canonical projection. Then it can be easily seen that

$$\pi|_{X_B(w^{\max})}: X_B(w_Q^{\max}) \to X_Q(w)$$

is a fibration with fiber $\simeq Q/B$, while

$$\pi|_{X_B(w^{\min})}: X_B(w_Q^{\min}) \to X_Q(w)$$

is birational. In particular, we have $\dim X_Q(w) = \dim X_B(w_Q^{\min})$.

2.7. *The Big Cell and the Opposite Big Cell*

The B-orbit Be_{w_0} in G/Q (w_0 being the unique element of maximal length in W) is called the *big cell* in G/Q. It is a dense open subset of G/Q, and it is identified with $R_u(Q)$, the unipotent radical of Q, namely the subgroup of B generated by $\{U_\alpha \mid \alpha \in R^+ \setminus R_Q^+\}$ (cf. [1]). Let B^- be the Borel subgroup of G opposite B, i.e., the subgroup of G generated by T and $\{U_\alpha \mid \alpha \in R^-\}$. The B^--orbit $B^- e_{\mathrm{id},Q}$ is called the *opposite big cell* in G/Q. This is again a dense open subset of G/Q, and it is identified with the unipotent subgroup of B^- generated by $\{U_\alpha \mid \alpha \in R^- \setminus R_Q^-\}$. Observe that both the big cell and the opposite big cell can be identified with $\mathbb{A}^{N_Q}$, where $N_Q = \#\{R^+ \setminus R_Q^+\}$.

For a Schubert variety $X(w) \subset G/Q$, $B^- e_{\mathrm{id}} \cap X(w)$ is called the *opposite cell* in $X(w)$ (by abuse of language). In general, it is not a cell (except for $w = w_0$). It is a nonempty affine open subvariety of $X(w)$ and a closed subvariety of the affine space $B^- e_{\mathrm{id}}$.

2.8. *Equations Defining a Schubert Variety*

Let L be an ample line bundle on G/Q. Consider the projective embedding $G/Q \hookrightarrow \mathrm{Proj}(H^0(G/Q, L))$. We recall (cf. [20]) that the homogeneous ideal of G/Q for this embedding is generated in degree 2, and any Schubert variety X in G/Q is scheme theoretically (even at the cone level) the intersection of G/Q with all the hyperplanes in $\mathrm{Proj}(H^0(G/Q, L))$ containing X.

For a maximal parabolic subgroup P_i, let us denote the ample generator of $\mathrm{Pic}\,(G/P_i)$ ($\simeq \mathbb{Z}$) by L_i.

Given a parabolic subgroup Q, let us denote $S \setminus S_Q$ by $\{\alpha_1, \dots, \alpha_t\}$, for some t. Let

$$R = \bigoplus_{\underline{a}} H^0\Big(G/Q, \bigotimes_i L_i^{a_i}\Big)$$

$$R_w = \bigoplus_{\underline{a}} H^0\Big(X_Q(w), \bigotimes_i L_i^{a_i}\Big),$$

where $\underline{a} = (a_1, \dots, a_t) \in \mathbb{Z}_+^t$. We recall (cf. [10]) that the natural map

$$\bigoplus S^{a_1}(H^0(G/Q, L_1)) \otimes \cdots \otimes S^{a_1}(H^0(G/Q, L_t)) \to R$$

is surjective, and its kernel is generated as an ideal by elements of total degree 2. Furthermore, the restriction map $R \to R_w$ is surjective, and its kernel is generated as an ideal by elements of total degree 1.

3. OPPOSITE CELLS IN SCHUBERT VARIETIES IN $SL(n)/B$

Let $G = SL(n)$, the special linear group of rank $n-1$. Let T be the maximal torus consisting of all the diagonal matrices in G, and let B be the Borel subgroup consisting of all the upper triangular matrices in G. It is well known that W can be identified with $\mathscr{S}_n$, the symmetric group on n letters.

Following [2], we denote the simple roots by $\epsilon_i - \epsilon_{i+1}$, $1 \le i \le n-1$ (note that $\epsilon_i - \epsilon_{i+1}$ is the character sending $\operatorname{diag}(t_1, \dots, t_n)$ to $t_i t_{i+1}^{-1}$). Then $R = \{\epsilon_i - \epsilon_j \mid 1 \le i, j \le n\}$, and the reflection $s_{\epsilon_i - \epsilon_{i+1}}$ may be identified with the transposition (i, j) in $\mathscr{S}_n$.

For $\alpha = \alpha_i$ $(= \epsilon_i - \epsilon_{i+1})$, we also denote $P_{\widehat{\alpha}}$ (resp. $W_{P_{\widehat{\alpha}}}^{\min}$) by just P_i (resp. W^i).

3.1. *The Partially Ordered Set $I_{d,n}$*

Let $Q = P_d$. Then

$$Q = \left\{ A \in G \,\middle|\, A = \begin{pmatrix} * & * \\ 0_{(n-d)\times d} & * \end{pmatrix} \right\},$$

$$W_Q = \mathscr{S}_d \times \mathscr{S}_{n-d}.$$

Hence

$$W_Q^{\min} = \big\{(a_1 \dots a_n) \in W \mid a_1 < \cdots < a_d,\ \ a_{d+1} < \cdots < a_n\big\}.$$

Thus $W_Q^{\min}$ may be identified with

$$I_{d,n} := \big\{\underline{i} = (i_1, \dots, i_d) \mid 1 \le i_1 < \cdots < i_d \le n\big\}.$$

Given $\underline{i}, \underline{j} \in I_{d,n}$, let $X_{\underline{i}}, X_{\underline{j}}$ be the associated Schubert varieties in G/P_d. We define $\underline{i} \geq \underline{j} \Leftrightarrow X_{\underline{i}} \supseteq X_{\underline{j}}$ (in other words, the partial order $\geq$ on $I_{d,n}$ is induced by the Chevalley–Bruhat order on the set of Schubert varieties, via the bijection in Section 2.6). In particular, we have

$$\underline{i} \geq \underline{j} \Leftrightarrow i_t \geq j_t \qquad \text{for all } 1 \leq t \leq d.$$

3.2. *The Chevalley–Bruhat Order on* $\mathcal{S}_n$

For $w_1, w_2 \in W$, we have

$$X(w_1) \subset X(w_2) \Leftrightarrow \pi_d(X(w_1)) \subset \pi_d(X(w_2)), \qquad \text{for all } 1 \leq d \leq n-1,$$

where π_d is the canonical projection $G/B \to G/P_d$. Hence we obtain that for $(a_1 \dots a_n), (b_1 \dots b_n) \in \mathcal{S}_n$,

$$(a_1 \dots a_n) \geq (b_1 \dots b_n) \Leftrightarrow (a_1 \dots a_d)\uparrow \geq (b_1 \dots b_d)\uparrow, \qquad \text{for all } 1 \leq d \leq n-1$$

(here, for a d-tuple $(t_1 \dots t_d)$ of distinct integers, $(t_1 \dots t_d)\uparrow$ denotes the ordered d-tuple obtained from $\{t_1, \dots, t_d\}$ by arranging its elements in ascending order).

3.3. *The Partially Ordered Set* $I_{a_1, \dots, a_k}$

Let Q be a parabolic subgroup in $SL(n)$. Let $1 \leq a_1 < \cdots < a_k \leq n$, such that $S_Q = S \setminus \{\alpha_{a_1}, \dots, \alpha_{a_k}\}$ (we follow [2] for indexing the simple roots). Then $Q = P_{a_1} \cap \cdots \cap P_{a_k}$, and $W_Q = \mathcal{S}_{a_1} \times \mathcal{S}_{a_2 - a_1} \times \cdots \times \mathcal{S}_{n-a_k}$. Let

$$I_{a_1, \dots, a_k} = \big\{(\underline{i}_1, \dots, \underline{i}_k) \in I_{a_1,n} \times \cdots \times I_{a_k,n} \mid \underline{i}_t \subset \underline{i}_{t+1} \text{ for all } 1 \leq t \leq k-1\big\}.$$

Then it is easily seen that $W_Q^{\min}$ may be identified with $I_{a_1, \dots, a_k}$.

The partial order on the set of Schubert varieties in G/Q (given by inclusion) induces a partial order $\geq$ on $I_{a_1, \dots, a_k}$, namely, for $\mathbf{i} = (\underline{i}_1, \dots, \underline{i}_k)$, $\mathbf{j} = (\underline{j}_1, \dots, \underline{j}_k) \in I_{a_1, \dots, a_k}$, $\mathbf{i} \geq \mathbf{j} \Leftrightarrow \underline{i}_t \geq \underline{j}_t$ for all $1 \leq t \leq k$.

3.4. *The Minimal and Maximal Representatives as Permutations*

Let $w \in W_Q$, and let $\mathbf{i} = (\underline{i}_1, \dots, \underline{i}_k)$ be the element in $I_{a_1, \dots, a_k}$ that corresponds to $w_Q^{\min}$. As a permutation, the element $w_Q^{\min}$ is given by $\underline{i}_1$, followed by $\underline{i}_2 \setminus \underline{i}_1$ arranged in ascending order, and so on, ending with $\{1, \dots, n\} \setminus \underline{i}_k$ arranged in ascending order. Similarly, as a permutation, the element $w_Q^{\max}$ is given by $\underline{i}_1$ arranged in descending order, followed by $\underline{i}_2 \setminus \underline{i}_1$ arranged in descending order, etc.

3.5. *The Opposite Big Cell in G/Q*

Let $Q = \bigcap_{t=1}^{k} P_{a_t}$. Let $a = n - a_k$, and let Q be the parabolic subgroup consisting of all the elements of G of the form

$$\begin{pmatrix} A_1 & * & * & \cdots & * & * \\ 0 & A_2 & * & \cdots & * & * \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & A_k & * \\ 0 & 0 & 0 & \cdots & 0 & A \end{pmatrix},$$

where A_t is a matrix of size $c_t \times c_t$, $c_t = a_t - a_{t-1}$, $1 \leq t \leq k$ (here $a_0 = 0$), A is a matrix of size $a \times a$, and $x_{ml} = 0$, $m > a_t$, $l \leq a_t$, $1 \leq t \leq k$. Denote by O^- the subgroup of G generated by $\{U_\alpha \mid \alpha \in R^- \setminus R_Q^-\}$. Then O^- consists of the elements of G of the form

$$\begin{pmatrix} I_1 & 0 & 0 & \cdots & 0 & 0 \\ * & I_2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ * & * & * & \cdots & I_k & 0 \\ * & * & * & \cdots & * & I_a \end{pmatrix},$$

where I_t is the $c_t \times c_t$ identity matrix, $1 \leq t \leq k$, I_a is the $a \times a$ identity matrix, and if $x_{ml} \neq 0$, with $m \neq l$, then $m > a_t$, $l \leq a_t$ for some t, $1 \leq t \leq k$. Furthermore, the restriction of the canonical morphism $f\colon G \to G/Q$ to O^- is an open immersion, and $f(O^-) \simeq B^- e_{\mathrm{id},Q}$. Thus $B^- e_{\mathrm{id},Q}$ is identified with O^-.

3.6. *Plücker Coordinates on the Grassmannian*

Let $G_{d,n}$ be the Grassmannian variety, consisting of d-dimensional subspaces of an n-dimensional vector space V. Let us identify V with k^n, and denote the standard basis of k^n by $\{e_i \mid 1 \leq i \leq n\}$. Consider the Plücker embedding $f_d\colon G_{d,n} \hookrightarrow \mathbb{P}(\wedge^d V)$, where $\wedge^d V$ is the dth exterior power of V. For $\underline{i} = (i_1, \ldots, i_d) \in I_{d,n}$, let $e_{\underline{i}} = e_{i_1} \wedge \cdots \wedge e_{i_d}$. Then the set $\{e_{\underline{i}} \mid \underline{i} \in I_{d,n}\}$ is a basis for $\wedge^d V$. Let us denote the basis of $(\wedge^d V)^*$ (the linear dual of $\wedge^d V$) dual to $\{e_{\underline{i}} \mid \underline{i} \in I_{d,n}\}$ by $\{p_{\underline{j}} \mid \underline{j} \in I_{d,n}\}$. Then $\{p_{\underline{j}} \mid \underline{j} \in I_{d,n}\}$ gives a system of coordinates for $\mathbb{P}(\wedge^d V)$. These are the so-called *Plücker coordinates*.

3.7. *Schubert Varieties in the Grassmannian*

Let $Q = P_d$. We have

$$G_{d,n} \simeq G/P_d.$$

Let $\underline{i} = (i_1, \ldots, i_d) \in I_{d,n}$. Then the T-fixed point $e_{\underline{i}, P_d}$ is simply the d-dimensional span of $\{e_{i_1}, \ldots, e_{i_d}\}$. Thus $X_{P_d}(\underline{i})$ is simply the Zariski closure of $B[e_{i_1} \wedge \cdots \wedge e_{i_d}]$ in $\mathbb{P}(\wedge^d V)$.

In view of the Bruhat decomposition for $X_{P_d}(\underline{i})$ (cf. Section 2.6), we have

$$p_{\underline{j}}\big|_{X_{P_d}(\underline{i})} \neq 0 \Leftrightarrow \underline{i} \geq \underline{j}.$$

3.8. *Evaluation of Plücker Coordinates on the Opposite Big Cell in G/P_d*

Consider the morphism $\phi_d\colon G \to \mathbb{P}(\wedge^d V)$, where $\phi_d = f_d \circ \theta_d$, θ_d being the natural projection $G \to G/P_d$. Then $p_{\underline{j}}(\phi_d(g))$ is simply the minor of g consisting of the first d columns and the rows with indices $j_1, \ldots, j_d$. Now, denote by Z_d the unipotent subgroup of G generated by $\{U_\alpha \mid \alpha \in R^- \setminus R^-_{P_d}\}$. We have, as in Section 3.5,

$$Z_d = \left\{ \begin{pmatrix} I_d & 0_{d\times(n-d)} \\ A_{(n-d)\times d} & I_{n-d} \end{pmatrix} \in G \right\}.$$

As in Section 3.5, we identify Z_d with the opposite big cell in G/P_d. Then, given $z \in Z_d$, the Plücker coordinate $p_{\underline{j}}$ evaluated at z is simply a certain minor of A, which may be explicitly described as follows. Let $\underline{j} = (j_1, \ldots, j_d)$, and let j_r be the largest entry $\leq d$. Let $\{k_1, \ldots, k_{d-r}\}$ be the complement of $\{j_1, \ldots, j_r\}$ in $\{1, \ldots, d\}$. Then this minor of A is given by column indices $k_1, \ldots k_{d-r}$ and row indices $j_{r+1}, \ldots, j_d$ (here the rows of A are indexed as $d+1, \ldots, n$). Conversely, given a minor of A, say, with column indices $b_1, \ldots, b_s$ and row indices $i_{d-s+1}, \ldots, i_d$, it is the evaluation of the Plücker coordinate $p_{\underline{i}}$ at z, where $\underline{i} = (i_1, \ldots, i_d)$ may be described as follows: $\{i_1, \ldots, i_{d-s}\}$ is the complement of $\{b_1, \ldots, b_s\}$ in $\{1, \ldots, d\}$, and $i_{d-s+1}, \ldots, i_d$ are simply the row indices (again, the rows of A are indexed as $d+1, \ldots, n$).

3.9. *Evaluation of the Plücker Coordinates on the Opposite Big Cell in G/Q*

Consider

$$f\colon G \to G/Q \hookrightarrow G/P_{a_1} \times \cdots \times G/P_{a_k} \hookrightarrow \mathbf{P}_1 \times \cdots \times \mathbf{P}_k,$$

where $\mathbf{P}_t = \mathbb{P}(\wedge^{a_t} V)$. Denoting the restriction of f to O^- also by just f, we obtain an embedding $f\colon O^- \hookrightarrow \mathbf{P}_1 \times \cdots \times \mathbf{P}_k$, O^- having been identified with the opposite big cell in G/Q. For $z \in O^-$, the multi-Plücker coordinates of $f(z)$ are simply all the $a_t \times a_t$ minors of z with column indices $\{1, \ldots, a_t\}$, $1 \leq t \leq k$.

3.10. *Equations Defining the Cones over Schubert Varieties in* $G_{d,n}$

Let $Q = P_d$. Given a d-tuple $\underline{i} = (i_1, \ldots, i_d) \in I_{d,n}$, let us denote the associated element of $W_{P_d}^{\min}$ by $\theta_{\underline{i}}$. For simplicity of notation, let us denote P_d by just P, and $\theta_{\underline{i}}$ by just θ. Then, by Section 3.7, $X_P(\theta)$ is simply the Zariski closure of $B[e_{i_1} \wedge \cdots \wedge e_{i_d}]$ in $\mathbb{P}(\wedge^d V)$. Now using Section 2.8, we obtain that the restriction map $R \to R_\theta$ is surjective, and the kernel is generated as an ideal by $\{p_{\underline{j}} \mid \underline{i} \not\geq \underline{j}\}$.

3.11. *Equations Defining Multicones over Schubert Varieties in* G/Q

Let Q be as in Section 3.5. Let $X_Q(w) \subset G/Q$. Denoting R, R_w as in Section 2.8, the kernel of the restriction map $R \to R_w$ is generated by the kernel of $R_1 \to (R(w))_1$; but now, in view of Section 3.7, this kernel is the span of

$$\{p_{\underline{i}} \mid \underline{i} \in I_{d,n}, d \in \{a_1, \ldots, a_k\},\ w^{(d)} \not\geq \underline{i}\},$$

where $w^{(d)}$ is the d-tuple corresponding to the Schubert variety that is the image of $X_Q(w)$ under the projection $G/Q \to G/P_{a_t}$, $1 \leq t \leq k$.

3.12. *Ideal of the Opposite Cell in* $X_Q(w)$

Let us denote $B^- e_{\mathrm{id},Q} \cap X_Q(w)$ by just A_w. Then as in Section 2.7, we identify $B^- e_{\mathrm{id},Q}$ with the unipotent subgroup O^- generated by $\{U_\alpha \mid \alpha \in R^- \setminus R_Q^-\}$ and consider A_w as a closed subvariety of O^-. In view of Section 3.11, we obtain that the ideal defining A_w in O^- is generated by

$$\{p_{\underline{i}} \mid \underline{i} \in I_{d,n}, d \in \{a_1, \ldots, a_k\},\ w^{(d)} \not\geq \underline{i}\}.$$

4. TWO LEMMAS RELATED TO THE EVALUATION OF PLÜCKER COORDINATES ON THE OPPOSITE CELL OF A SCHUBERT VARIETY IN G/Q

Let $G = SL(n)$, $1 \leq a_1 < \cdots < a_h \leq n$, $Q = P_{a_1} \cap \cdots \cap P_{a_h}$. Let O^- be the opposite big cell in G/Q. Let $X = (x_{ba})$, $1 \leq b$, $a \leq n$, be a generic $n \times n$ matrix and let H be the one-sided ladder in X defined by the outside corners $(a_i + 1, a_i)$, $1 \leq i \leq h$. Clearly, $\mathbb{A}(H) \simeq O^-$. Let $X^- = (x^-_{ba})$, $1 \leq b$, $a \leq n$, where

$$x^-_{ba} = \begin{cases} x_{ba}, & \text{if } (b, a) \in H \\ 1, & \text{if } b = a \\ 0, & \text{otherwise.} \end{cases}$$

Note that, given $\tau \in W^{a_i}$, for some i, $1 \leq i \leq h$, the function $p_\tau|_{O^-}$ represents the determinant of the $a_i \times a_i$ submatrix T of X^- whose row indices are $\{\tau(1), \ldots, \tau(a_i)\}$ and column indices are $\{1, \ldots, a_i\}$.

Let $H_i = \{x_{ba} \mid a_i + 1 \leq b \leq n, 1 \leq a \leq a_i\}$, $1 \leq i \leq h$.

LEMMA 4.1. *Let M be a $t \times t$ matrix contained in H_i, for some i, $1 \leq i \leq h$, with row indices $r_1 < \cdots < r_t$. Then $\det M$ belongs to the ideal of $k[H]$ generated by $p_\phi|_{O^-}$, with $\phi \in W^{a_i}$ such that $\{\phi(1), \ldots, \phi(a_i)\} \cap \{a_i + 1, \ldots, n\} = \{r_1, \ldots, r_t\}$.*

Proof. Denote by $c_1 < \cdots < c_t$ the column indices of M. Let $\tau = (\{1, \ldots, a_i\} \setminus \{c_1, \ldots, c_t\}) \cup \{r_1, \ldots, r_t\}$. Then $\tau \in W^{a_i}$, and $p_\tau|_{O^-} = \det T$, where T is the $a_i \times a_i$ submatrix of X^- with row indices $\{\tau(1), \ldots, \tau(a_i)\}$ and column indices $\{1, \ldots, a_i\}$. Using Laplace expansion with respect to the last t rows of T, we obtain

$$\det T = \sum \pm \det N_{c'_1, \ldots, c'_t} \det M_{c'_1, \ldots, c'_t}, \tag{$*$}$$

the sum being taken over all subsets with t elements $\{c'_1, \ldots, c'_t\}$ of $\{1, \ldots, a_i\}$, where $N_{c'_1, \ldots, c'_t}$ is the $(a_i - t) \times (a_i - t)$ submatrix of X^- with row indices $\{1, \ldots, a_i\} \setminus \{c_1, \ldots, c_t\}$ and column indices $\{1, \ldots, a_i\} \setminus \{c'_1, \ldots, c'_t\}$, and $M_{c'_1, \ldots, c'_t}$ is the $t \times t$ submatrix of X^- with row indices $\{r_1, \ldots, r_t\}$ and column indices $\{c'_1, \ldots, c'_t\}$. Note that $M_{c_1, \ldots, c_t} = M$, and $N_{c_1, \ldots, c_t}$ is a lower triangular matrix, with all diagonal entries equal to 1, and hence $\det M$ appears in $(*)$, and its coefficient is ± 1. Also note that $N_{c'_1, \ldots, c'_t}$ is obtained from $N_{c_1, \ldots, c_t}$ by replacing the columns with indices $c'_1, \ldots, c'_t$ by the columns with indices $c_1, \ldots, c_t$.

Let $\geq$ denote the partial order on I_{t, a_i} as in Section 3.1, namely $(d_1, \ldots, d_t) \geq (c_1, \ldots, c_t)$ if $d_j \geq c_j$ for all $1 \leq j \leq t$. We prove the lemma by decreasing induction with respect to the order $\geq$ on the t-tuple $(c_1, \ldots, c_t)$ consisting of the column indices of M.

If $c_j > a_{i-1}$ for all $1 \leq j \leq t$, then for $\{c'_1, \ldots, c'_t\} \neq \{c_1, \ldots, c_t\}$ we have $\det N_{c'_1, \ldots, c'_t} = 0$, since at least one of $c_1, \ldots, c_t$ is an index for a column in $N_{c'_1, \ldots, c'_t}$, and all entries of this column are 0. Thus, in this case $(*)$ reduces to $\det T = \pm \det M$, i.e., $\det M = \pm p_\tau|_{O^-}$, with $\tau \in W^{a_i}$ such that $\{\tau(1), \ldots, \tau(a_i)\} \cap \{a_i + 1, \ldots, n\} = \{r_1, \ldots, r_t\}$.

Assume now that the assertion is true for all matrices with row indices $r_1 < \cdots < r_t$ and column indices $d_1 < \cdots < d_t$ such that $(d_1, \ldots, d_t) > (c_1, \ldots, c_t)$ (i.e., such that $d_j \geq c_j$ for all $1 \leq j \leq t$ and $(d_1, \ldots, d_t) \neq (c_1, \ldots, c_t)$). We shall now prove it for the matrix M with row indices $r_1 < \cdots < r_t$ and column indices $c_1 < \cdots < c_t$. Consider a typical $N_{c'_1, \ldots, c'_t}$ in $(*)$. If there exists a j such that $c'_j < c_j$, then the column with index c_j is replacing the column with index c'_j while obtaining $N_{c'_1, \ldots, c'_t}$ from $N_{c_1, \ldots, c_t}$; hence $N_{c'_1, \ldots, c'_t}$ is still lower triangular, but the diagonal entry in the column with index c_j is 0, which implies that $\det N_{c'_1, \ldots, c'_t} = 0$. Consequently we obtain

$$\det T = \pm \det M + \sum \pm \det N_{c'_1, \ldots, c'_t} \det M_{c'_1, \ldots, c'_t},$$

and hence

$$\det M = \pm p_\tau|_{O^-} + \sum \pm \det N_{c'_1, \ldots, c'_t} \det M_{c'_1, \ldots, c'_t},$$

the sum being taken over all $(c_1, \ldots c'_t) \in I_{t, a_i}$ such that $(c'_1, \ldots, c'_t) > (c_1, \ldots, c_t)$. The required result now follows by induction hypothesis. ∎

LEMMA 4.2. *Let $1 \leq t \leq a \leq a_i$, $1 \leq s \leq n$, and $\tau \in W^{a_i}$ such that $\tau(a - t + 1) \geq s$. Then $p_\tau|_{O^-}$ belongs to the ideal of $k[H]$ generated by t-minors in X^- with row indices $\geq s$ and column indices $\leq a$.*

Proof. Let T be the $a_i \times a_i$ submatrix of X^- with row indices $\{\tau(1), \ldots, \tau(a_i)\}$ and column indices $\{1, \ldots, a_i\}$. Then $p_\tau|_{O^-} = \det T$. Using Laplace expansion with respect to the first a columns, we have $\det T = \sum_p \det A_p \det B_p$, where A_p (resp. B_p) is an $a \times a$ (resp. $(a_i - a) \times (a_i - a)$) matrix. Clearly, all the column indices of a typical A_p are $\leq a$, and since $\tau(a - t + 1) \geq s$, at least t of the row indices of A_p are $\geq s$. Using Laplace expansion for A_p with respect to t rows with indices $\geq s$, we obtain $\det A_p = \sum_q \det C_q \det D_q$, where C_q (resp. D_q) is a $t \times t$ (resp. $(a - t) \times (a - t)$) matrix, the row indices of C_q are $\geq s$, and column indices of C_q are $\leq a$. The required result follows from this. ∎

5. LADDER DETERMINANTAL VARIETIES AND SCHUBERT VARIETIES

Let $L \subset X$ be a one-sided ladder in X defined by the outside corners (b_i, a_i), $1 \leq i \leq h$, $1 \leq b_1 < \cdots < b_h < n$, $1 < a_1 < \cdots < a_h \leq n$, where X is a generic $n \times n$ matrix $X = (x_{ba})$, with n large enough such that L is situated below the main diagonal, i.e., $b_i \geq a_i + 1$, $1 \leq i \leq h$. Let $G = SL(n)$, $Q = P_{a_1} \cap \cdots \cap P_{a_h}$. Let O^- be the opposite big cell in G/Q. Let H be the one-sided ladder defined by the outside corners $(a_i + 1, a_i)$, $1 \leq i \leq h$. Let $\mathbf{s}, \mathbf{t} \in \mathbb{Z}^l_+$, satisfying (L1), (L2), and (L3), as in Section 1, with $m = n$. Let notations be as in Section 1. Let Z be the variety in $\mathbb{A}(H) \simeq O^-$ defined by the vanishing of the t_i-minors in $L(i)$, $1 \leq i \leq l$. Note that $Z \simeq D_{\mathbf{s},\mathbf{t}}(L) \times \mathbb{A}(H \setminus L) \simeq D_{\mathbf{s},\mathbf{t}}(L) \times \mathbb{A}^r$, where $r = \dim SL(n)/Q - |L|$.

We shall now define an element $w \in W_Q^{\min}$, such that the variety Z identifies with the opposite cell in the Schubert variety $X(w)$ in G/Q. We define $w \in W_Q^{\min}$ by specifying $w^{(a_i)} \in W^{a_i}$, $1 \leq i \leq h$, where $\pi_i(X(w)) = X(w^{(a_i)})$ under the projection π_i: $G/Q \to G/P_{a_i}$.

Define $w^{(a_i)}$, $1 \le i \le h$, inductively, as the (unique) maximal element in W^{a_i} such that

(1) $w^{(a_i)}(a_i - t_j + 1) = s_j - 1$ for all $j \in \{1, \dots, l\}$ such that $s_j \ge b_i$, and $t_j \ne t_{j-1}$ if $j > 1$.

(2) If $i > 1$, then $w^{(a_{i-1})} \subset w^{(a_i)}$.

Note that $w^{(a_i)}$, $1 \le i \le h$, is well defined in W^i, and w is well defined as an element in $W_Q^{\min}$.

5.1. Let us denote the distinct elements in $\{t_1, \dots, t_l\}$ by $t_1 = t_{i_1} > t_{i_2} > \cdots > t_{i_m} = t_l$, where $t_{i_k - 1} > t_{i_k}$ for $2 \le k \le m$. For $2 \le k \le m$, let $I_k = [e_{i_k}, s_{i_k} - 1]$, where $e_{i_k} = s_{i_k} - (t_{i_{k-1}} - t_{i_k})$. Let $I_1 = [b_1 - (a_1 - t_1 + 1),\ b_1 - 1]$, $I_{m+1} = [n - t_l + 2, n]$ (here for $p, q \in \mathbb{Z}$, $p < q$, $[p, q]$ denotes the set $\{p, p+1, \dots, q\}$).

REMARK 5.2. Fix j, $1 \le j \le h$. Let $b_j = s_c$, for some c, $1 \le c \le l$. Let i_k be the smallest index such that $s_{i_k} > b_j$. Then in $w^{(a_j)}$, $b_j - 1$ appears at the $(a_i - t_c + 1)$th place and is followed by the blocks $I_k, I_{k+1}, \dots, I_{m+1}$.

LEMMA 5.3. *We have*

(1) $w^{(a_1)} = I_1 \cup I_2 \cup \cdots \cup I_{m+1}$.

(2) $I_j \subset w^{(a_i)}$, $1 \le j \le m+1$, $1 \le i \le h$.

(3) *The entries in* $w^{(a_i)} \setminus w^{(a_{i-1})}$ *are* $\le b_i - 1$, $1 \le i \le h$.

All the assertions are clear from the definition of w.

LEMMA 5.4. *Fix* j, $1 \le j \le l$.

(1) *We have* $s_j \notin I_r$, $1 \le r \le m+1$.

(2) *Let* $t_j = t_{i_{k-1}}$, *for some* k, $2 \le k \le m+1$. *Then* $e_{i_k} > s_j$

(*here*, $e_{i_{m+1}} = n - t_l + 2$).

Proof. If $k = m+1$, then $t_j = t_l$, $e_{i_{m+1}} = n - t_l + 2 > s_j$ (since $t_j < n - s_j + 1$). Furthermore, $s_j \ge s_{i_m}$, and hence $s_j \notin I_r$ for any $1 \le r \le m+1$. Let then $k \le m$. We have $s_{i_k} - s_j > t_j - t_{i_k} = t_{i_{k-1}} - t_{i_k}$. This implies $e_{i_k} > s_j$. Hence $s_j \notin I_r$, $r \ge k$. Also the fact that $s_j \ge s_{i_{k-1}}$ implies that $s_j \notin I_r$, $r \le k-1$. ∎

REMARK 5.5. Consider a block of consecutive integers in $w^{(a_i)}$, $1 \le i \le h$, ending with $s_j - 1$ at the $(a_k - t_j + 1)$th place, for some $k \le i$. Then either $k = i$ or $k = j^*$; in other words, k is the largest integer in $\{1, \dots, i\}$ such that $b_k \le s_i$. In particular, if $j^* \le i$, then $k = j^*$.

THEOREM 5.6. *The variety* Z $(= D_{\mathbf{s},\mathbf{t}}(L) \times \mathbb{A}^r)$ *identifies with the opposite cell in* $X(w)$, *i.e.*, $Z = X(w) \cap O^-$ (*scheme theoretically*).

Proof. Let $f = \det M$, where M is a $t_i \times t_i$ matrix contained in $L(i)$ for some $1 \le i \le l$, be a generator of $I(Z)$. Let $k = i^*$, i.e., k is the largest integer such that $b_k \le s_i$. Then M is contained in H_k. By Lemma 4.1, f can be written in the form $f = \sum g_\phi p_\phi|_{O^-}$, with $\phi \in W^{a_k}$ such that $\{\phi(1), \dots, \phi(a_k)\} \cap \{a_k + 1, \dots, n\} = \{r_1, \dots, r_{t_i}\}$, and $g_\phi \in k[H]$ (here $r_1, \dots, r_{t_i}$ are the row indices of M). In particular, we have $\phi(a_k - t_i + 1) = r_1$. Since M is contained in $L(i)$, we have $r_1 \ge s_i$, and hence $\phi(a_k - t_i + 1) \ge s_i$. We have $w^{(a_k)}(a_k - t_i + 1) = s_i - 1$, and hence $\phi(a_k - t_i + 1) > w^{(a_k)}(a_k - t_i + 1)$. This shows that $\phi \not\le w^{(a_k)}$, and therefore $p_\phi \in I(X(w) \cap O^-)$. Thus $f \in I(X(w) \cap O^-)$.

Let now g be a generator of the ideal $I(X(w) \cap O^-)$, i.e., $g = p_\tau|_{O^-}$, with $\tau \in W^{a_i}$ for some i, $1 \le i \le h$, such that $\tau \not\le w^{(a_i)}$. Since $w^{(a_i)}$ consists of several blocks of consecutive integers ending with $s_m - 1$ at the $(a_k - t_m + 1)$th place, for some $m \in \{1, \dots, l\}$, where $k \in \{1, \dots, i\}$ is the largest index such that $b_k \le s_m$, and a last index ending with n at the a_ith place, it follows that $\tau(a_k - t_m + 1) \ge s_m$ for some m, where $k \in \{1, \dots, i\}$ is the largest index such that $s_m \ge b_k$. Using Lemma 4.2, we deduce that $p_\tau|_{O^-}$ belongs to the ideal of $k[H]$ generated by t_m-minors in L with row indices $\ge s_m$ and column indices $\le a_k$. Thus $p_\tau|_{O^-}$ belongs to the ideal generated by t_m-minors contained in $L(m)$, which shows that $g \in I(Z)$. ∎

Since the Schubert varieties are irreducible, normal, and Cohen–Macaulay and have rational singularities (cf. [10, 18–20]), as a consequence of Theorem 5.6 we obtain

THEOREM 5.7. *The variety $D_{\mathbf{s},\mathbf{t}}(L)$ is irreducible, normal, and Cohen–Macaulay and has rational singularities.*

6. THE DIMENSION OF $D_{\mathbf{s},\mathbf{t}}(L)$

Let $X = (x_{ba})$, $1 \le b \le m$, $1 \le a \le n$ be an $m \times n$ matrix of indeterminates.

6.1. *The Partial Order among Minors*

We shall denote the determinant of the $r \times r$ submatrix of X whose row indices are $i_1 < \cdots < i_r$ and column indices are $j_1 < \cdots < j_r$ by $[i_1, \dots, i_r | j_1, \dots, j_r]$. We introduce a partial order on the set of all minors of X as follows: $[i_1, \dots, i_r | j_1, \dots, j_r] \le [i'_1, \dots, i'_s | j'_1, \dots, j'_s]$ if

$$r \ge s \qquad \text{and} \qquad \begin{aligned} &i_r \ge i'_s,\ i_{r-1} \ge i'_{s-1}, \dots, i_{r-s+1} \ge i'_1, \\ &j_1 \le j'_1,\ j_2 \le j'_2, \dots, j_s \le j'_s. \end{aligned}$$

We say that an ideal I of $k[X]$ is *cogenerated* by a given minor M if I is generated by the minors in the set $\{M' \mid M' \text{ a minor of } X \text{ such that } M' \not\geq M\}$.

6.2. *The Monomial Order $\prec$ and Gröbner Bases*

We introduce a total order on the variables as follows:

$$x_{m1} > x_{m2} > \cdots > x_{mn} > x_{m-1\,1} > x_{m-1\,2} > \cdots$$
$$> x_{m-1\,n} > \cdots > x_{11} > x_{12} > \cdots > x_{1n}.$$

This induces a total order, namely the lexicographic order, on the set of monomials in $k[X] = k[x_{11}, \ldots, x_{mn}]$, denoted by $\prec$. The largest monomial (with respect to $\prec$) present in a polynomial $f \in k[X]$ is called the *initial term* of f and is denoted by $\mathrm{in}(f)$. Note that the initial term (with respect to $\prec$) of a minor of X is equal to the product of its elements on the skew diagonal.

Given an ideal $I \subset k[X]$, a set $G \subset I$ is called a *Gröbner basis* of I (with respect to the monomial order $\prec$) if the ideal $\mathrm{in}(I)$ generated by the initial terms of the elements in I is generated by the initial terms of the elements in G. Note that a Gröbner basis of I generates I as an ideal.

We recall the following (see [9]).

THEOREM 6.3. *Let $M = [i_1, \ldots, i_r | j_1, \ldots, j_r]$ be a minor of X, and let I be the ideal of $k[X]$ cogenerated by M. For $1 \leq t \leq r+1$, let G_t be the set of all t-minors $[i'_1, \ldots, i'_r | j'_1, \ldots, j'_r]$ satisfying the conditions*

$$\begin{gathered} i'_t \leq i_r, \quad i'_{t-1} \leq i_{r-1}, \quad \ldots, \quad i'_2 \leq i_{r-t+2}, \\ j'_{t-1} \geq j_{t-1}, \quad \ldots, \quad j'_2 \geq j_2, j'_1 \geq j_1 \end{gathered} \tag{1}$$

$$\text{if } t \leq r, \qquad \text{then } i'_1 > i_{r-t+1} \quad \text{or} \quad j'_t < j_t. \tag{2}$$

Then the set $G = \bigcup_{i=1}^{r+1} G_i$ is a Gröbner basis for the ideal I with respect to the monomial order $\prec$.

6.4. *The Ideal $I_{\mathbf{s},\mathbf{t}}(X)$ and the Set $\mathcal{G}$*

The matrix X can be viewed as a one-side ladder with a unique outside corner, namely $(1, n)$. Let $\mathbf{s}, \mathbf{t} \in \mathbb{Z}_+^l$ satisfy (L1), as in Section 1 (where $b_1 = 1$). Let $I_{\mathbf{s},\mathbf{t}}(X)$ be as in Section 1, for $L = X$. In other words, $I_{\mathbf{s},\mathbf{t}}(X)$ is the ideal of $k[X]$ generated by the t_i-minors in $X_i = \{x_{ba} \mid s_i \leq b \leq m\}$, $1 \leq i \leq l$. For $1 \leq i < l$, let $\mathcal{G}_i$ be the set consisting of the t_i minors in X_i such that the number of rows contained in X_j is less than t_j, for all j, $i < j \leq l$, and let $\mathcal{G}_l$ be the set consisting of the t_l minors in X_l. Let $\mathcal{G} = \bigcup_{i=1}^{l} \mathcal{G}_i$. Clearly, $I_{\mathbf{s},\mathbf{t}}(X)$ is generated by $\mathcal{G}$.

PROPOSITION 6.5. *Let* $\mathbf{s}, \mathbf{t} \in \mathbb{Z}_+^l$ *satisfy* $(L1)$, *and let* $\mathcal{G}$ *be as above. Then* $\mathcal{G}$ *is a Gröbner basis of* $I_{\mathbf{s},\mathbf{t}}(X)$, *with respect to the monomial order* $\prec$.

Proof. Let $M_{\mathbf{s},\mathbf{t}}$ be the minor of X of size $t_1 - 1$ given by the last $t_i - t_{i+1}$ rows of $X_i \setminus X_{i+1}$, $1 \le i < l$, and the last $t_l - 1$ rows of X_l, and the first $t_1 - 1$ columns of X. First we show that the ideal $I_{\mathbf{s},\mathbf{t}}(X)$ is cogenerated by $M_{\mathbf{s},\mathbf{t}}$. Let $M_{\mathbf{s},\mathbf{t}} = [i_1, \dots, i_{t_1-1} | j_1, \dots, j_{t_1-1}]$, and $\mathcal{F} = \{M' \mid M' \not\geq M_{\mathbf{s},\mathbf{t}}\}$. Note that $M' \ge M_{\mathbf{s},\mathbf{t}}$ if and only if M' contains at most $t_i - 1$ rows in X_i, $1 \le i \le l$. Thus $\mathcal{F} = \bigcup_{i=1}^{l} \mathcal{F}_i$, where $\mathcal{F}_i = \{M' \mid M' \text{ contains at least } t_i \text{ rows in } X_i\}$. Now $\mathcal{F}_i \subset I_{\mathbf{s},\mathbf{t}}(X)$, $1 \le i \le l$, and hence $\langle \mathcal{F} \rangle \subset I_{\mathbf{s},\mathbf{t}}(X)$. On the other hand, $\mathcal{G}_i \subset \mathcal{F}_i$, $1 \le i \le l$, and $\langle \mathcal{G} \rangle = I_{\mathbf{s},\mathbf{t}}(X)$. Therefore $I_{\mathbf{s},\mathbf{t}}(X) = \langle \mathcal{F} \rangle$, i.e., $I_{\mathbf{s},\mathbf{t}}(X)$ is cogenerated by $M_{\mathbf{s},\mathbf{t}}$.

The inequalities regarding j's in condition (1) of Theorem 6.3 are redundant in our case (since $j_t = t$, $1 \le t \le t_1 - 1$); also, condition (2) reduces to the condition that if $t \le r$, then $i'_1 > i_{r-t+1}$ (since $j_t = t$, and hence $j'_t \ge j_t$ for all t, $1 \le t \le t_1 - 1$). Therefore, in our case the conditions (1) and (2) are equivalent to

$$i'_t \le i_{t_1-1}, \quad i'_{t-1} \le i_{t_1-2}, \quad \dots, \quad i'_2 \le i_{t_1-t+1},$$
$$\text{and} \qquad \text{if } t \le t_1 - 1, \quad \text{then } i'_1 > i_{t_1-t}.$$

Note that the above inequalities imply $i_{t_1-t+1} \ge i'_2 > i'_1 > i_{t_1-t}$; now, if $t \notin \{t_1, \dots, t_l\}$, then this is not possible, since $i_{t_1-t+1} = i_{t_1-t} + 1$. Hence $G_t = \emptyset$ for $t \in \{1, \dots, t_1\} \setminus \{t_1, \dots, t_l\}$. It is easily seen that $G_{t_i} = \mathcal{G}_i$ for $1 \le i \le l$. Therefore Theorem 6.3 implies that $\mathcal{G}$ is a Gröbner basis for $I_{\mathbf{s},\mathbf{t}}(X)$ with respect to the monomial order $\prec$. ∎

We recall the following well known lemma.

LEMMA 6.6. *Let* $k[X]$ *be the polynomial ring in the set of indeterminates* X, *let* I *be an ideal of* $k[X]$, *and let* G *be a Gröbner basis of* I *with respect to a certain monomial order. Let* $L \subset X$ *such that*

$$\textit{if } f \in G \textit{ and } \operatorname{in}(f) \in k[L], \quad \textit{then } f \in k[L].$$

Then the set $G \cap k[L]$ *is a Gröbner basis of the ideal* $I \cap k[L]$.

Proof. Let $g \in I \cap k[L]$. Since G is a Gröbner basis of I, there exists $f \in G$ such that $\operatorname{in}(g) = \langle \operatorname{in}(f) \rangle$. Since $g \in k[L]$, we have $\operatorname{in}(g) \in k[L]$, and hence $\operatorname{in}(f) \in k[L]$. By hypothesis, $f \in k[L]$, and hence $f \in G \cap k[L]$. Therefore, the initial terms of the elements of $G \cap k[L]$ generate the ideal $\operatorname{in}(I \cap k[L])$. ∎

As a direct consequence, we obtain the following.

PROPOSITION 6.7. *Let* $L \subset X$ *be a one-sided ladder and let* $\mathbf{s}, \mathbf{t} \in \mathbb{Z}_+^l$, *satisfying* $(L1)$. *Then* $I_{\mathbf{s},\mathbf{t}}(L) = I_{\mathbf{s},\mathbf{t}}(X) \cap k[L]$, *and* $\mathcal{G}_L = \mathcal{G} \cap k[L]$ *is a Gröbner basis of* $I_{\mathbf{s},\mathbf{t}}(L)$ *with respect to the monomial order* $\prec$.

Proof. By Proposition 6.5, $\mathscr{G}$ is a Gröbner basis of $I_{\mathbf{s},\mathbf{t}}(X)$. By Lemma 6.6, $\mathscr{G}_L$ is a Gröbner basis of the ideal $I_{\mathbf{s},\mathbf{t}}(X) \cap k[L]$. On the other hand it is easily seen that $\mathscr{G}_L$ generates $I_{\mathbf{s},\mathbf{t}}(L)$, and the result follows. ∎

6.8. *The set* $\mathscr{C}$

We construct a set $\mathscr{C}_{\mathbf{s},\mathbf{t}}(X) \subset X$ as follows. Let $\mathscr{C}_l(X)$ be the submatrix obtained from X_l by deleting the first $t_l - 1$ columns and the last $t_l - 1$ rows. For $i < l$, let $\mathscr{C}_i(X)$ be the matrix obtained from $\tilde{X}_i = X_i \setminus X_{i+1}$ by deleting the first $t_i - 1$ columns and the last $t_i - t_{i+1}$ rows. Now let $\mathscr{C}_{\mathbf{s},\mathbf{t}}(X) = \bigcup_{i=1}^{l} \mathscr{C}_i(X)$.

For a one-sided ladder $L \subset X$, and $\mathbf{s}, \mathbf{t} \in \mathbb{Z}_+^l$ satisfying (L1), we define $\mathscr{C}_i(L) = \mathscr{C}_i(X) \cap L$, $\mathscr{C}_{\mathbf{s},\mathbf{t}}(L) = \mathscr{C}_{\mathbf{s},\mathbf{t}}(X) \cap L$.

Note that in a solid minor in $\mathscr{G}_L$ (i.e., a minor with consecutive row indices and consecutive column indices), the smallest (for the order in 6.2) element belongs to $\mathscr{C}_{\mathbf{s},\mathbf{t}}(L)$, and conversely, an element $\alpha \in \mathscr{C}_{\mathbf{s},\mathbf{t}}(L)$ determines uniquely a solid minor in $\mathscr{G}_L$ having α as the smallest element. Hence the number of elements in $\mathscr{C}_{\mathbf{s},\mathbf{t}}(L)$ is equal to the number of solid minors in the set $\mathscr{G}_L$.

The following is a generalization of Proposition 8 in [7].

PROPOSITION 6.9. *Let $L \subset X$ be a one-sided ladder, and let $\mathbf{s}, \mathbf{t} \in \mathbb{Z}_+^l$ satisfying* $(L1)$. *Then*

$$\operatorname{codim}_{\mathbb{A}(L)} D_{\mathbf{s},\mathbf{t}}(L) = |\mathscr{C}_{\mathbf{s},\mathbf{t}}(L)|.$$

Proof. By Proposition 6.7, the ideal $I_{\mathbf{s},\mathbf{t}}(L)$ and the ideal $J_{\mathbf{s},\mathbf{t}}(L)$ of its initial terms determine graded quotient rings of $k[L]$ having the same Hilbert series, and hence the codimension of the variety $D_{\mathbf{s},\mathbf{t}}(L)$ is equal to the height of the monomial ideal $J_{\mathbf{s},\mathbf{t}}(L)$. In general, the height of a monomial ideal J in a polynomial ring $k[x_1, \dots, x_N]$ is equal to the minimal cardinality of a set $\mathscr{C} \subset \{x_1, \dots, x_N\}$ of variables such that

each monomial in a set of monomial generators for J contains a variable from $\mathscr{C}$. $(*)$

Let $J = J_{\mathbf{s},\mathbf{t}}(L)$ and $\mathscr{C} = \mathscr{C}_{\mathbf{s},\mathbf{t}}(L)$. Then it is easy to see that $\mathscr{C}$ satisfies (*), the set of monomial generators being the set of the initial terms of all the t_i-minors in L_i, $1 \leq i \leq l$. Let us denote $\Delta_k = \{x_{ba} \in L \mid b + a = k + 1\}$, $k \geq 1$. Then $L = \dot{\bigcup}_{k \geq 1} \Delta_k$, and $\mathscr{C} = \dot{\bigcup}_{k \geq 1} (\mathscr{C} \cap \Delta_k)$.

Now let $\mathscr{C}' \subset \{x_{ba} \mid x_{ba} \in L\}$ be a set such that $|\mathscr{C}'| < |\mathscr{C}|$. Then there exists a k such that $|\mathscr{C}' \cap \Delta_k| < |\mathscr{C} \cap \Delta_k|$ (in particular, $\mathscr{C} \cap \Delta_k \neq \emptyset$). Let

$i \in \{1, \ldots, l\}$ be the largest such that $\Delta_k \cap \mathscr{C} \subset L_i$. Then

$$|\mathscr{C}' \cap (\Delta_k \cap L_i)| \leq |\mathscr{C}' \cap \Delta_k| < |\mathscr{C} \cap \Delta_k| = |\Delta_k \cap L_i| - (t_i - 1).$$

Therefore there exist t_i distinct variables in $(\Delta_k \cap L_i) \setminus \mathscr{C}'$. Thus the initial term of the t_i-minor in L_i having these elements on the skew diagonal does not contain any variable in $\mathscr{C}'$, and hence $\mathscr{C}'$ does not satisfy (*).

Therefore $\mathscr{C}$ is a set of minimal cardinality among the sets satisfying (*), and the required result follows. ∎

7. THE SINGULAR LOCUS OF $D_{\mathbf{s},\mathbf{t}}(L)$

Let $X = (x_{ba})$, $1 \leq b < m$, $1 < a \leq n$ be a $m \times n$ matrix of indeterminates. Let $L \subset X$ be an one-sided ladder defined by the outside corners $\omega_i = x_{b_i a_i}$, $1 \leq i \leq h$, $1 \leq b_1 < \cdots < b_h \leq m$, $1 \leq a_1 < \cdots < a_h \leq n$. Let $\mathbf{s}, \mathbf{t} \in \mathbb{Z}_+^l$ satisfy (L1), (L2), and (L3) of Section 1. We preserve the notations of Section 1. Let $V = D_{\mathbf{s},\mathbf{t}}(L)$, $\mathscr{C} = \mathscr{C}_{\mathbf{s},\mathbf{t}}(L)$.

For $1 \leq i \leq l$, let $V_i \subset \mathbb{A}(L)$ be the variety defined by the vanishing of the t_j-minors in $L(j)$, with $j \in \{1, \ldots, l\} \setminus \{i\}$, and the $(t_i - 1)$-minors in $L(i)$.

THEOREM 7.1. *With notations as above, we have*

$$\operatorname{Sing} V = \bigcup_{i=1}^{l} V_i.$$

Proof. For simplicity of notation, we identify the variable x_{ba} with the element (b, a).

First, we prove that $V_i \subset \operatorname{Sing} V$, for all $1 \leq i \leq l$. Let $x \in V_i$ for some $1 \leq i \leq l$. Let $\mathscr{J}$ be the jacobian matrix associated to the variety $V \subset \mathbb{A}(L)$, evaluated at x. Then the rows of $\mathscr{J}$ are indexed by t_j-minors in $L(j)$, $1 \leq j \leq l$, and the columns are indexed by the elements $\alpha \in L$. The (M, α)th entry in $\mathscr{J}$ is equal to $\pm(\det M')(x)$, where M' is the matrix obtained from M by deleting the row and the column containing α, if α appears in M, and 0 otherwise.

We distinguish two cases.

(I) $s_i \in \{b_1, \ldots, b_h\}$. Let $s_i = b_j$, for some $1 \leq j \leq h$. It is easily seen that

$$\omega_j \in \mathscr{C}_{\mathbf{s},\mathbf{t}}(L)$$

(since $s_{i+1} - s_i > t_i - t_{i+1}$ and $a_j \geq t_i$). Now consider the one-sided ladder L' obtained from L by deleting the element ω_j, i.e., the one-sided ladder defined by the outside corners

$$\omega_1 = (b_1, a_1), \quad \ldots, \quad \omega_{j-1} = (b_{j-1}, a_{j-1}), \quad \omega_{j^-} = (b_j, a_j - 1),$$
$$\omega_{j^+} = (b_j + 1, a_j), \quad \omega_{j+1} = (b_{j+1}, a_{j+1}), \quad \ldots, \quad \omega_l = (b_l, a_l),$$

where ω_{j^-} is present only if $a_j - 1 > a_{j-1}$, and ω_{j^+} is present only if $b_j + 1 < b_{j+1}$.

Since $x \in V_i$, a row of $\mathcal{J}$ indexed by a t_i-minor involving $\omega_j = x_{b_j a_j}$ is 0. Furthermore, the column of $\mathcal{J}$ indexed by ω_j is 0. Let $\mathcal{J}'$ be the matrix obtained from $\mathcal{J}$ by deleting the column indexed by ω_j and the rows indexed by t_i-minors containing ω_j. Then

$$\operatorname{rank} \mathcal{J} = \operatorname{rank} \mathcal{J}',$$

since $\mathcal{J}'$ is obtained from $\mathcal{J}$ by deleting zero rows and columns. Let $x' = (x_\alpha)_{\alpha \in L'}$. Then $x' \in D_{\mathbf{s},\mathbf{t}}(L')$, and $\mathcal{J}'$ is the jacobian matrix associated to the variety $D_{\mathbf{s},\mathbf{t}}(L') \subset \mathbb{A}(L')$, evaluated at x'. Thus

$$\operatorname{rank} \mathcal{J}' \leq \operatorname{codim}_{\mathbb{A}(L')} D_{\mathbf{s},\mathbf{t}}(L').$$

Now, using Proposition 6.9 we obtain

$$\begin{aligned} \operatorname{codim}_{\mathbb{A}(L')} D_{\mathbf{s},\mathbf{t}}(L') &= \left|\mathscr{C}_{\mathbf{s},\mathbf{t}}(L')\right| = \left|\mathscr{C}_{\mathbf{s},\mathbf{t}}(L) \setminus \{\omega_j\}\right| < \left|\mathscr{C}_{\mathbf{s},\mathbf{t}}(L)\right| \\ &= \operatorname{codim}_{\mathbb{A}(L)} D_{\mathbf{s},\mathbf{t}}(L). \end{aligned}$$

Hence $\operatorname{rank} \mathcal{J}' < \operatorname{codim}_{\mathbb{A}(L)} V$, which implies $\operatorname{rank} \mathcal{J} < \operatorname{codim}_{\mathbb{A}(L)} V$, i.e., $x \in \operatorname{Sing} V$.

(II) $s_i \notin \{b_1, \ldots, b_h\}$. We have $i > 1$ and $t_{i-1} > t_i$. Let $k = i^*$, i.e., k is the largest integer such that $b_k < s_i$. Define $\mathbf{s}' = (s_1, \ldots, s_{i-1}, \widehat{s_i}, s_{i+1}, \ldots, s_l)$, $\mathbf{t}' = (t_1, \ldots, t_{i-1}, \widehat{t_i}, t_{i+1}, \ldots, t_l)$. Let $\mathscr{C} = \mathscr{C}_{\mathbf{s},\mathbf{t}}(L)$, $\mathscr{C}' = \mathscr{C}_{\mathbf{s}',\mathbf{t}'}(L)$, and

$$\mathscr{C} = \bigcup_{j \in \{1, \ldots, l\}} \mathscr{C}_j, \qquad \mathscr{C}' = \bigcup_{j \in \{1, \ldots, l\} \setminus \{i\}} \mathscr{C}'_j,$$

as defined in Section 6.2. Then $\mathscr{C}_j = \mathscr{C}'_j$ for $j \notin \{i-1, i\}$, and

$$\begin{aligned} |\mathscr{C}| - |\mathscr{C}'| &= |\mathscr{C}_{i-1}| + |\mathscr{C}_i| - |\mathscr{C}'_{i-1}| \\ &= \left[(s_i - s_{i-1}) - (t_{i-1} - t_i)\right]\left[a_k - (t_{i-1} - 1)\right] \\ &\quad + \left[(s_{i+1} - s_i) - (t_i - t_{i+1})\right]\left[a_k - (t_i - 1)\right] \\ &\quad - \left[(s_{i+1} - s_{i-1}) - (t_{i-1} - t_{i+1})\right]\left[a_k - (t_{i-1} - 1)\right] \\ &= \left[(s_{i+1} - s_i) - (t_i - t_{i+1})\right](t_{i-1} - t_i) > 0 \end{aligned}$$

(here $s_{i+1} = m+1$, $t_{i+1} = 1$, if $i = l$). Therefore

$$\left|\mathscr{C}_{\mathbf{s}',\mathbf{t}'}(L)\right| < \left|\mathscr{C}_{\mathbf{s},\mathbf{t}}(L)\right|.$$

Since $x \in V_i$, a row indexed by a t_i-minor contained in $L(i)$ is 0. Let $\mathscr{J}'$ be the matrix obtained from $\mathscr{J}$ by deleting the rows indexed by t_i-minors contained in $L(i)$. Then

$$\operatorname{rank} \mathscr{J} = \operatorname{rank} \mathscr{J}'.$$

Now, $x \in D_{\mathbf{s}',\mathbf{t}'}(L)$, and $\mathscr{J}'$ is the Jacobian matrix associated to the variety $D_{\mathbf{s}',\mathbf{t}'}(L) \subset \mathbb{A}(L)$, evaluated at x. Thus

$$\operatorname{rank} \mathscr{J}' \leq \operatorname{codim}_{\mathbb{A}(L)} D_{\mathbf{s}',\mathbf{t}'}(L).$$

Now, using Proposition 6.9 we obtain

$$\operatorname{codim}_{\mathbb{A}(L)} D_{\mathbf{s}',\mathbf{t}'}(L) = |\mathscr{C}_{\mathbf{s}',\mathbf{t}'}(L)| < |\mathscr{C}_{\mathbf{s},\mathbf{t}}(L)| = \operatorname{codim}_{\mathbb{A}(L)} D_{\mathbf{s},\mathbf{t}}(L).$$

Hence $\operatorname{rank} \mathscr{J}' < \operatorname{codim}_{\mathbb{A}(L)} V$, which implies $\operatorname{rank} \mathscr{J} < \operatorname{codim}_{\mathbb{A}(L)} V$, i.e., $x \in \operatorname{Sing} V$.

Now we prove that $\operatorname{Sing} V \subset \bigcup_{i=1}^{l} V_i$. Let $\mathscr{C} = \mathscr{C}_{\mathbf{s},\mathbf{t}}(L)$, $\mathscr{C} = \bigcup_{i=1}^{l} \mathscr{C}_i$, as defined in Section 6.8.

We introduce a total order on the set of minors of L of size r, with $r \geq 1$ fixed, as follows: $[i_1, \dots, i_r | j_1, \dots, j_r] < [i'_1, \dots, i'_r | j'_1, \dots, j'_r]$ if there exists $1 \leq k \leq r$ such that

$$\begin{array}{lllllll} \text{either} & i_1 = i'_1, & \dots, & i_{k-1} = i'_{k-1}, & i_k < i'_k, & & \\ \text{or} & i_1 = i'_1, & \dots, & i_r = i'_r, & j_1 = j'_1, & \dots, & j_{k-1} = j'_{k-1}, \quad j_k < j'_k \end{array}$$

(this is simply the lexicographic order on $\{i_1, \dots, i_r, j_1, \dots, j_r\}$). Let $x \in V \setminus \bigcup_{i=1}^{l} V_i$. For each $1 \leq i \leq l$, let M_i be the largest $(t_i - 1)$-minor in $L(i)$ such that $(\det M_i)(x) \neq 0$. Let $\mathscr{T}_l$ be the set of elements in L_l not in the rows or the columns given by the rows and the columns of M_l. Clearly, $|\mathscr{T}_l| = |\mathscr{C}_l|$. By (decreasing) induction on i, suppose that, for some i, $1 < i \leq l$, the sets $\mathscr{T}_i, \dots, \mathscr{T}_l$ have been constructed, such that

(1)$_i$ $\mathscr{T}_j \subset L(j)$, $i \leq j \leq l$.

(2)$_i$ The sets $\mathscr{T}_i, \dots, \mathscr{T}_l$ are pairwise disjoint.

(3)$_i$ $|\mathscr{T}_j| = |\mathscr{C}_j|$, $i \leq j \leq l$.

(4)$_i$ $\mathscr{T}_j$ contains no elements appearing in the rows or in the columns of L given by the rows and the columns of M_j, $i \leq j \leq l$.

$(5)_i$ There exist $t_i - 1$ rows in $L(i)$ not containing any element from $\mathcal{T}_i \cup \cdots \cup \mathcal{T}_l$.

We define the set $\mathcal{T}_{i-1}$ as follows. Let r be the number of the rows of M_{i-1} contained in $\tilde{L}(i-1) = L(i-1) \setminus L(i)$. We distinguish two cases.

(I) $t_{i-1} - t_i \geq r$. In this case $\mathcal{T}_{i-1}$ is obtained from $\tilde{L}(i-1)$ by deleting the rows given by the rows of M_{i-1}, and $t_{i-1} - t_i - r$ other rows, followed by the deletion of the $t_{i-1} - 1$ columns given by the columns of M_{i-1}. Then properties $(1)_{i-1} - (4)_{i-1}$ are obvious; the $t_{i-1} - t_i$ rows of $\tilde{L}(i-1)$, which were deleted while defining $\mathcal{T}_{i-1}$, and the $t_i - 1$ rows of $L(i)$ in $(5)_i$, intersected with $L(i-1)$, give $t_{i-1} - 1$ rows of $L(i-1)$ not containing any elements in $\mathcal{T}_{i-1} \cup \mathcal{T}_i \cup \cdots \cup \mathcal{T}_l$, so that we have $(5)_{i-1}$.

(II) $t_{i-1} - t_i < r$. In this case $\mathcal{T}_{i-1}$ is obtained from $\tilde{L}(i-1)$ by deleting the r rows given by the rows of M_{i-1}, then adding $r - t_{i-1} + t_i$ rows from the $t_i - 1$ rows of $L(i)$ in $(5)_i$ that are not rows of M_{i-1}, intersected with $L(i-1)$ (this is possible, since there are $t_{i-1} - 1 - r$ rows of M_{i-1} in $L(i)$, and hence at least $(t_i - 1) - (t_{i-1} - 1 - r) = r - t_{i-1} + t_i$ rows from the $t_i - 1$ rows of $L(i)$ in $(5)_i$ are not rows of M_{i-1}), followed by the deletion of the $t_{i-1} - 1$ columns given by the columns of M_{i-1}. Again, the properties $(1)_{i-1} - (4)_{i-1}$ are obvoius; the r rows of M_{i-1} that were deleted from $\tilde{L}(i-1)$, and the $(t_i - 1) - (r - t_{i-1} + t_i)$ rows from the $t_i - 1$ rows in $(5)_i$ that were not used while defining $\mathcal{T}_{i-1}$, intersected with $L(i-1)$, give $t_{i-1} - 1$ rows of $L(i-1)$ not containing any elements in $\mathcal{T}_{i-1} \cup \mathcal{T}_i \cup \cdots \cup \mathcal{T}_l$, so that we have $(5)_{i-1}$.

Thus, using induction, we obtain the disjoint sets $\mathcal{T}_j \subset L(j)$, $1 \leq j \leq l$, such that $|\mathcal{T}_j| = |\mathcal{C}_j|$, and $\mathcal{T}_j$ contains no elements in the rows or columns of L given by the rows and columns of M_j.

For $\tau \in \mathcal{T}_i \subset \mathcal{T}$, $1 \leq i \leq l$, let M^τ be the t_i-minor obtained from M_i by adding the row and the column containing τ. Obviously, $M^\tau \neq M^{\tau'}$ for τ, $\tau' \in \mathcal{T}$, with $\tau \neq \tau'$.

We now take a total order on $\mathcal{T}$, namely $(b, a) > (b', a')$ if either $b > b'$, or $b = b'$ and $a > a'$.

Let us fix $\tau \in \mathcal{T}$, say $\tau \in \mathcal{T}_i$ for some i, $1 \leq i \leq l$. Then the (M^τ, τ)th entry in $\mathcal{J}$ is equal to $\pm(\det M_i)(x)$, so it is nonzero. Now let $\sigma \in \mathcal{T}$, $\sigma < \tau$. If σ is not an entry of M^τ, then the (M^τ, σ)th entry of $\mathcal{J}$ is equal to 0. Assume now that σ is the (r, s)th entry of M^τ. Then the (M^τ, σ)th entry of $\mathcal{J}$ is equal to $\pm(\det M')(x)$, where M' is the $(t_i - 1) \times (t_i - 1)$ matrix obtained from M^τ by deleting the rth row and the sth column. Let $\tau = (b, a)$, $\sigma = (b', a')$. If $b' < b$, then the indices of the first $r - 1$ rows of M' and M_i are the same, while the index of the rth row of M' is $> b'$, which is the index of the rth row of M_i. Thus, $M' > M_i$, and by the maximality of M_i, we obtain $(\det M')(x) = 0$. If $b' = b$, then $a' < a$. The indices of all the rows and those of the first $s - 1$ columns in M' and M_i are the same, while the index of the sth column in M' is $> a'$, which is the index of the sth column of M_i. Thus $M' > M_i$, and the maximality of M_i implies that $(\det M')(x) = 0$. Thus, for $\sigma < \tau$, the (M^τ, σ)th entry in $\mathcal{J}$ is 0.

Let $\mathcal{J}'$ be the submatrix of $\mathcal{J}$ given by the rows indexed by M^τ's and the columns indexed by τ's, with $\tau \in \mathcal{T}$. We suppose that both rows and columns of $\mathcal{J}'$ are indexed by the elements in $\mathcal{T}$, and we arrange them increasingly, with respect to the total order on $\mathcal{T}$ defined above. Then $\mathcal{J}'$ is upper triangular, and all the diagonal entries are nonzero. Thus $\det \mathcal{J}' \neq 0$, and this implies that

$$\operatorname{rank} \mathcal{J}' = |\mathcal{T}| = |\mathcal{C}| = \operatorname{codim}_{\mathbb{A}(L)} D_{\mathbf{s},\mathbf{t}}(L).$$

Consequently $\operatorname{rank} \mathcal{J} = \operatorname{codim}_{\mathbb{A}(L)} V$, i.e., $x \notin \operatorname{Sing} V$. ∎

8. THE IRREDUCIBLE COMPONENTS OF $\operatorname{Sing} V$ AND $\operatorname{Sing} X(w)$

We preserve the notations of Section 5.

Let us fix $j \in \{1, \dots, l\}$, and let $Z_j = V_j \times \mathbb{A}(H \setminus L)$. We shall now define $\theta_j \in W_Q^{\min}$ such that the variety Z_j identifies with the opposite cell in the Schubert variety $X(\theta_j)$ in G/Q.

Note that $w^{(a_r)}(a_r - t_j + 1) = s_j - 1$, and $s_j - 1$ is the end of a block of consecutive integers in $w^{(a_r)}$, where $r = j^*$ is the largest integer such that $b_r \leq s_j$. Furthermore, the beginning of this block is ≥ 2 (if the block started with 1, we would have $a_r - t_j + 1 = s_j - 1 \geq b_r - 1 \geq a_r$, which is not possible, since $t_j \geq 2$). Let $u_j + 1$ be the beginning of this block, where $u_j \geq 1$. Then it is easily seen that if $s_j - 1$ is the end of a block in $w^{(a_i)}$, $1 \leq i \leq h$, then the beginning of the block is $u_j + 1$. For each i, $1 \leq i \leq h$, such that $u_j \notin w^{(a_i)}$, let v_i be the smallest entry in $w^{(a_i)}$ that is bigger than $s_j - 1$. Note that $v_i = w^{(a_i)}(a_k - t_j + 2)$, where $k \in \{1, \dots, i\}$ is the largest index such that $b_k \leq s_j$.

Define $\theta_j^{(a_i)}$, $1 \leq i \leq h$, as follows.

If $s_j - 1 \notin w^{(a_i)}$ (which is equivalent to $j > 1$, $t_{j-1} = t_j$, and $i < r$), let $\theta_j^{(a_i)} = w^{(a_i)} \setminus \{v_i\} \cup \{s_j - 1\}$.

If $s_j - 1 \in w^{(a_i)}$ and $u_j \notin w^{(a_i)}$, then $\theta_j^{(a_i)} = w^{(a_i)} \setminus \{v_i\} \cup \{u_j\}$.

If $s_j - 1$ and $u_j \in w^{(a_i)}$, then $\theta_j^{(a_i)} = w^{(a_i)}$ (note that in this case $i > r$).

Note that θ_j is well defined as an element in $W_Q^{\min}$, and $\theta_j \leq w$.

REMARK 8.1. An equivalent description of θ_j is the following. Let $t_{i_k} < t_j \leq t_{i_{k-1}}$.

(I) If $j \notin \{i_1, \dots, i_m\}$ (i.e., $j > 1$ and $t_{j-1} = t_j$), then for $i < r$, $\theta_j^{(a_i)} = w_j^{(a_i)} \setminus \{e_{i_k}\} \cup \{s_j - 1\}$;

for $i = r$, $\theta_j^{(a_r)} = w_j^{(a_r)} \setminus \{e_{i_k}\} \cup \{u_j\}$, where u_j is the largest entry in $\{1, \ldots, s_j - 1\} \setminus w^{(a_r)}$;

for $i > r$ and $u_j \in w^{(a_i)}$, $\theta_j^{(a_i)} = w_j^{(a_i)}$;

for $i > r$ and $u_j \notin w^{(a_i)}$, $\theta_j^{(a_i)} = w_j^{(a_i)} \setminus \{v_i\} \cup \{u_j\}$, where v_i is the smallest entry in $w^{(a_i)} \setminus \theta_j^{(a_{i-1})}$.

(II) If $j \in \{i_1, \ldots, i_m\}$ (i.e., $t_{j-1} > t_j$ if $j > 1$), then

for $i \le r$, $\theta_j^{(a_i)} = w_j^{(a_i)} \setminus \{e_{i_k}\} \cup \{u_j\}$, where u_j is the largest entry in $\{1, \ldots, s_j - 1\} \setminus w^{(a_r)}$;

for $i > r$ and $u_j \in w^{(a_i)}$, $\theta_j^{(a_i)} = w_j^{(a_i)}$;

for $i > r$ and $u_j \notin w^{(a_i)}$, $\theta_j^{(a_i)} = w_j^{(a_i)} \setminus \{v_i\} \cup \{u_j\}$, where v_i is the smallest entry in $w^{(a_i)} \setminus \theta_j^{(a_{i-1})}$.

THEOREM 8.2. *The subvariety $Z_j \subset Z$ identifies with the opposite cell in $X(\theta_j)$, i.e., $Z_j = X(\theta_j) \cap O^-$ (scheme theoretically).*

Proof. Let $f = \det M$, where M is either a t_i-minor contained in $L(i)$, $i \in \{1, \ldots, h\} \setminus \{j\}$, or a $(t_j - 1)$-minor contained in $L(j)$, be a generator of $I(Z_j)$. In the former case we have $f \in I(Z)$, and Theorem 5.6 implies that $f \in I(X(w) \cap O^-) \subset I(X(\theta_j) \cap O^-)$. In the latter case, M is contained in H_k, where $k \in \{1, \ldots, h\}$ is the largest entry such that $b_k \le s_j$. By Lemma 4.1, f can be written in the form $f = \sum g_\phi p_\phi|_{O^-}$, with $\phi \in W^{a_k}$ such that $\{\phi(1), \ldots, \phi(a_k)\} \cap \{a_k + 1, \ldots, n\} = \{r_1, \ldots, r_{t_j - 1}\}$, and $g_\phi \in k[H]$ (here $r_1, \ldots, r_{t_j-1}$ are the row indices of M). In particular we have $\phi(a_k - t_j + 2) = r_1$. Since M is contained in $L(j)$, we deduce that $r_1 \ge s_j$, and hence $\phi(a_k - t_j + 2) \ge s_j$. We have $\theta_j{}^{(a_k)}(a_k - t_j + 2) = s_j - 1$, and hence $\phi(a_k - t_j + 2) > \theta_j{}^{(a_k)}(a_k - t_j + 2)$. This shows that $\phi \not\le \theta_j{}^{(a_k)}$, and therefore $p_\phi \in I(X(\theta) \cap O^-)$. Thus $f \in I(X(\theta) \cap O^-)$.

Now let $g = p_\tau|_{O^-}$, with $\tau \in W^{a_i}$ for some i, $1 \le i \le h$, such that $\tau \not\le \theta^{(a_i)}$, be a generator of the ideal $I(X(\theta_j) \cap O^-)$. Since $\theta_j{}^{(a_i)}$ consists of several blocks of consecutive integers ending with $s_m - 1$ at the $(a_k - t_m + 1)$th place, for some $m \in \{1, \ldots, l\} \setminus \{j\}$, where $k \in \{1, \ldots, i\}$ is the largest entry such that $b_k \le s_m$, a possible block ending with $s_j - 1$ at the $(a_k - t_j + 2)$th place, where $k \in \{1, \ldots, i\}$ is the largest entry such that $b_k \le s_j$, and a last block ending with n at the a_ith place, it follows that either $\tau(a_k - t_m + 1) \ge s_m$, for some $m \ne j$, where $k \in \{1, \ldots, i\}$ is the largest entry such that $s_m \ge b_k$, or $\tau(a_k - t_j + 2) \ge s_j$, where $k \in \{1, \ldots, i\}$ is the largest entry such that $s_j \ge b_k$. In the first case we have $\tau \not\le w$, and hence $p_\tau|_{O^-} \in I(X(w) \cap O^-) = I(Z) \subset I(Z_j)$. Suppose now that $\tau(a_k -$

$t_j + 2) \geq s_j$, where $k \in \{1, \ldots, i\}$ is the largest entry such that $s_j \geq b_k$. Using Lemma 4.2, we deduce that $p_\tau|_{O^-}$ belongs to the ideal of $k[H]$ generated by $(t_j - 1)$-minors with row indices $\geq s_j$ and column indices $\leq a_k$. Thus $p_\tau|_{O^-}$ belongs to the ideal generated by $(t_j - 1)$-minors contained in $L(j)$, which implies that $g \in I(Z_j)$. ∎

THEOREM 8.3. *The irreducible components of* $\operatorname{Sing} D_{\mathbf{s},\mathbf{t}}(L)$ *are precisely the* V_j*'s*, $1 \leq j \leq l$.

Proof. In view of Theorem 8.2, we obtain that V_j, $1 \leq j \leq l$, is irreducible, and the required result follows from Theorem 7.1. ∎

Let $X(w^{\max})$ (resp. $X(\theta_j^{\max})$, $1 \leq j \leq l$) be the pull-back in $SL(n)/B$ of $X(w)$ (resp. $X(\theta_j)$, $1 \leq j \leq l$) under the canonical projection π: $SL(n)/B \to SL(n)/Q$. Then using Theorems 7.1, 5.6, and 8.2, we obtain

THEOREM 8.4. *The irreducible components of* $\operatorname{Sing} X(w^{\max})$ *are precisely* $X(\theta_j^{\max})$, $1 \leq j \leq l$.

9. A CONJECTURE ON THE IRREDUCIBLE COMPONENTS OF A SCHUBERT VARIETY IN $SL(n)/B$

Let $G = SL(n)$. In this section we state a conjecture that is a refinement of the conjecture in [12] on the irreducible components of the singular locus of a Schubert variety and prove the conjecture for a certain class of Schubert varieties, namely the pull-backs $\pi^{-1}(X_Q(w))$ under π: $G/B \to G/Q$, where w and Q are as in Section 5.

For $\tau \in W$, let P_τ (resp. Q_τ) be the maximal element of the set of parabolic subgroups that leave $\overline{B\tau B}$ (in G) stable under multiplication on the left (resp. right).

We recall the following two well-known results (for a proof, see [11], for example).

LEMMA 9.1. *Let* α *be a simple root, and let* P_α *be the rank* 1 *parabolic subgroup with* $S_{P_\alpha} = \{\alpha\}$. *Let* $\tau \in W$. *Then* $\overline{B\tau B}$ *is stable under multiplication on the right* (*resp. left*) *by* P_α *if and only if* $\tau(\alpha) \in R^-$ (*resp.* $\tau^{-1}(\alpha) \in R^-$).

COROLLARY 9.2. *With notations as in* 2.2, *we have*

$$S_{P_\tau} = \{\alpha \in S \mid \tau^{-1}(\alpha) \in R^-\},$$
$$S_{Q_\tau} = \{\alpha \in S \mid \tau(\alpha) \in R^-\}.$$

DEFINITION 9.3. *Given parabolic subgroups* P, Q, *we say that* $\overline{B\tau B}$ *is* P-Q *stable if* $P \subset P_\tau$ *and* $Q \subset Q_\tau$.

LEMMA 9.4. *Let $G = SL(n)$. Let $\tau \in \mathcal{S}_n$, say $\tau = (a_1, \ldots, a_n)$. Let $\alpha = \epsilon_i - \epsilon_{i+1}$. Then*

(1) *$\tau(\alpha) \in R^-$ if and only if $a_i > a_{i+1}$.*

(2) *$\tau^{-1}(\alpha) \in R^-$ if and only if $i+1$ occurs before i in τ.*

Proof. We have $\tau(\alpha) = \epsilon_{a_i} - \epsilon_{a_{i+1}}$ and $\tau^{-1}(\alpha) = \epsilon_j - \epsilon_k$, where $a_j = i$ and $a_k = i+1$. The results follow from this. ∎

Let $\eta \in W$. We shall denote $X_B(\eta)$ by just $X(\eta)$. We first recall the criterion given in [12] for $X(\eta)$ to be singular.

THEOREM 9.5. *Let $\eta = (a_1 \ldots a_n) \in \mathcal{S}_n$. Then $X(\eta)$ is singular if and only if there exist i, j, k, m, $1 \le i < j < k < m \le n$, such that*

$$\textit{either } a_k < a_m < a_i < a_j \qquad \textit{or} \qquad a_m < a_j < a_k < a_i .$$

9.6. *The Set F_η*

Let $\eta = (a_1 \ldots a_n) \in \mathcal{S}_n$. Let E_η be the set of all $\tau' \le \eta$ such that either (1) or (2) below holds.

(1) There exist i, j, k, m, $1 \le i < j < k < m \le n$, such that

(a) $a_k < a_m < a_i < a_j$.

(b) If $\tau' = (b_1 \ldots b_n)$, then there exist i', j', k', m', $1 \le i' < j' < k' < m' \le n$, such that $b_{i'} = a_k$, $b_{j'} = a_i$, $b_{k'} = a_m$, $b_{m'} = a_j$.

(c) If τ (resp. η') is the element obtained from η (resp. τ') by replacing a_i, a_j, a_k, a_m respectively by a_k, a_i, a_m, a_j (resp. $b_{i'}, b_{j'}, b_{k'}, b_{m'}$ respectively by $b_{j'}, b_{m'}, b_{i'}, b_{k'}$), then $\tau' \ge \tau$ and $\eta' \le \eta$.

(2) There exist i, j, k, m, $1 \le i < j < k < m \le n$, such that

(a) $a_m < a_j < a_k < a_i$.

(b) If $\tau' = (b_1 \ldots b_n)$, then there exist i', j', k', m', $1 \le i' < j' < k' < m' \le n$, such that $b_{i'} = a_j$, $b_{j'} = a_m$, $b_{k'} = a_i$, $b_{m'} = a_k$.

(c) If τ (resp. η') is the element obtained from η (resp. τ') by replacing a_i, a_j, a_k, a_m respectively by a_j, a_m, a_i, a_k (resp. $b_{i'}, b_{j'}, b_{k'}, b_{m'}$ respectively by $b_{k'}, b_{i'}, b_{m'}, b_{j'}$), then $\tau' \ge \tau$ and $\eta' \le \eta$.

Let $F_\eta = \{\tau \in E_\eta \mid \overline{B\tau B} \text{ is } P_\eta\text{-}Q_\eta \text{ stable}\}$.

CONJECTURE. *The singular locus of $X(\eta)$ is equal to $\bigcup_\lambda X(\lambda)$, where λ runs over the maximal (under the Bruhat order) elements of F_η.*

9.7. Let $\eta = (a_1 \dots a_n) \in \mathcal{S}_n$. Let $\operatorname{Sing} X(\eta) \neq \emptyset$. Let (a, b, c, d) be four distinct entries in $\{1, \dots, n\}$ such that $a < b < c < d$. An occurrence in η of the form d, b, c, a, where $d = a_i$, $b = a_j$, $c = a_k$, $a = a_m$, $i < j < k < m$, will be referred to as a *Type I bad occurrence in* η. An occurrence in η of the form (c, d, a, b), where $c = a_i$, $d = a_j$, $a = a_k$, $b = a_m$, $i < j < k < m$, will be referred to as a *Type II bad occurrence in* η. Let (d, b, c, a) (resp. (c', d', a', b')) be a bad occurrence of Type I (resp. Type II), where $a < b < c < d$ (resp. $a' < b' < c' < d'$). Let θ, θ' be both $\leq w$. Furthermore, let b, a, d, c (resp. a', c', b', d') appear in that order in θ (resp. θ'). By abuse of language, we shall refer to (b, a, d, c) (resp. (a', c', b', d')) as a bad occurrence in θ (resp. θ') corresponding to the bad occurrence (d, b, c, a) (resp. (c', d', a', b')) in η.

Let $\tau \in W_Q^{\min}$. We have $\pi^{-1}(X_Q(\tau)) = X_B(\tau^{\max})$, where $\tau^{\max}$, as a permutation, is given by $\tau^{(a_1)}$ arranged in descending order, followed by $\tau^{(a_2)} \setminus \tau^{(a_1)}$ arranged in descending order, etc. We shall refer to the set $\tau^{(a_i)} \setminus \tau^{(a_{i-1})}$, $1 \leq i \leq l+1$, arranged in descending order, as the ith block in $\tau^{\max}$ (here, $\tau^{(a_0)} = \emptyset$, and $\tau^{(a_{l+1})}$ is the set $\{1, \dots, n\} \setminus \tau^{(a_l)}$ arranged in descending order).

For the rest of this section, w and Q will be as in Section 5.

REMARK 9.8. Set $b_{h+1} - 1 = n - t_l + 1$. All of the entries in the ith block in $w^{\max}$ are $\leq b_i - 1$, $2 \leq i \leq h+1$. In particular, for $1 \leq j \leq l$, s_j occurs after $s_j - 1$ in $w^{\max}$ (in view of Lemma 5.4).

LEMMA 9.9. *We have*

(1) $Q_{w^{\max}} = Q$.

(2) *Let* $I_{w^{\max}} = \{\epsilon_i - \epsilon_{i+1} \mid i = s_j - 1, 1 \leq j \leq l\}$. *Then* $S_{P_{w^{\max}}} = S \setminus I_{w^{\max}}$.

The assertions are clear from the description of $w^{\max}$ in view of Lemma 9.4 and Remark 9.8.

LEMMA 9.10. *Let* $P = P_{w^{\max}}$, $Q = Q_{w^{\max}}$. *Then* $\overline{B\theta_j^{\max}B}$ *is* P-Q *stable.*

Proof. The Q-stability of $\overline{B\theta_j^{\max}B}$ on the right is obvious. Regarding the P-stability of $\overline{B\theta_j^{\max}B}$ on the left, let x denote either e_{i_k} or v_i, where $i > j$, $u_j \notin w^{(a_i)}$ (notations are as in Section 8). Then $x - 1$ occurs after x in $w^{\max}$. It is clear from the definition of $\theta_j^{\max}$ that $x - 1$ also occurs after x in $\theta_j^{\max}$. For any other entry $y \neq x$, $s_j - 1$, if $y - 1$ occurs after y in $w^{\max}$, then it does so in $\theta_j^{\max}$ also. The result now follows from this. ∎

LEMMA 9.11. *Fix* j, $1 \leq j \leq h$. *Let* C *be a block of consecutive integers in* $w^{(a_j)}$ *ending with* $s_k - 1$ *at the* $(a_j - t_k + 1)$*th place* (*for some* k) *and beginning with* x_k. *Let the block preceding* C *end with* $s_i - 1$ *for some* i.

Suppose $k^* \leq j$. Then for $\alpha = \epsilon_y - \epsilon_{y+1}$, where $y \in [s_i, x_k]$, the rank 1 parabolic subgroup P_α is contained in $P(= P_{w^{\max}})$.

Proof. The result follows (in view of Lemma 9.9) from the fact that $[s_i, x_k]$ does not contain $s_t - 1$ for any t, $1 \leq t \leq l$. ∎

We first show the above conjecture to be true for $X(w^{\max})$ for the case $t_1 = \cdots = t_l$, since the exposition in this case is much neater (and simpler) than in the general case. Let then $t_1 = \cdots = t_l = t$, say. In this case, we have $b_i - 1 \in w^{(a_i)} \setminus w^{(a_{i-1})}$, $2 \leq i \leq l$. Furthermore, $h = l$, and $\{s_j, 1 \leq j \leq l\} = \{b_i, 1 \leq i \leq h\}$.

LEMMA 9.12. *Any bad occurrence in $w^{\max}$ is of Type I.*

Proof. Let $w^{\max} = (a_1 \ldots a_n)$. Assume that (c, d, a, b) is a bad occurrence of Type II in $w^{\max}$, where $a < b < c < d$. Clearly, c and d (resp. a and b) cannot both appear in the same block, in view of the description of $w^{\max}$. Let then c, d, a, b appear in the rth, ith, jth, kth blocks, respectively, where $r < i \leq j < k$. This implies that $a < b < c < d \leq b_i - 1$ (cf. Remark 9.8). But now, a and b are both $< b_i - 1$, and they both appear after $b_i - 1$; furthermore, a appears before b in $w^{\max}$, which is not possible by the construction of $w^{\max}$ (note that $a < b$). The required result follows from this. ∎

REMARK 9.13. Of course, there are several bad occurrences in $w^{\max}$ of Type I. For example, fix some j, $1 \leq j \leq h$. Observe that b_j appears after $b_j - 1$ (cf. Remark 9.8), and u_j appears after b_j in $w^{\max}$ (notations are as in Section 8). Take d to be any entry in $\{n - t + 2, \ldots, n\}$, $b = b_j - 1$, $c = b_j$, $a = u_j$. Then d, b, c, a occur in the 1st, jth, kth, and mth blocks, respectively, where $m \geq k > j$. This provides an example of a Type I bad occurrence in $w^{\max}$.

LEMMA 9.14. *Let d, b, c, a be a Type I bad occurrence in $w^{\max}$, where $a < b < c < d$. Assume that b belongs to the i-th block, for some i (note that $i \leq h$, since $b < c$). Then*

(1) $c < n - t + 2$.

(2) $b \leq b_i - 1$.

(3) $d \geq n - t + 2$.

Proof. Let d, b, c, a occur in the rth, ith, jth, kth blocks, respectively, in $w^{\max}$, where $r \leq i < j \leq k$. The hypothesis that $b < c$ implies that $j > 1$. Hence we obtain $c \leq b_j - 1$ (cf. Remark 9.8), and (1) follows. Now, if $i \geq 2$, then assertion (2) follows from Remark 9.8. If $i = 1$, then assertion (2) follows from the fact that $b < c < n - t + 2$.

CLAIM. $d > b_i - 1$.

Proof. Assume that $d \le b_i - 1$. Then the assumption implies $c < b_i - 1$ (since $c < d$). Now both c and b are $< b_i - 1$, and b belongs to the ith block in $w^{\max}$. This implies that c should occur before b, which is not possible. Hence our assumption is wrong, and the claim follows. ∎

Note that the claim and Remark 9.8 imply that $d \ge n - t + 2$, and d appears in the first block.

LEMMA 9.15. *Fix j, $1 \le j \le h$. Then $\theta_j^{\max}$ is the unique maximal element of the set $\{\tau \in W \mid \tau \le w^{\max}, \tau^{(a_j)}(a_j - t + 2) \le b_j - 1\}$.*

The proof is clear from the definition of $\theta_j^{\max}$.

PROPOSITION 9.16. *The maximal elements in $F_{w^{\max}}$ are precisely $\theta_j^{\max}$, $1 \le i \le h$ (here $F_{w^{\max}}$ is as in Section 9.6).*

Proof. We first observe that $\theta_j^{\max} \in F_{w^{\max}}$; for, corresponding to the bad occurrence $d = n - t + 2$, $b = b_j - 1$, $c = b_j$, $a = u_j$ (cf. Remark 9.13), we have the bad occurrence (b, a, d, c) (note that b, a, d, c occur in that order in $\theta_j^{\max}$). Let us denote $\theta_j^{\max}$ by τ'. Let w' (resp. τ) be the element of $\mathscr{S}_n$ obtained from τ' (resp. w) by replacing b, a, d, c (resp. d, b, c, a), respectively, by d, b, c, a (resp. b, a, d, c). Then clearly $\tau \le \tau'$ and $w' \le w$. Furthermore, $\overline{B\theta_j^{\max}B}$ is P-Q stable (cf. Lemma 9.10). Thus $\theta_j^{\max} \in F_{w^{\max}}$

Now let $\tau' \in F_{w^{\max}}$. In particular, we have $\tau' \in W_Q^{\max}$.

We have a bad occurrence in τ', which has to be of the form (b, a, d, c), $a < b < c < d$, corresponding to the occurrence (d, b, c, a) in $w^{\max}$ (cf. Lemma 9.12). Let b, a, d, c occur in the pth, qth, rth, and sth blocks, respectively, in τ', where $p \le q < r \le s$ (note that $\tau' \in W_Q^{\max}$).

We have

$$w'^{(a_q)}(a_q - t + 1) \le w^{(a_q)}(a_q - t + 1) = b_q - 1$$

(here w' is as in Section 9). Furthermore, $\tau'^{(a_q)}$ is obtained from $w'^{(a_q)}$ by replacing d by a, where $a(< b) < n - t + 2 \le d$ (cf. Lemma 9.14). Hence we obtain $a \le b_q - 1$ (since $\tau'^{(a_q)} \le w^{(a_q)}$), and

$$\tau'^{(a_q)}(a_q - t + 2) \le w'^{(a_q)}(a_q - t + 1) \le b_q - 1.$$

This implies $\tau' \le \theta_q^{\max}$ (cf. Lemma 9.15). ∎

THEOREM 9.17. *The conjecture 9 holds for $X(w^{\max})$.*

Proof. In view of Theorem 8.4, $X(\theta_j^{\max})$, $1 \le j \le h$, are precisely the irreducible components of $X(w^{\max})$. On the other hand, we have (cf. Proposition 9.16) that the maximal elements in $F_{w^{\max}}$ are precisely $\theta_j^{\max}$, $1 \le j \le h$. Hence the irreducible components of $\operatorname{Sing} X(w^{\max})$ are precisely $\{X(\theta) \mid \theta$ a maximal element of $F_{w^{\max}}\}$. Thus the conjecture holds for $X(w^{\max})$. ∎

Now we prove the conjecture for $X(w^{\max})$ in the general case.

LEMMA 9.18. *Fix j, $1 \le j \le l$. Let $j^* = r$. Then $\theta_j^{\max}$ is the unique maximal element of the set $\{\tau \in W \mid \tau \le w^{\max},\ \tau^{(a_r)}(a_r - t_j + 2) \le s_j - 1\}$.*

The proof is clear from the definition of θ_j.

LEMMA 9.19. *A bad occurrence in $w^{\max}$ has to be of Type I.*

Proof. If possible, let c, d, a, b, where $a < b < c < d$, occur in the ith, jth, kth, and pth blocks, respectively, in $w^{\max}$. Now $c < d$ implies that $i < j$. Hence $j > 1$. Hence $d \le b_j - 1$ (cf. Remark 9.8), and this implies that $b < d \le b_j - 1 \le b_k - 1$. But then a cannot appear before b (by definition of $w^{\max}$). ∎

REMARK 9.20. Of course, there are several Type I bad occurrences. For example, take j, $1 \le j \le l$. Let $j^* = r$. With notations as in Lemma 5.4, let $d = e_{i_k}$. We have (cf. Lemma 5.4) $d > s_j$. Also, in view of Remark 9.8, s_j is not an entry in $w^{(a_i)}$, $i \le r$, and s_j appears after $s_j - 1$ in $w^{\max}$. From the definition of $w^{\max}$, it is clear that u_j appears after s_j in $w^{\max}$ (notations are as in Section 8). Take $d = e_{i_k}$, $b = s_j - 1$, $c = s_j$, $a = u_j$.

LEMMA 9.21. *Let d, b, c, a be a Type I bad occurrence in $w^{\max}$. Then*

(1) *$d \in I_r$, for some r, $1 \le r \le m + 1$.*

(2) *$a, c \notin I_r$, for any r, $1 \le r \le m + 1$.*

Proof. Let d, b, c, a belong to the ith, jth, kth, and pth blocks, respectively, in $w^{\max}$. Assertion (2) is immediate, since $p, k > 1$. Note that assertion (1) is equivalent to the assertion that $i = 1$. If $j = 1$, then $i = 1$, and (1) follows (cf. Lemma 5.3). Then let $j > 1$. This implies $b \le b_j - 1 < c$. Suppose $i > 1$. Then we would obtain that $d \le b_i - 1 \le b_j - 1 < c$, which is not possible. Hence $i = 1$, and (1) follows. ∎

REMARK 9.22. With notations as in Lemma 9.21, we have in fact $d \in I_r$ for some $r \ge 2$. This is clear if $j \ge 2$ (since $b \le b_j - 1 < c < d$). If $j = 1$, then we have $b_1 - 1 < c < d$. Thus we get that $r \ge 2$.

PROPOSITION 9.23. *The maximal elements of $F_{w^{\max}}$ are precisely $\theta_j^{\max}$.*

Proof. Let us denote j^* by r. Then with d, b, c, a as in Remark 9.20, we have that b, a, d, c occur in that order in $\theta_j^{\max}$. Let us denote $\theta_j^{\max}$ by τ'. Let w' (resp. τ) be the element of $\mathscr{S}_n$ obtained from τ' (resp. w) by replacing b, a, d, c (resp. d, b, c, a), respectively, by d, b, c, a (resp. b, a, d, c). Then clearly $\tau \le \tau'$, and $w' \le w$. Furthermore, $\overline{B\theta_j^{\max}B}$ is P-Q stable (cf. Lemma 9.10). Thus $\theta_j^{\max} \in F_{w^{\max}}$. Now let $\tau' \in F_{w^{\max}}$. Let b, a, d, c be a bad occurrence in τ'. Furthermore, let b, a, d, c appear in the pth, qth, rth, and sth blocks, respectively, in τ' (note that $\tau' \in W_Q^{\max}$). Let $b_q = s_z$ for some

z, $1 \le z \le l$. If $a \le b_q - 1$, and $d > b_q - 1$, then as in the proof of Proposition 9.16, we obtain $\tau'^{(a_q)}(a_q - t_z + 2) \le b_q - 1 (= s_z - 1)$. This implies $\tau' \le \theta_z^{\max}$ (note that $z^* = q$).

We now distinguish the following two cases.

Case 1. $d \le b_q - 1$. Let $d \in I_k (= [e_{i_k}, s_{i_k} - 1])$ for some $k \ge 2$ (cf. Remark 9.22). Let $j = i_k^*$. We first observe that $j \le q$. For, if $i_k = i_k^* (= j)$, then $j \le q$ (since $d \le b_q - 1$). If $i_k > i_k^*$, then again in view of Lemma 5.4, we have $s_{i_k^*} < d \le b_q - 1$, and hence $b_j - 1 < b_q - 1$ (note that $s_{i_k^*} = b_j$). Hence we get $j < q$. Thus in either case we have $j \le q$.

We further divide this case into the following two subcases.

Subcase 1 (a). $j < i_k$. Now, I_k appears in $w^{(a_j)}$ as a block of consecutive integers (cf. Remark 5.2), and $s_{i_k} - 1$ appears at the $(a_j - t_{i_k} + 1)$th place. Let the block in $w^{(a_j)}$ preceding this block end with $s_i - 1$ at the $(a_u - t_i + 1)$th place, for some u and i. Then $u = j$ necessarily (since $j < i_k$), and hence $i^* = u = j$. Now, in view of Lemmas 9.9 and 9.11 for $\alpha = \epsilon_y - \epsilon_{y+1}$, where $y \in [s_i, d-1]$, the rank 1 parabolic subgroup P_α is contained in $P (= P_{w^{\max}})$. This, together with the fact that $d \notin \tau'^{(a_j)}$, implies that $[s_i, d] \cap \tau'^{(a_j)} \neq \emptyset$ (in view of the P-stability on the left of $X(\tau')$ (cf. Lemma 9.4)). Hence we obtain that $\tau'^{(a_j)}(a_j - t_i + 2) \le s_i - 1$, where $i^* = j$. This implies $\tau' \le \theta_i^{\max}$ (cf. Lemma 9.18).

Subcase 1 (b). $j = i_k$. Note that $j > 1$. (cf. Remark 9.22). Consider $w^{(a_{j-1})}$. Now I_k appears in $w^{(a_{j-1})}$ as a block (cf. Remark 5.2, since $i_k^* > j - 1$), and d belongs to this block. Furthermore, $s_{i_k} - 1$ appears at the $(a_{j-1} - t_{i_k} + 1)$th place. Let the block in $w^{(a_{j-1})}$ preceding this block end with $s_i - 1$ at the $(a_{j-1} - t_i + 1)$th place for some i. Then $i^* = j - 1$, necessarily (since $j = i_k$). Furthermore, for $\alpha = \epsilon_y - \epsilon_{y+1}$, where $y \in [s_i, d-1]$, the rank 1 parabolic subgroup P_α is contained in P (in view of Lemma 9.9, since $[s_i, d-1]$ does not contain $s_t - 1$ for any t, $1 \le t \le l$). Now, the fact that $d \notin \tau'^{(a_q)}$ implies that $\tau'^{(a_{j-1})} \cap [s_i, d] = \emptyset$ (in view of P-stability on the left of $X(\tau')$). Hence we obtain $\tau'^{(a_{j-1})}(a_{j-1} - t_i + 2) \le s_i - 1$, where $i^* = j - 1$. This implies $\tau' \le \theta_i^{\max}$ (cf. Lemma 9.18).

Case 2. $a > b_q - 1$. Let d, b, c, a appear in the ith, jth, kth, and xth blocks, respectively, in $w^{\max}$, where $i \le j < k \le x$. Let u be the smallest index such that $a \le s_u - 1$. We have $q \le u^*$ (since $q > u^*$ would imply $a \le s_u - 1 < b_q - 1$, which is not true).

Claim. $x > u^*$. If $j \ge 2$, then we have $b_q - 1 < a < b \le b_j - 1$ (cf. Remark 9.8). Hence we obtain $u^* \le j$, from which the claim follows (since $x > j$).

If $j = 1$, let $b \in I_v$ for some $v \ge 2$ (cf. Lemma 5.3; note that $b_q - 1 < a < b$ implies $b > b_1 - 1$). We have $b_q - 1 < a < b \le s_{i_v} - 1$. Hence we obtain

$s_u - 1 \le s_{i_v} - 1$, and $u^* \le i_v^*$. Now, we have $b_k - 1 \ge c > s_{i_v} - 1 \ge s_{i_v^*} - 1$ (by the definition of $w^{\max}$). This implies $c \notin w^{(a_{i_v^*})}$, and hence $k > i_v^* \ge u^*$. The claim now follows from this (since $x \ge k$). Thus we obtain $q \le u^* < x$. Now the fact that $a \in \tau'^{(a_q)}$ implies $a \in \tau'^{(a_{u^*})}$. This, together with the P-stability on the left of $X(\tau')$, implies that $[a, s_u - 1] \subset \tau'^{(a_{u^*})}$ (note that $s_j - 1 \notin [a, s_u - 1]$, for any $j \neq u$, and hence for $\alpha = \epsilon_y - \epsilon_{y+1}$, where $y \in [a, s_u - 2]$, the rank 1 parabolic subgroup P_α is contained in P). From this, we obtain $\tau'^{(a_{u^*})}(a_{u^*} - t_u + 2) \le s_u - 1$ (since $\tau'^{(a_{u^*})} \le w^{(a_{u^*})}$, and $a \notin w^{(a_{u^*})}$ (note that $x > u^*$)). This implies $\tau' \le \theta_u^{\max}$ (cf. Lemma 9.18).

THEOREM 9.24. *Conjecture* 9 *holds for* $X(w^{\max})$.

Proof. As in the proof of Theorem 9.17, the result follows from Theorem 8.4 and Proposition 9.23. ∎

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