Finite $p$-groups with a minimal non-abelian subgroup of index $p$ (I)

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For an odd prime $p$, we classify finite $p$-groups with a unique minimal non-abelian subgroup of index $p$. In fact, such groups have a maximal quotient which is a 3-group of maximal class. This paper is a part of classification of finite $p$-groups with a minimal non-abelian subgroup of index $p$, and partly solves a problem proposed by Berkovich.

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1. Introduction

Finite $p$-groups are an important class of finite groups. After the classification of finite simple groups was finally completed, the study of finite $p$-groups becomes more and more active. Many leading group theorists, for example, Glauberman, Janko, etc., have turned their attentions to the study of finite $p$-groups. As Janko mentioned in the foreword of [4], to study $p$-groups with “large” abelian subgroups is another approach to finite $p$-groups. A well-known important result is the classification of finite $p$-groups with a cyclic subgroup of index $p$, which was obtained by Burnside [8]. Tuan [19] and Nazarova et al. [14] studied finite $p$-groups with an abelian subgroup of index $p$. Another important concept in finite $p$-groups is minimal non-abelian $p$-groups. A non-abelian group $G$ is

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said to be minimal non-abelian if every proper subgroup of \( G \) is abelian. Finite minimal non-abelian groups were classified in [13], and in more detail for finite \( p \)-groups in [17]. Berkovich and Janko [3] introduced a new concept — \( A_t \)-groups, which is a more general concept than that of minimal non-abelian \( p \)-groups. For a positive integer \( t \), a finite \( p \)-group is called an \( A_t \)-group if all subgroups of index \( p^t \) are abelian, and at least one subgroup of index \( p^{t-1} \) is not abelian. Obviously, \( A_1 \)-groups are just the minimal non-abelian \( p \)-groups. For small \( t \), \( A_t \)-groups can be considered to have “large” abelian subgroups. Many scholars studied and classified in [13], and in more detail for finite \( p \)-groups were classified in [17]. Berkovich and Janko [3]

The starting point of this paper is the \( A_2 \)-groups. By the definition of \( A_2 \)-groups, \( G \) is an \( A_2 \)-group if and only if \( G \) is a finite \( p \)-group all of whose proper subgroups are abelian or minimal non-abelian, and \( G \) has at least one minimal non-abelian subgroup of index \( p \). It is natural to classify \( p \)-groups with a minimal non-abelian subgroup of index \( p \). In fact, Berkovich in [4] proposed the following

\textbf{Problem 239.} Classify the \( p \)-groups containing an \( A_1 \)-subgroup of index \( p \).

It turns out that solving Problem 239 involves a lot of work. Roughly we may divide this problem into two parts.

Part 1. Classify the finite \( p \)-groups with at least two \( A_1 \)-subgroups of index \( p \).

Part 2. Classify the finite \( p \)-groups with a unique \( A_1 \)-subgroup of index \( p \).

Since groups in Part 1 have more \( A_1 \)-subgroups of index \( p \) than groups in Part 2, such groups have small nilpotency class (at most 3). Hence the methods of Part 1 are quite different from those of Part 2. In fact, completing Part 1 is more complicated and troublesome than that of Part 2, and completing Part 2 is more technical and tricky than that of Part 1.

In this paper, for an odd \( p \), we solve completely Part 2, i.e., we classify the finite \( p \)-groups with a unique \( A_1 \)-subgroup of index \( p \). Such groups have a maximal quotient group which is a 3-group of maximal class.

In our coming several papers [1,2,15,16], we shall provide complete solution of Part 1, i.e., we shall classify the finite \( p \)-groups with at least two \( A_1 \)-subgroups of index \( p \), and Part 2 for \( p = 2 \). Hence Problem 239 is completely solved.

As a direct application of the classification of the \( p \)-groups with an \( A_1 \)-subgroup of index \( p \), recently, Zhang et al. [24] have classified \( A_3 \)-groups. This solves an old problem which is given in [5] by Janko and Berkovich, i.e.,

\textbf{Problem 1278 (Old problem).} Classify \( A_3 \)-groups.

2. Preliminaries

Let \( G \) be a finite \( p \)-group. We use \( c(G) \), \( \exp(G) \) and \( d(G) \) to denote the nilpotency class, the exponent and the minimal number of generators of \( G \) respectively. For any positive integer \( s \), we define \( \Omega_s(G) = \langle a \in G \mid a^{p^s} = 1 \rangle \) and \( \bar{\Omega}_s(G) = \langle a^{p^s} \mid a \in G \rangle \). Let

\[
G > G' = G_2 > G_3 > \cdots > G_{c+1} = 1
\]

denote the lower central series of \( G \), where \( c = c(G) \). The Frattini subgroup \( \Phi(G) \) of \( G \) is equal to \( G' \bar{\Omega}_1(G) \). We use \( C_p^m, C_{pm}^r \) and \( H \ast K \) to denote the cyclic group of order \( p^m \), the direct product of \( n \) cyclic groups of order \( p^m \), and a central product of \( H \) and \( K \) respectively. \( M \leq G \) means \( M \) is maximal in \( G \).

We use \( M_p(m, n) \) to denote groups \( \langle a, b \mid a^{p^m} = b^{p^m} = 1, a^b = a^{1+p^{m-1}} \rangle \), where \( m \geq 2 \). We use \( M_p(m, n, 1) \) to denote groups \( \langle a, b, c \mid a^{p^m} = b^{p^n} = c^{p} = 1, [a, b] = c, [c, a] = [c, b] = 1 \rangle \), where \( m \geq n \), and if \( p = 2 \), then \( m + n \geq 3 \). For other notation and terminology the reader is referred to [10].

\textbf{Lemma 2.1.} (See [17].) Let \( G \) be a minimal non-abelian \( p \)-group. Then \( G \) is \( Q_8 \), \( M_p(m, n) \), or \( M_p(m, n, 1) \).
Lemma 2.2. (See [21, Lemma 2.2].) Suppose that $G$ is a finite non-abelian $p$-group. Then the following conditions are equivalent:

1. $G$ is minimal non-abelian;
2. $d(G) = 2$ and $|G'| = p$;
3. $d(G) = 2$ and $\Phi(G) = Z(G)$.

A finite group $G$ is called metacyclic if it has a cyclic normal subgroup $N$ such that $G/N$ is also cyclic.

Lemma 2.3. (See [11].) If $p > 2$, then $G$ is metacyclic if and only if $|G : \mathcal{U}_1(G)| \leq p^2$.

A $p$-group $G$ is said to be regular if for every $x, y \in G$, there exists $c \in \mathcal{U}_1((x, y)^i)$ such that $x^p y^p = (xy)^p c$. A $p$-group $G$ is said to be absolutely regular if $|G : \mathcal{U}_1(G)| < p^2$. An absolutely regular $p$-group is regular. A $p$-group $G$ is said to be $p$-abelian if $x^p y^p = (xy)^p$ for every $x, y \in G$. The following lemma is obvious.

Lemma 2.4. Let $G$ be a regular $p$-group. If $\exp(G') = p$, then $G$ is $p$-abelian.

Lemma 2.5. (See [4, Theorem 7.1, p. 98].) Let $G$ be a $p$-group.

1. Regularity is inherited by sections;
2. If $c(G) < p$, then $G$ is regular;
3. If $G_{p-1}$ is cyclic, then $G$ is regular.

Lemma 2.6. (See [4, Theorem 7.9, p. 105].) If $G$ is a regular two-generator 3-group, then $G'$ is cyclic.

Lemma 2.7. (See [4, Theorem 7.2, p. 99].) Suppose that $G$ is a regular $p$-group. Then $\exp(\Omega_n(G)) \leq p^n$, $\mathcal{U}_n(G) = \{x^p \mid x \in G\}$, $|\Omega_n(G)| = |G : \mathcal{U}_n(G)|$, and $[x^p, y^p] = 1 \Leftrightarrow [x, y]^{p^k n} = 1$.

A non-abelian group $G$ is said to be metabelian if $G'$ is abelian. The following formulae are useful in this paper.

Proposition 2.8. (See [20].) Let $G$ be a metabelian group and $a, b \in G$. For any positive integers $i$ and $j$, let

$$[ia, jb] = [a, b, a, \ldots, a, b, \ldots, b].$$

Then:

1. For any positive integers $m$ and $n$,

$$[a^m, b^n] = \prod_{i=1}^{m} \prod_{j=1}^{n} [ia, jb]^{\binom{n}{i} \binom{m}{j}}.$$ 

2. Let $n$ be a positive integer. Then

$$(ab^{-1})^n = a^n \left( \prod_{i+j \leq n} [ia, jb]^{\binom{n}{i+j}} \right) b^{-n}.$$
A group $G$ of order $p^n$ is of maximal class if $c(G) = n - 1 \geq 3$. If $G$ is of maximal class, then $G_1 := C_G(G_2/G_4)$ is maximal in $G$, and is called fundamental subgroup of $G$. In [6], Blackburn determined 3-groups of maximal class and $p$-groups of maximal class satisfying $G_1$ is abelian.

**Lemma 2.9.** (See [10, III, 14.6(b) and 14.22].) Suppose that $G$ is a $p$-group of maximal class and $|G| = p^n$ where $n \geq p + 2$. Then $|G : \Omega_1(G)| = p^{n-1}$, and maximal subgroups of $G$ except $G_1$ are also of maximal class.

**Lemma 2.10.** (See [4, Exercise 10, p. 121].) Let $G$ be a 3-group of maximal class. Then the fundamental subgroup $G_1$ is metacyclic. Moreover, $G_1$ is abelian or minimal non-abelian.

**Lemma 2.11.** (See [6] or [22, Section 8.3].) Let $G$ be a 3-group of maximal class. If $G_1$ is abelian, then $G$ is one of the following non-isomorphic groups:

1. $|G| = 3^{2e+1}$, where $e \geq 2$.
   (1a) $\langle s_1, s_2, \beta | s_1^{3^e} = s_2^3 = \beta^3 = 1, [s_1, \beta] = s_2, [s_2, \beta] = s_2^{-3}s_1^{-3}, [s_1, s_2] = 1 \rangle$. In this case, $G_1 = \langle s_1, s_2 \rangle$ and $g^3 = 1$ for all $g \in G \setminus G_1$;
   (1b) $\langle s_1, s_2, \beta | s_1^{3^e} = s_2^3 = \beta^3 = s_2^{-3}s_1^{-3}, [s_1, \beta] = s_2, [s_2, \beta] = s_2^{-3}s_1^{-3}, [s_1, s_2] = 1 \rangle$. In this case, $G_1 = \langle s_1, s_2 \rangle$ and $G G_1 = \langle s_1^{3^e-1} \rangle$ for all $g \in G \setminus G_1$;
   (1c) $\langle s_1, s_2, \beta, \alpha | s_1^{3^e} = s_2^3 = \beta^3 = 1, \alpha^3 = s_1^{-3}s_2^{-1}s_1^{3e-1}, [\alpha, \beta] = s_1, [s_1, \beta] = s_2, [s_2, \beta] = s_2^{-3}s_1^{-3}, [s_1, \alpha] = [s_1, s_2] = 1 \rangle$. In this case, $G_1 = \langle \alpha, s_1 \rangle$.

2. $|G| = 3^{2e}$, where $e \geq 2$.
   (2a) $\langle s_1, s_2, \beta | s_1^{3^e} = s_2^3 = \beta^3 = 1, [s_1, \beta] = s_2, [s_2, \beta] = s_2^{-3}s_1^{-3}, [s_1, s_2] = 1 \rangle$. In this case, $G_1 = \langle s_1, s_2 \rangle$ and $g^3 = 1$ for all $g \in G \setminus G_1$;
   (2b) $\langle s_1, s_2, \beta | s_1^{3^e} = s_2^3 = \beta^3 = s_1^{-3}s_2^{-1}, [s_1, \beta] = s_2, [s_2, \beta] = s_2^{-3}s_1^{-3}, [s_1, s_2] = 1 \rangle$. In this case, $G_1 = \langle s_1, s_2 \rangle$ and $G G_1 = \langle s_1^{3^e-1} \rangle$ for all $g \in G \setminus G_1$;
   (2c) $\langle s_1, s_2, \beta, \alpha | s_1^{3^e} = s_2^3 = \beta^3 = 1, \alpha^3 = s_1^{-3}s_2^{-1}s_2^{-3}s_1^{3e-2}, [\alpha, \beta] = s_1, [s_1, \beta] = s_2, [s_2, \beta] = s_2^{-3}s_1^{-3}, [s_1, \alpha] = [s_1, s_2] = 1 \rangle$ where $\nu = 1, 2$. In this case, $G_1 = \langle \alpha, s_1, s_2 \rangle$. In particular, if $e > 2$ or $\nu = 2$, then $G_1 = \langle \alpha, s_1 \rangle$.

**Lemma 2.12.** (See [6] or [22, Section 8.4].) Let $G$ be a 3-group of maximal class. If $G_1$ is non-abelian, then $|G| \geq 3^3$ and $G$ is one of the following non-isomorphic groups:

1. $|G| = 3^{2e}$ and $G = \langle s_1, s_2, s | s_1^{3^e} = s_2^{3^e-1}, s^3 = s_1^{3e-1}, [s_1, s] = s_2, [s_2, s] = s_2^{-3}s_1^{-3}, [s_2, s_1] = s_2^{-3}s_1^{-3} \rangle$ where $\delta = 0, 1, 2$;
2. $|G| = 3^{2e+1}$ and $G = \langle s_1, s_2, s | s_1^{3^e} = s_2^{3^e}, s^3 = s_1^{3e-1}, [s_1, s] = s_2, [s_2, s] = s_2^{-3}s_1^{-3}, [s_2, s_1] = s_2^{-3}s_1^{-3} \rangle$ where $\delta = 0, 1, 2$.

We also need following lemmas:

**Lemma 2.13.** (See [4, Exercise 6, p. 27].) Let $G$ be a non-abelian $p$-group. Then the number of abelian subgroups of index $p$ in $G$ is $0$ or $p + 1$.

**Lemma 2.14.** (See [19] or [10, Aufgabe 2, p. 259].) Suppose that a finite non-abelian $p$-group $G$ has an abelian normal subgroup $A$, and $G/A = [bA]$ is cyclic. Then the map $a \mapsto [a, b]$, $a \in A$, is an epimorphism from $A$ to $G'$, and $G' \cong A/A \cap Z(G)$. In particular, if a non-abelian $p$-group $G$ has an abelian maximal subgroup, then $|G| = p|G'||Z(G)|$.

**Lemma 2.15.** (See [7, Theorem 4.1].) Suppose that $G$ is a group of order $p^n$, where $p$ is odd and $n \geq 5$, and that all normal subgroups of $G$ of order $p^r$ have two generators, where $r$ is fixed and $3 \leq r \leq n - 2$. Then either
(1) $G$ is metacyclic,
(2) $G$ is a 3-group of maximal class, or
(3) for $r = 3$, $G$ is one of the following groups:

(3a) $G = \langle a, b, c \mid a^{p^{n-2}} = b^p = c^p = 1, [a, b] = c, [c, b] = a^{\nu p^{n-3}}, [c, a] = 1 \rangle$ where $\nu = 1$ or a fixed square non-residue modulo $p$,

(3b) $G = M_p(1, 1, 1) \ast C_{p^{n-2}}$.

**Lemma 2.16.** Suppose that $p$ is an odd prime, $G$ is a finite $p$-group, $N \leq G$ such that $|N| = p$ and $G/N$ is minimal non-abelian.

(1) If $G/N \cong M_p(m, n)$ or $G/N \cong M_p(m, 1, 1)$, then $G$ has two distinct minimal non-abelian subgroups of index $p$;

(2) If $G/N \cong M_p(m, n)$, where $m \geq n \geq 2$, then $G$ has no minimal non-abelian subgroup of index $p$.

**Proof.** By Lemma 2.2, $d(G/N) = 2$ and $|G/N'| = p$. Since $N \leq G'$, we get $d(G) = d(G/N) = 2$, $(G/N)' = G'/N$ and hence $|G'| = p^2$. Let $\mathcal{M} = \{H \mid G \neq H' \neq 1\}$ and $\mathcal{N} = \{H \mid G \neq H' = 1\}$. Since $d(G) = 2$, we have $|\mathcal{M}| + |\mathcal{N}| = p + 1$. Since $G$ is not minimal non-abelian, we have $|\mathcal{N}| \neq p + 1$. It follows from Lemma 2.13 that $|\mathcal{N}| \leq 1$. Hence $|\mathcal{M}| \geq p \geq 3$.

For all $H \in \mathcal{M}$, we have $H/N \leq G/N$. Since $G/N$ is minimal non-abelian, we deduce that $H/N$ is abelian and hence $H' \leq N$. Since $H' \neq 1$, we get $H' = N$. Thus $d(H) = d(H/N)$.

(1) If $G/N \cong M_p(m, n)$, then $d(H/N) \leq 2$ for all $H \leq G$. Hence $d(H) = 2$ for all $H \in \mathcal{M}$. By Lemma 2.2, $H$ is minimal non-abelian for all $H \in \mathcal{M}$. Since $|\mathcal{M}| \geq 3$, $G$ has two distinct minimal non-abelian subgroups of index $p$.

If $G/N \cong M_p(m, 1, 1)$, then there exists $M \leq G$ such that $M/N = \Omega_{m-1}(G/N)$. Since $d(H/N) \leq 2$ for all $H/N \leq M$, we get $d(H) = 2$ for all $H \in \mathcal{M} \setminus \{M\}$. By Lemma 2.2, $H$ is minimal non-abelian for all $H \in \mathcal{M} \setminus \{M\}$. Since $|\mathcal{M}| \geq 3$, $G$ has two distinct minimal non-abelian subgroups of index $p$.

(2) If $G/N \cong M_p(m, 1, 1)$, where $m \geq n \geq 2$, then, since $d(\Phi(G/N)) = 3$, $d(H/N) = 3$ for all $H/N \leq G/N$. Hence $d(H) = 3$ for all $H \in \mathcal{M}$. By Lemma 2.2, $H$ is not minimal non-abelian for all $H \in \mathcal{M}$. Thus $G$ has no minimal non-abelian subgroups of index $p$. □

Following lemma can be deduced from [4, Theorem 10.10]. We give a simple proof.

**Lemma 2.17.** Suppose that $G$ is not cyclic and of order $p^n$, where $p > 2$ and $n \geq 5$. Then $G$ has a unique metacyclic subgroup of index $p$ if and only if $G$ is a 3-group of maximal class.

**Proof.** “$\Leftarrow$”: Let $G$ be a 3-group of maximal class. By Lemma 2.3 and Lemma 2.9, $G_1$ is the unique metacyclic subgroup of index $p$ of $G$.

“$\Rightarrow$”: Let $M$ be the unique metacyclic subgroup of index $p$. Suppose that $G$ is not a 3-group of maximal class. We shall deduce a contradiction.

Firstly we claim that $G$ has a normal subgroup $E$ which is isomorphic to $C^3_p$. Otherwise, by Lemma 2.15 (consider $r = 3$), $G$ is metacyclic or $G = M_p(1, 1, 1) \ast C_{p^{n-2}}$, or $G = \langle a, b, c \mid a^{p^{n-2}} = b^p = c^p = 1, [a, b] = c, [c, b] = a^{\nu p^{n-3}}, [c, a] = 1 \rangle$ where $\nu = 1$ or a fixed square non-residue modulo $p$. For every case, we can find two distinct metacyclic subgroups of index $p$, a contradiction.

By the cyclic extension theory, we may assume that $M = \langle x, y \mid x^{p^n} = 1, y^{p^{n-1}} = x^p, x^y = x^{1+p^l} \rangle$ where $(i, p) = 1$ and $l \geq 1$. Then $\Phi(M) = \Omega_1(M) = (x^p, y^p)$. Since $p \geq 3$, we have $|\Omega_1(M)| = p^2$.

Since $M \cap E \subseteq \Omega_1(M)$, we have $E \not\subseteq M$. Taking $e \in E \setminus M$, we have $e^p = 1$, $E = \Omega_1(M) \times \langle e \rangle$ and $G = M(e)$. Since $E \leq G$ and $M \leq G$, we deduce that $[M, \langle e \rangle] \leq M \cap E = \Omega_1(M)$.

If $m = 1$, then $M$ is abelian. Since $x \in \Omega_1(M) \leq E$, we get $[x, e] = 1$. Let $N = \langle x, ye \rangle$. Then $N$ is metacyclic and maximal in $G$, which is contrary to the uniqueness of $M$.

If $m = 2$, then $M$ is abelian or minimal non-abelian. Since $M' \leq \Omega_1(M)$ and $[M, \langle e \rangle] \leq M \cap E = \Omega_1(M)$, we deduce that $G/\Omega_1(M)$ is abelian. It follows that $G' \leq \Omega_1(M)$. Since $n \geq 5$, we have $\Omega_1(M) \leq \Phi(M) \leq Z(M)$. It follows that $\Omega_1(M) \leq Z(G)$. Hence $G' \leq Z(G)$ (i.e., $c(G) = 2$). Let $N =$
\(\langle x, y \rangle\). Then \(\overline{\Omega}_1(M) = \langle x^p, y^p \rangle \leq \overline{\Omega}_1(N)\). It follows that \(N\) is maximal in \(G\) and \(|N : \overline{\Omega}_1(N)| = p^2\). By Lemma 2.3, \(N\) is metacyclic, which is again contrary to the uniqueness of \(M\).

If \(m \geq 3\), then \(M' \cap \Omega_1(M) \leq \langle x^p, x \rangle\). By Proposition 2.8, \((x^{-1})^p = x^p\). It follows that \((x^{-1})^p = x^p\). Let \(N = \langle x^{-1}, y \rangle\), then \(\overline{\Omega}_1(M) = \langle x^p, y^p \rangle \leq \overline{\Omega}_1(N)\). It follows that \(N\) is maximal in \(G\) and \(|N : \overline{\Omega}_1(N)| = p^2\). By Lemma 2.3, \(N\) is metacyclic, which is also contrary to the uniqueness of \(M\). \(\Box\)

3. Finite \(p\)-groups with a unique \(A_1\)-subgroup of index \(p\)

Lemma 3.1. Suppose that \(p\) is an odd prime, \(G\) is a finite \(p\)-group and \(M\) is the unique \(A_1\)-subgroup of index \(p\) of \(G\). Then (1) \(|G| \geq p^5\), (2) \(|G| = 2\), (3) \(M/M'\) is the unique abelian maximal subgroup of \(G/M'\).

**Proof.** (1) Since \(M\) is non-abelian, we have \(|G| \geq p^4\). Let \(M = \{H \leq G \mid H' \neq 1\}\) and \(N = \{H \leq G \mid H' = 1\}\). Then \(|M| \geq 1\).

If \(|G| = p^5\), then \(H\) is minimal non-abelian for all \(H \in M\), and so \(|M| = 1\). If \(d(G) = 2\), then \(|M| + |N| = p + 1\). By Lemma 2.13, \(|N| \leq 1\) and \(|M| \geq p\), a contradiction. If \(d(G) \geq 3\), then \(|M| + |N| \geq p^2 + p + 1\). Again by Lemma 2.13, \(|N| \leq p + 1\) and \(|M| \geq p^2\), again a contradiction. Hence \(|G| \geq p^5\).

(2) If \(d(G) \neq 2\), then \(d(G) = 3\), since \(d(M) = 2\). It follows that \(d(G) = \Phi(G) = p^3\). Since \(\Phi(M) \leq \Phi(G)\) and \(\Phi(M)/\Phi(M) = p^3\), we get \(\Phi(G) = \Phi(M)\). Let \(d \in \Phi(M)\). Then \(G = M\). By Lemma 2.2, \(\Phi(M) = Z(M)\). Since \(d \in \Phi(G) = \Phi(M) = Z(M)\), we get \(d \in Z(G)\). Since \(M'\) char \(M \leq G\), we have \(M' \leq G\).

Since \(|M| = p\), we get \(M' \leq Z(G)\).

Since \(G \leq \Phi(G) = \Phi(M) = Z(M)\), we deduce that \(G\) is metabelian. By Proposition 2.8, we have \([h, d]^p = [h, d]^p [h, d] \equiv [h, d]^p [h, d] \equiv [h, d]^p = 1 \mod G_4\) for all \(h \in M\). It follows that \([\Phi(G), G] = [\Phi(M), M] = [\Phi(M), d] = [\overline{\Omega}_1(M'), d] = [\overline{\Omega}_1(M), d] = 1 \mod G_4\). Since \(G_3 = [G_3, G] \subseteq [\Phi(G), G] \subseteq G_4\), we have \(G_3 = 1\) and \([\Phi(G), G] = 1\). Hence \(c(G) = 2\) and \(\Phi(G) \leq Z(G)\). For all \(g, h \in G\) we have \([h, g]^p = [h, g]^p = 1\) and \((gh)^p = g^p h^p\). It follows that \(2p\) divides \(\Phi(G)\), and \(G\) is \(p\)-abelian.

Assume \(N \leq G\), \(d(N) = 2\) and \(N' \neq 1\). Then \(|N'| = p\) since \(c(G) = 2p\). It follows from Lemma 2.2 that \(N\) is non-maximal non-abelian. If \(N\) is maximal in \(G\), then we have \(N = M\) by the uniqueness of \(M\).

In the following, we shall deduce a contradiction by finding an \(N\) such that \(N \neq M, N' \neq 1\), \(d(N) = 2\) and \(N\) is maximal in \(G\).

Case 1. \(M\) is metacyclic.

Let \(M = \langle a, b \rangle \mid a^m = b^n = 1, \{a, b\} = a_{m-1}^b \cong M_3(m, n)\). We may assume that \(d^p = a^pb^{p^2}\) since \(d^p \in \Phi(M) = \langle a, b \rangle \times \langle b^p \rangle\). Since \(G\) is \(p\)-abelian, we have \((d^{-1}a^{-1}b)^p = 1\). Replacing \(d\) with \((a^{-1}b)^{-1}d\), we get \(G = \langle a, b \rangle \times \langle d \rangle\), where \(d^p = 1\). Let \(N = \langle a, c \rangle \times \langle b \rangle\), where \(|a| = w\) and \(|b| = 1\). Then \(|N| = m = d(N), N' = 1\) and \(N\) is maximal in \(G\), a contradiction.

Case 2. \(M\) is not metacyclic.

Let \(M = \langle a, b \rangle \mid a^m = b^n = c^n = 1, \{a, b\} = c, \{c, a\} = \{c, b\} = 1 \cong M_3(m, n, 1)\) where \(m \geq n\). We have \(m \geq 2\) since \(|G| \geq p^3\). Since \(d^p \in \Phi(M) = \langle a, b \rangle, c \rangle\), we may assume that \(d^p = a^pb^{p^2}c^k\). Since \(G\) is \(p\)-abelian, we have \((d^{-1}a^{-1}b)^p = c^k\). Replacing \(d\) with \((a^{-1}b)^{-1}d\), we have \(G = \langle a, b \rangle \langle d \rangle\), where \(d^p = c^k\).

Subcase 2a: \(k = 0\) or \(m \geq 3\).

Since \(\{a, b\} = c \notin \overline{\Omega}_1(M)\), there exists \(w\) such that \(w \mid w\) and \(|a, b| \neq \overline{\Omega}_1(M)\). Let \(N = \langle a, b \rangle \langle d \rangle\). Then \(\overline{\Omega}_1(N) = \langle a^p, b^p \rangle \times \langle b^p \rangle\) since \(G\) is \(p\)-abelian. Since \(k = 0\) or \(m \geq 3\), we have \(\overline{\Omega}_1(M) = \langle a^p, b^p \rangle \times \langle b^p \rangle\). Hence \(\overline{\Omega}_1(N) = \langle a^p, b^p \rangle \times \langle b^p \rangle\). It follows that \(|a, b| \neq \overline{\Omega}_1(M)\) that \(|a, b| \neq \overline{\Omega}_1(N)\). Since \(|a, b| \neq \overline{\Omega}_1(M)\) that \(|a, b| \neq \overline{\Omega}_1(N)\). Then \(N \neq M\), \(d(N) = 2\), \(N' \neq 1\) and \(N\) is maximal in \(G\), a contradiction.
Subcase 2b: \( k \neq 0 \), \( m = 2 \) and \( n = 1 \).

In this subcase, \( |G| = p^5 \). Let \( N = \langle a, bd^w \rangle \), where \( p \nmid w \) and \( [a, bd^w] \neq 1 \). Then \( N \nmid M \), \( d(N) = 2 \), \( N' \neq 1 \) and \( N \) is maximal in \( G \), a contradiction.

Subcase 2c: \( k \neq 0 \) and \( m = n = 2 \).

If \( [d, b] \not\in (b^p, c) \), letting \( N = \langle d, b \rangle \), then \( N \nmid M \), \( d(N) = 2 \), \( N' \neq 1 \) and \( N \) is maximal in \( G \), a contradiction. If \( [d, b] \in (b^p, c) \), then we may assume that \( [d, b] = b^p \). Let \( N = \langle ad^w, b \rangle \) where \( p \nmid w \) and \( p | 1 + cw \). Then \( N \nmid M \) and \( d(N) = 2 \). Since \( [ad^w, b] = c^{1+w}b^{wp} \notin U_1(N) \), we have \( N' \neq 1 \) and \( N \) is maximal in \( G \), a contradiction.

(3) If \( G/M' \) has two abelian maximal subgroups, then \( G/M' \) has \( p + 1 \) abelian maximal subgroups by Lemma 2.13. Since \( d(G) = 2 \), \( G/M' \) is minimal non-abelian. By Lemma 2.16, \( G \) cannot have a unique \( A_1 \)-subgroup of index \( p \), a contradiction. \( \square \)

According to Lemma 3.1, we need research two-generator \( p \)-groups with a unique abelian maximal subgroup \( M \) such that \( d(M) = 2 \).

**Lemma 3.2.** Suppose that \( p \) is an odd prime, \( G \) is a finite \( p \)-group and \( M \) is the unique abelian maximal subgroup of \( G \). If \( G \) is regular, \( |G| = p^n \) and \( d(G) = d(M) = 2 \), then \( p \geq 5 \) and \( G = \langle a, b, c \mid a^{p^{n-2}} = b^p = c^p = 1, [a, b] = c, [c, b] = a^{mp-3}, [c, a] = 1 \rangle \), where \( (i, p) = 1 \).

**Proof.** Taking \( b \in G \setminus M \) and \( a \in M \setminus \Phi(G) \), then \( G = \langle a, b \rangle \). Since \( M \) is abelian and \( b^p \in Z(G) \), by Lemma 2.7, for all \( d \in M \) we have \( [d^p, b] = [d, b^p] = 1 \). It follows that \( \Phi(M) = U_1(M) \leq Z(G) \).

We claim that \( \Phi(M) = Z(G) \). Otherwise, \( |G/Z(G)| = |M/\Phi(M)| = p^2 \). It follows that \( Z(G) = \Phi(G) \).

By Lemma 2.2, \( G \) is minimal non-abelian, which is contrary to the uniqueness of \( M \).

By Lemma 2.14, \( G' \equiv M/M \cap Z(G) = M/\Phi(M) \cong C_p \). It follows from Lemma 2.6 that \( p \geq 5 \). By Lemma 2.4, \( G \) is \( p \)-abelian. Since \( b^p \in Z(G) = \Phi(M) \), there \( d \in M \) such that \( b^p = d^p \). Replacing \( b \) with \( bd^{-1} \), we have \( b^p = 1 \).

Let \( [a, b] = c \). Then \( c^p = 1 \). Since \( G' \equiv C_p \), we have \( c(G) = 3 \) and \( c \notin Z(G) = \Phi(M) \). Hence \( c \in \Phi(G) \setminus \Phi(M) \). Noting \( a \in M \setminus \Phi(G) \), \( M = \langle a, c \rangle \cong C_{p^{n-2}} \times C_p \). Since \( [c, a] = 1 \), we have \( 1 \neq [c, b] \in \Omega_1(Z(G)) = \Omega_1(\Phi(M)) = \langle a^{p^{n-3}} \rangle \). Thus we may assume that \( [c, b] = a^{mp^{n-3}} \), where \( (i, p) = 1 \). \( \square \)

**Lemma 3.3.** Suppose that \( p \) is an odd prime, \( G \) is a finite \( p \)-group and \( M \) is the unique \( A_1 \)-subgroup of index \( p \) of \( G \). Let \( L = \langle a, b, c \mid a^{p^{n-2}} = b^p = c^p = 1, [a, b] = c, [c, b] = a^{mp^{n-3}}, [c, a] = 1 \rangle \), where \( (i, p) = 1 \). If \( G \) is not a 3-group of maximal class, then \( G/M' \neq L \).

**Proof.** Assume that \( \tilde{G} = G/M' = \langle \tilde{a}, \tilde{b}, \tilde{c} \rangle \cong L \). Since \( \langle \tilde{a}, \tilde{c} \rangle \) is the unique abelian maximal subgroup of \( G \), \( M/M' = \langle \tilde{a}, \tilde{c} \rangle \). It follows that \( M = \langle a, c \rangle \). Let \( [c, a] = d \). Then \( d^p = 1 \). Since \( \langle d \rangle = M' \leq Z(G) \), we get \( G = \langle a, b, c, d \rangle \) with the following relations:

\[
a^{p^{n-2}} = d^t, \quad b^p = d^s, \quad c^p = d^f, \quad [a, b] = cd^u, \quad [c, a] = d, \quad [c, b] = a^{mp^{n-3}} d^v.
\]

Replacing \( c \) with \( cd^u \), we have \( [a, b] = c \). Since \( c^p \in Z(G) \), we get \( [c, b]^p = [c^p, b] = 1 \). It follows that \( a^{p^{n-2}} = 1 \).

If \( p \geq 5 \), then \( G \) is regular by Lemma 2.5(2). By Lemma 2.7, \( [a^p, b] = [a, b^p] = 1 \) and hence \( [c, b] \in Z(G) \). It follows that \( c(G) = 3 \) and \( G_3 = \langle d, a^{p^{n-3}} \rangle \).

If \( p = 3 \), then \( d^p \in Z(G) \) since \( [d^3, b] \in \langle d \rangle \). If \( n \geq 5 \), then \( [c, b] \in Z(G) \). It follows that \( c(G) = 3 \) and \( G_3 = \langle d, a^{p^{n-3}} \rangle \). If \( n = 4 \), then we also have \( c(G) = 3 \) and \( G_3 = \langle d, a^{p^{n-3}} \rangle \) since \( G \) is not a 3-group of maximal class.

In summary, we have \( c(G) = 3 \) and \( G_3 = \langle d, a^{p^{n-3}} \rangle \). Let \( N = \langle ab^w, c \rangle \), where \( p \nmid w \) and \( [c, ab^w] \notin \langle a^{p^{n-3}} \rangle \). Then \( N \neq M \) and \( N \) is minimal non-abelian and is maximal in \( G \), which is contrary to the uniqueness of \( M \). \( \square \)
Corollary 3.4. Suppose that $p$ is an odd prime, $G$ is a finite $p$-group and $M$ is the unique $A_1$-subgroup of index $p$ of $G$. Then $G/M'$ is irregular. In particular, $G$ is irregular and $p = 3$.

Proof. It follows from Lemma 3.1, Lemma 3.2 and Lemma 3.3 that $G/M'$ is irregular. Hence $G$ is irregular. Since $|M : \bar{\Omega}_1(M)| \leq p^3$ and $\bar{\Omega}_1(M) \leq \bar{\Omega}_1(G)$, we have $|G : \bar{\Omega}_1(G)| \leq p^4$. It $p \geq 5$, then $G$ is absolutely regular, a contradiction. Hence $p = 3$. □

Corollary 3.5. Suppose that $p$ is an odd prime, $G$ is a finite $p$-group and $M$ is the unique $A_1$-subgroup of index $p$ of $G$. If $|G| \leq p^3$, then $G$ is one of the following non-isomorphic groups:

1. Groups of order $3^5$, which are of maximal class and have no abelian maximal subgroup;
2. $(a, b, c | a^2 = b^3 = c^3 = 1, [b, a] = c, [c, a] = a^3, [c, b] = b^{-3})$.

Proof. By Lemma 3.1, $|G| = p^5, d(G) = 2$. By Corollary 3.4, $p = 3$. Checking the list of groups of order $3^5$, we get groups listed in the Corollary. □

Theorem 3.6. Suppose that $p$ is an odd prime, $G$ is a finite $p$-group with $|G| \geq p^6$. Then $G$ has a unique $A_1$-subgroup of index $p$ if and only if $G$ has no abelian maximal subgroup and has a maximal quotient which is a 3-group of maximal class and contains an abelian maximal subgroup.

Proof. "⇒": Let $M$ be the unique $A_1$-subgroup of index $p$ of $G$, $\bar{G} = G/M'$ and $\bar{M} = M/M'$. By Lemma 3.1, $d(G) = d(\bar{G}) = 2$ and $M$ is the unique abelian maximal subgroup of $\bar{G}$. By Corollary 3.4, $\bar{G}$ is irregular and $p = 3$.

Let $\bar{E}$ be a normal elementary abelian subgroup of $\bar{G}$. We claim that $|E| \leq 3^2$. Otherwise, $\bar{E} \supseteq \bar{M}$ since $d(\bar{M}) = 2$. Then $\bar{G} = \bar{M}E$ and hence $\bar{G} \leq \bar{M} \cap \bar{E} \leq Z(\bar{G})$. It follows that $c(\bar{G}) = 2$ and hence $\bar{G}$ is regular, a contradiction. By Lemma 2.15 and Lemma 3.3, $\bar{G}$ is a $3$-group of maximal class. The remains are obvious.

"⇐": Let $N \trianglelefteq G$, $|N| = 3$ and $G/N$ is a $3$-group of maximal class having an abelian maximal subgroup $M/N$. By hypothesis, $M$ is non-abelian. It follows that $M' = N$. By Lemma 2.10, $d(M/N) = 2$. Hence $d(M) = 2$. It follows from Lemma 2.2 that $M$ is minimal non-abelian. By Lemma 2.9, maximal subgroups of $G/N$ except $M/N$ are of maximal class. Hence for every $H \not\leq G$ and $H \neq M$, $H$ is not minimal non-abelian. Thus $M$ is the unique $A_1$-subgroup of index $p$ of $G$. □

Theorem 3.7. Suppose that $p$ is an odd prime, $G$ is a finite $p$-group, $|G| \geq p^6$ and $M$ is the unique $A_1$-subgroup of index $p$ of $G$. Then $M$ is metacyclic if and only if $G$ is a $3$-group of maximal class in which $G_1$ is non-abelian.

Proof. "⇐": If $G$ is a $3$-group of maximal class in which $G_1$ is non-abelian, then, by Lemma 2.10, $G_1$ is metacyclic and minimal non-abelian. Since $M$ is the only maximal subgroup of $G_1$ which is minimal non-abelian, we deduce that $M = G_1$ is metacyclic.

"⇒": By Theorem 3.6, $G/M'$ is a $3$-group of maximal class. By Lemma 2.9, maximal subgroups of $G/M'$ except $M/M'$ are of maximal class. Hence for every $H \not\leq G$ and $H \neq M$, $H$ is not metacyclic. It follows that $M$ is the unique metacyclic subgroup of index $p$ of $G$. By Lemma 2.17, $G$ is a $3$-group of maximal class. □

In the following, we determine finite $p$-groups in which the unique $A_1$-subgroup of index $p$ is not metacyclic.

Theorem 3.8. Suppose that $G$ is a finite $3$-group, $M$ is the unique $A_1$-subgroup of index $3$. If $M$ is not metacyclic and $|G| = 3^{2e+2}$ where $e \geq 2$, then $G$ is one of the following non-isomorphic groups (where $k = 0, 1$ or $2$):

1. $(s_1, s_2, \beta, x | s_1^{3^e} = s_2^{3^e} = x^3 = 1, \beta^3 = x^k, [s_1, \beta] = s_2, [s_2, \beta] = s_2^{-3}s_1^{-3}, [s_1, s_2] = x, [x, \beta] = 1)$;
(2) $\langle s_1, s_2, \beta, x | s_1^3 = s_2^3 = x^3 = 1, \beta^3 = s_2^{s_1^{-1}}x^k, [s_1, \beta] = s_2, [s_2, \beta] = s_2^{-3}s_1^{-3}, [s_1, s_2] = x, [x, s_1] = [x, \beta] = 1 \rangle$;
(3) $\langle s_1, s_2, \alpha, \beta, x | s_1^3 = s_2^{x^3} = x^3 = 1, \beta^3 = x^k, \alpha^3 = s_1^{-3}s_2^{-1}s_1^{-1}, [\alpha, \beta] = s_1, [s_1, \alpha] = x, [s_1, \beta] = s_2, [s_2, \beta] = s_2^{-3}s_1^{-3}, [s_1, s_2] = [x, \alpha] = [x, \beta] = 1 \rangle$;
(4) $\langle s_1, s_2, \alpha, \beta, x | s_1^3 = s_2^{x^3} = x^3 = 1, \beta^3 = x^k, \alpha^3 = s_1^{-1}s_2^{-1}s_1^{-1}x, [\alpha, \beta] = s_1, [s_1, \alpha] = x, [s_1, \beta] = s_2, [s_2, \beta] = s_2^{-3}s_1^{-3}, [s_1, s_2] = [x, \alpha] = [x, \beta] = 1 \rangle$.

**Proof.** Firstly, we prove that $G$ is one of the groups listed in theorem. By Theorem 3.6, $\tilde{G} = G/M'$ is a 3-group of maximal class having an abelian maximal subgroup. Since $|G| = 3^{2e+1}$, $G$ is isomorphic to one of the groups listed in Lemma 2.11(1).

**Case 1.** $\tilde{G}$ is isomorphic to the group of Type (1a) in Lemma 2.11.

Let $\tilde{G} = \langle \tilde{s}_1, \tilde{s}_2, \tilde{\beta} | \tilde{s}_1^3 = \tilde{\beta}^3 = 1, [\tilde{s}_1, \tilde{\beta}] = \tilde{s}_2, [\tilde{s}_2, \tilde{\beta}] = \tilde{s}_2^{-3}\tilde{s}_1^{-3}, [\tilde{s}_1, \tilde{s}_2] = 1 \rangle$. Then $\tilde{M} = \tilde{G}_1 = \langle \tilde{s}_1, \tilde{s}_2 \rangle$ is the unique abelian maximal subgroup of $\tilde{G}$, and for all $\tilde{g} \in \tilde{G} \setminus \tilde{G}_1$, $\tilde{g}^3 = 1$. Let $[s_1, s_2] = x$. Then $x \in Z(G)$. Since $M = \langle s_1, s_2 \rangle$ is not metacyclic, we get $s_1^3 = s_2^3 = 1$. Hence $G = \langle s_1, s_2, \beta, x \rangle$ with the following relations:

$$s_1^3 = s_2^3 = x^3 = 1, \quad \beta^3 = x^k, \quad [s_1, \beta] = s_2x^j, \quad [s_2, \beta] = s_2^{-3}s_1^{-3}x^i, \quad [s_1, s_2] = x.$$ 

Since $s_1^3, \beta^3 = 1$, we may assume that $(s_1^3, \beta^3) = x^k$ without losing generality. Replacing $\beta$ with $s_1\beta$, we have $[s_1, \beta] = s_2^{-3}s_1^{-3}$. Replacing $s_2$ with $s_2x^i$, we have $[s_1, \beta] = s_2$. Thus we get groups of Type (1) in theorem.

**Case 2.** $G/M'$ is isomorphic to the group of Type (1b) in Lemma 2.11.

Let $G/M' = \langle s_1, s_2, \beta | s_1^3 = s_2^3 = \beta^3 = 1, [s_1, \beta] = s_2, [s_2, \beta] = s_2^{-3}s_1^{-3}, [s_1, s_2] = 1 \rangle$. Then $\tilde{M} = \tilde{G}_1 = \langle \tilde{s}_1, \tilde{s}_2 \rangle$ is the unique abelian maximal subgroup of $\tilde{G}$, and for all $\tilde{g} \in \tilde{G} \setminus \tilde{G}_1$, $\tilde{g}^3 = 1$. Let $[s_1, s_2] = x$. Then $x \in Z(G)$. Since $M = \langle s_1, s_2 \rangle$ is not metacyclic, we get $s_1^3 = s_2^3 = 1$. Hence $G = \langle s_1, s_2, \beta, x \rangle$ with the following relations:

$$s_1^3 = s_2^3 = x^3 = 1, \quad \beta^3 = s_2^x, \quad [s_1, \beta] = s_2x^j, \quad [s_2, \beta] = s_2^{-3}s_1^{-3}x^i, \quad [s_1, s_2] = x.$$ 

For all integer $i$, we have $(s_1^3, \beta^3) = s_2^{3x^3}x^k$. Hence we may assume that $(s_1^3, \beta^3) = s_2^{3x^3}x^k$, where $(w, 3) = 1$ without losing generality. Replacing $s_1$, $\beta$ and $s_2$ with $s_1^w$, $s_1\beta$ and $s_2^w$, $s_1^w \beta$ respectively, we have $[s_1, \beta] = s_2$ and $[s_2, \beta] = s_2^{-3}s_1^{-3}$. Thus we get groups of Type (2) in theorem.

**Case 3.** $G/M'$ is isomorphic to the group of Type (1c) in Lemma 2.11.

Let $G/M' = \tilde{G} = \langle \tilde{s}_1, \tilde{s}_2, \tilde{\alpha}, \tilde{\beta} | \tilde{s}_1^3 = \tilde{\beta}^3 = 1, [\tilde{s}_1, \tilde{\alpha}] = \tilde{s}_2, [\tilde{s}_1, \tilde{\beta}] = \tilde{s}_2, [\tilde{s}_2, \tilde{\beta}] = \tilde{s}_2^{-3}s_1^{-3}, [\tilde{s}_1, \tilde{s}_2] = 1 \rangle$. Then $M = \tilde{G}_1 = \langle \tilde{s}_1, \tilde{s}_2 \rangle$ is the unique abelian maximal subgroup of $\tilde{G}$. Let $[s_1, \alpha] = x$. Then $x \in Z(G)$. Since $M = \langle s_1, \beta \rangle$ is not metacyclic, we get $s_1^3 = \alpha^3 = 1$. Hence $G = \langle s_1, s_2, \beta, \alpha, x \rangle$ with the following relations:

$$s_1^3 = s_2^{-3}s_1^{-3} = x^3 = 1, \quad \beta^3 = x^k, \quad [\alpha, \beta] = s_1x^u, \quad [s_1, \beta] = s_2x^v, \quad [s_2, \beta] = s_2^{-3}s_1^{-3}x^i, \quad [s_1, \alpha] = x.$$ 

By Proposition 2.8, $1 = [\alpha, \beta^3] = [\alpha, \beta^3][\alpha, \beta][\alpha, \beta, \beta] = x^k$. Hence $[s_2, \beta] = s_2^{-3}s_1^{-3}$. Replacing $s_1$ and $s_2$ with $s_1x^u$ and $s_2x^i$ respectively, we have $[\alpha, \beta] = s_1$ and $[s_1, \beta] = s_2$.

If $\alpha^3 = s_1^{-1}s_2^{-1}s_1^{-1}$, then we get groups of Type (3) in theorem. If $\alpha^3 = s_1^{-3}s_2^{-1}s_1^{-1}$, then we get groups of Type (4) in theorem. If $\alpha^3 = s_1^{-1}s_2^{-1}s_1^{-1}x^2$, then, replacing $\alpha, s_1$ and $s_2$ with $\alpha^2, s_1^2x$ and $s_2^2$ respectively, we also get groups of Type (4) in theorem.
Next, we show that they are not mutually isomorphic. For distinct $k$, $G/\mathcal{U}_1(M)$ are non-isomorphic groups of maximal class with order $3^k$. Hence distinct parameter $k$ gives non-isomorphic groups in theorem. Since $G/M'$ is isomorphic to the group of Type (1a) in Lemma 2.11 for Type (1), the group of Type (1b) for Type (2) and the group of Type (1c) for Type (3) and (4), we only need prove that groups of Type (3) is not isomorphic to groups of Type (4). Let $N$ be the maximal subgroup of $G$ such that $N/\mathcal{U}_1(M)$ is the unique abelian maximal subgroup of $G/\mathcal{U}_1(M)$. Then $N = \langle s_1, \beta, x \rangle$ if $G$ is of Type (3) and $N = \langle s_1, \alpha \beta^{-1}, x \rangle$ if $G$ is of Type (4). By calculation, we get $N/M'$ is a group of Type (2a) in Lemma 2.11 for Type (3) and is a group of Type (2b) in Lemma 2.11 for Type (4). Hence groups of Type (3) is not isomorphic to groups of Type (4).

It is trivial to check that groups in theorem satisfy conditions in theorem. □

By a similar argument as that of Theorem 3.8, we have the following

**Theorem 3.9.** Suppose that $G$ is a finite 3-group, $M$ is the unique $A_1$-subgroup of index 3. If $M$ is not metacyclic and $|G| = 3^{2k+1}$, where $e \geq 3$, then $G$ is one of the following non-isomorphic groups (where $k = 0, 1$ or 2, $v = 1$ or 2):

1. $\langle s_1, s_2, \beta \mid s_1^{s-1} = x^3 = 1, \beta^3 = \alpha^k, [s_1, \beta] = s_2, [s_2, \beta] = s_2^{-3}s_1^{-3}, [s_1, s_2] = x, [x, s_1] = [x, s_2] = 1 \rangle$;
2. $\langle s_1, s_2, \beta \mid s_1^{s-1} = x^3 = 1, \beta^3 = s_1^{-3}s_2^{-3}s_1^{-3}, [s_1, \beta] = s_2, [s_2, \beta] = s_2^{-3}s_1^{-3}, [s_1, s_2] = x, [x, s_1] = [x, s_2] = 1 \rangle$;
3. $\langle s_1, s_2, \beta, \alpha \mid s_1^{s-1} = x^3 = 1, \beta^3 = \alpha^k, \alpha^3 = s_1^{-3}s_2^{-1}s_1^{-3}, [\alpha, \beta] = s_1, [s_1, \beta] = s_2, [s_2, \beta] = s_2^{-3}s_1^{-3}, [s_1, \alpha] = x, [x, \beta] = [x, \alpha] = [s_1, s_2] = 1 \rangle$;
4. $\langle s_1, s_2, \beta, \alpha \mid s_1^{s-1} = x^3 = 1, \beta^3 = \alpha^k, \alpha^3 = s_1^{-3}s_2^{-1}s_1^{-3}s_2^{-1}, [\alpha, \beta] = s_1, [s_1, \beta] = s_2, [s_2, \beta] = s_2^{-3}s_1^{-3}, [s_1, \alpha] = x, [x, \beta] = [x, \alpha] = [s_1, s_2] = 1 \rangle$.

In summary, we have following

**Main Theorem.** Suppose that $G$ is a finite $p$-group with a unique $A_1$-subgroup of index $p$. Then $G$ is one of groups listed in Lemma 2.12, Corollary 3.5, Theorem 3.8 and Theorem 3.9.

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**References**


