



New identities among gauge theory amplitudes

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ABSTRACT

Color-ordered amplitudes in gauge theories satisfy non-linear identities involving amplitude products of different helicity configurations. We consider the origin of such identities and connect them to the Kawai–Lewellen–Tye (KLT) relations between gravity and gauge theory amplitudes. Extensions are made to one-loop order of the full $\mathcal{N} = 4$ super Yang–Mills multiplet.

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In a recent paper [1] we have proven a series of non-linear identities among gauge theory amplitudes at tree level. These identities were discovered accidentally in the process of giving a field theory proof of the Kawai–Lewellen–Tye (KLT) relations [2] between gravity and gauge theory amplitudes. The KLT-relations express particular combinations of products of two n -point color-ordered gauge theory amplitudes explicitly in terms of an n -point graviton amplitude. It is crucial here that the helicities in the product match: The graviton helicities ± 2 arise because the corresponding gauge theory helicities are chosen, correspondingly, as $(\pm 1) \times (\pm 1)$. There is thus a direct connection to the factorization properties of helicities of massless external states. What happens if helicities do not match? In Ref. [1] we proved for all n that if the helicity of just one leg is flipped, the particular combinations of products that enter the right-hand side of the KLT-relations *vanish*. Viewed from the gauge theory side this gives a series of quite unusual identities among color-ordered amplitudes.

One of the surprises in amplitude calculations in recent years was the discovery by Bern, Carrasco and Johansson (BCJ) of a new set of linear relations among color-ordered gauge theory amplitudes [3] (see also the extension to scalar and fermionic matter in [4]). These identities were shown to follow also from the field theory limit of string theory [5,6] and in Ref. [5] it was proven that this implies that the minimal basis of color-order gauge theory amplitudes is of size $(n-3)!$, rather than the $(n-2)!$ that would be inferred from just photon decoupling and Kleiss–Kuijf relations [7,8]. Very recently, a purely field theoretic proof of all these identities has been presented [9], see also [10]. An alternative understanding of the BCJ-relations has also been achieved through a study of the generalized Jacobi-like identities they imply for residues of poles [11]. Squaring-relations of these pole structures yield alternative forms of KLT-relations. As shown recently by Tye and Zhang in the second paper of Ref. [11], this actually follows from the field theory limit of the heterotic string. Such squaring relations even seem to hold beyond tree level [12].

While the new identities we present in this Letter do not have the power of BCJ-relations, and in particular by necessity cannot reduce the $(n-3)!$ growth of the amplitude basis, the identities themselves are so striking and unusual that they deserve further study. As we will show below, some of the identities carry over to at least one-loop level. All of these identities are on the gauge theory side alone, with no relation to gravity. Nevertheless, an understanding of these identities from the point of view of string theory could be most interesting.

2. New form of KLT and gauge theory identities

We will here briefly review the results of Ref. [1] and phrase them in a new form which is more practical. Let $A_n(1, 2, \dots, n)$ and $\tilde{A}_n(1, 2, \dots, n)$ denote n -point color-ordered gauge theory amplitudes of fixed helicity. Furthermore, let

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$$\mathcal{S}[i_1, \dots, i_k | j_1, \dots, j_k]_{p_1} \equiv \prod_{t=1}^k \left(s_{i_t 1} + \sum_{q>t}^k \theta(i_t, i_q) s_{i_t i_q} \right), \quad (1)$$

where $s_{12\dots i} \equiv (p_1 + \dots + p_i)^2$ and $\theta(i_a, i_b)$ is zero if i_a sequentially comes before i_b in $\{j_1, \dots, j_k\}$. Otherwise it is unity. The function \mathcal{S} has the following symmetry $\mathcal{S}[i_1, \dots, i_k | j_1, \dots, j_k] = \mathcal{S}[j_k, \dots, j_1 | i_k, \dots, i_1]$. We also introduce a dual $\tilde{\mathcal{S}}$ defined by

$$\tilde{\mathcal{S}}[i_1, \dots, i_k | j_1, \dots, j_k]_{p_n} \equiv \prod_{t=1}^k \left(s_{j_t n} + \sum_{q<t}^k \theta(j_q, j_t) s_{j_t j_q} \right), \quad (2)$$

where again $\theta(j_a, j_b)$ is zero if j_a sequentially comes before j_b in $\{i_1, \dots, i_k\}$. Otherwise it is unity.

A main result of [1] was the proof of the following equation (we will throughout use the shorthand notation: $\gamma_{2,n-1}$ for the ordering of legs 2, 3, ..., $n-1$ in the amplitude \tilde{A}_n and likewise $\beta_{2,n-1}$ for the order in A_n).

$$X_n^{(n_+, n_-)} = \sum_{\gamma, \beta} \frac{\tilde{A}_n(n, \gamma_{2,n-1}, 1) \mathcal{S}[\gamma_{2,n-1} | \beta_{2,n-1}]_{p_1} A_n(1, \beta_{2,n-1}, n)}{s_{12\dots(n-1)}}, \quad (3)$$

where n_+ (n_-) denotes the number of positive (negative) helicity legs in A_n which is changed to negative (positive) helicity legs in \tilde{A}_n and we sum over all permutations of legs in sets γ and β . When no helicities are changed we obtain the gravity amplitude, i.e. $X_n^{(0,0)} = (-1)^n M_n(1, 2, \dots, n)$ with $M_n(1, 2, \dots, n)$ denoting the n -point gravity amplitude. However, just by flipping one helicity on one of the two gluon amplitudes we have $X_n^{(1,0)} = X_n^{(0,1)} = 0$. These are quite surprising identities for gluon amplitudes of definite helicities.

On-shell, the identities implied by (3) have both vanishing numerator and denominator, and are therefore ill-defined. However, as explained in Ref. [1], one should understand the expression in terms of a regularization that takes the n 'th leg off-shell. In detail, this can be achieved, for instance, by shifting momenta as follows: $p_1 \rightarrow p_1 - xq$, and $p_n \rightarrow p_n + xq$. Here x is an arbitrary parameter and $q^2 = 0 = p_1 \cdot q$ but $p_n \cdot q \neq 0$. This clearly preserves overall energy-momentum conservation, keeps the external leg 1 on-shell, but makes $p_n^2 = s_{12\dots(n-1)} \neq 0$. The expression (3) is then well-defined, and one obtains the correct result as the limit of $x \rightarrow 0$. How to systematically take this limit was explained in [1] (see also [13]).

Although Eq. (3) has the advantage of being manifestly symmetric in $(n-2)!$ legs, it is for practical purposes more convenient to use an equivalent form [13] which has fewer terms and does not require regularization. Such a general form is given by

$$X_n^{(n_+, n_-)} = - \sum_{\sigma \in S_{n-3}} \sum_{\tilde{\alpha} \in S_{j-2}} \sum_{\beta \in S_{n-1-j}} \tilde{A}_n(\tilde{\alpha}(\sigma_{2,j-1}), 1, n-1, \tilde{\beta}(\sigma_{j,n-2}), n) \mathcal{S}[\tilde{\alpha}(\sigma_{2,j-1}) | \sigma_{2,j-1}]_{p_1} \times \tilde{\mathcal{S}}[\sigma_{j,n-2} | \tilde{\beta}(\sigma_{j,n-2})]_{p_{n-1}} A_n(1, \sigma_{2,j-1}, \sigma_{j,n-2}, n-1, n). \quad (4)$$

The particular KLT-expression that was conjectured in Ref. [14] is equivalent to (4) in the special case of $j = [n/2 + 1]$. However, Eq. (4) is more general and valid for any j . Particularly interesting forms arise when we take either the left or the right j -set empty. Then we get two highly symmetric relations:

$$X_n^{(n_+, n_-)} = - \sum_{\gamma, \beta \in S_{n-3}} \tilde{A}_n(n-1, n, \gamma_{2,n-2}, 1) \mathcal{S}[\gamma_{2,n-2} | \beta_{2,n-2}]_{p_1} A_n(1, \beta_{2,n-2}, n-1, n), \quad (5)$$

and

$$X_n^{(n_+, n_-)} = - \sum_{\gamma, \beta \in S_{n-3}} A_n(1, \beta_{2,n-2}, n-1, n) \tilde{\mathcal{S}}[\beta_{2,n-2} | \gamma_{2,n-2}]_{p_{n-1}} \tilde{A}_n(1, n-1, \gamma_{2,n-2}, n). \quad (6)$$

It is interesting to observe how Eq. (5) resembles the numerator of Eq. (3). The difference lies only in the number of legs being permuted. Note that both of the two gauge amplitudes are expanded in a minimal basis of $(n-3)!$ amplitudes. However, the basis of amplitudes for A_n is not the same as that of \tilde{A}_n . This is a very simple form of the KLT-relations and the new gauge theory identities.

3. Flipping several helicities

As noted, one can generate new identities among gauge theory amplitudes by simply flipping the helicity of one leg in the n -point KLT relation. In this section we consider the cases with more than one flipped helicity. We begin by writing out some explicit examples in the form of Eq. (5).

In the 4-point case $\mathcal{S}[2|2] = s_{12}$ so that

$$X_4^{(n_+, n_-)} = -s_{12} A_4(1, 2, 3, 4) \tilde{A}_4(3, 4, 2, 1), \quad (7)$$

which gives trivial zeros when $n_+ \neq n_-$. For example, with $(n_+, n_-) = (0, 1)$ we get

$$0 = -s_{12} A_4(1^-, 2^-, 3^+, 4^+) \tilde{A}_4(3^+, 4^+, 2^+, 1^-), \quad (8)$$

which expresses nothing but the standard MHV helicity selection rule.

In the 5-point case we have

$$X_5^{(n_+, n_-)} = -s_{12} A_5(1, 2, 3, 4, 5) [s_{13} \tilde{A}_5(4, 5, 2, 3, 1) + (s_{13} + s_{23}) \tilde{A}_5(4, 5, 3, 2, 1)] - s_{13} A_5(1, 3, 2, 4, 5) [s_{12} \tilde{A}_5(4, 5, 3, 2, 1) + (s_{12} + s_{23}) \tilde{A}_5(4, 5, 2, 3, 1)]. \quad (9)$$

Explicit calculations show that we get zeros in cases like (e.g. with $(n_+, n_-) = (1, 0)$)

$$0 = s_{12}A_5(1^-, 2^-, 3^+, 4^+, 5^+) [s_{13}\tilde{A}_5(4^+, 5^+, 2^-, 3^-, 1^-) + (s_{13} + s_{23})\tilde{A}_5(4^+, 5^+, 3^-, 2^-, 1^-)] \\ + s_{13}A_5(1^-, 3^+, 2^-, 4^+, 5^+) [s_{12}\tilde{A}_5(4^+, 5^+, 3^-, 2^-, 1^-) + (s_{12} + s_{23})\tilde{A}_5(4^+, 5^+, 2^-, 3^-, 1^-)]. \quad (10)$$

In contrast to the 4-point case, this is already a new non-trivial identity. We also get zero when we do helicity flips in the category of $(n_+, n_-) = (2, 1)$, $(n_+, n_-) = (2, 0)$, or $(n_+, n_-) = (3, 2)$.

Finally, we give the explicit expression for the 6-point case,

$$X_6^{(n_+, n_-)} = -s_{12}s_{13}A_6(1, 2, 3, 4, 5, 6) [s_{14}\tilde{A}_6(5, 6, 2, 3, 4, 1) + (s_{14} + s_{34})\tilde{A}_6(5, 6, 2, 4, 3, 1) + (s_{14} + s_{34} + s_{24})\tilde{A}_6(5, 6, 4, 2, 3, 1)] \\ - s_{12}(s_{13} + s_{23})A_6(1, 2, 3, 4, 5, 6) [s_{14}\tilde{A}_6(5, 6, 3, 2, 4, 1) + (s_{14} + s_{24})\tilde{A}_6(5, 6, 3, 4, 2, 1) \\ + (s_{14} + s_{24} + s_{34})\tilde{A}_6(5, 6, 4, 3, 2, 1)] + \mathcal{P}(2, 3, 4), \quad (11)$$

where we similarly get vanishing relations in all non-trivial cases where $n_+ \neq n_-$. From these simple examples we seem to extract the following general rule:

$$n_+ \neq n_- \implies X_n^{(n_+, n_-)} = 0. \quad (12)$$

This is what we will show below.

4. Proof of new gauge theory identities with several flipped helicities

To simplify the proof of (12) we use the form of Eq. (3). The reason is that Eq. (3) is manifestly symmetric in $(n-2)!$ legs. Choosing the two remaining legs as 1 and n , this is ideally suited for a proof based on BCFW-recursion [15].

We do the proof by induction. We thus assume that we have verified the rule (12) for the identities up to $n-1$ points and now we want to show that this implies the rule at n points. We thus look at the n -point identity and imagine having changed n_+ of the positive helicity legs and $n_- \neq n_+$ of the negative helicity legs in \tilde{A}_n (compared to A_n). Doing a BCFW-shift in the legs 1 and n , we can then consider following contour integral ($C_\infty = 0$),

$$0 = \oint \frac{dz}{z} X_n^{(n_+, n_-)}(z) = X_n^{(n_+, n_-)}(0) + (\text{residues for } z \neq 0). \quad (13)$$

We treat separately the following two classes of contributions (for each residue):

- (A) We have a pole appearing in only one of the amplitudes \tilde{A}_n or A_n .
- (B) We have a pole that is present in both amplitudes \tilde{A}_n and A_n .

Starting with case (A), we first note that one needs to consider only \tilde{A}_n having the pole. (If the pole instead sits in A_n there is a similar argument, by symmetry.)

Looking back at Eq. (3), we can compute the residue of the pole $s_{\hat{1}2\dots k}$ as $-\lim_{z \rightarrow z_{\hat{1}2\dots k}} [s_{\hat{1}2\dots k}(z) X_n^{(n_+, n_-)}(z)] / s_{12\dots k}$, where $z_{\hat{1}2\dots k}$ is the z -value that makes $s_{\hat{1}2\dots k}$ go on-shell. From this we get

$$\frac{(-1)}{s_{\hat{1}2\dots n-1}} \sum_{\gamma, \sigma, \beta} \frac{\sum_h \tilde{A}_{n-k+1}(\hat{n}, \gamma, -\hat{P}^h) \tilde{A}_{k+1}(\hat{P}^{-h}, \sigma, \hat{1})}{s_{12\dots k}} \times \mathcal{S}[\gamma\sigma | \beta_{2, n-1}]_{p_1} A_n(\hat{1}, \beta_{2, n-1}, \hat{n}), \quad (14)$$

where we can rewrite $\mathcal{S}[\gamma\sigma | \beta_{2, n-1}]_{p_1}$ as $\mathcal{S}[\sigma | \rho_{2, k}]_{p_1} \times$ (something independent of σ). Here $\rho_{2, k}$ stands for the relative ordering of legs 2, 3, ..., k in β . We thus conclude that Eq. (14) is identically zero (at $z = z_{\hat{1}2\dots k}$) since

$$\sum_{\sigma} \tilde{A}_{k+1}(\hat{P}^{-h}, \sigma, \hat{1}) \mathcal{S}[\sigma | \rho_{2, k}]_{p_1} = 0, \quad (15)$$

independently of the helicity configuration in $\tilde{A}_{k+1}(\hat{P}^{-h}, \sigma, \hat{1})$. This holds also with the regularization discussed above. All terms coming from the class (A) above will therefore not contribute to any residues.

For the class (B), consider again Eq. (3) and the $s_{12\dots k}$ pole contribution which is now present in both \tilde{A}_n and A_n . The residue takes the form

$$\frac{(-1)}{s_{\hat{1}2\dots(n-1)}} \sum_{\gamma, \beta, \sigma, \alpha} \left[\frac{\sum_h \tilde{A}(\hat{n}, \gamma, \hat{P}^{-h}) \tilde{A}(-\hat{P}^h, \sigma, \hat{1})}{s_{12\dots k}} \right] \mathcal{S}[\gamma\sigma | \alpha\beta]_{p_1} \left[\frac{\sum_h A(\hat{1}, \alpha, -\hat{P}^h) A(\hat{P}^{-h}, \beta, \hat{n})}{s_{\hat{1}2\dots k}} \right], \quad (16)$$

where the sets $\beta \equiv \beta_{k+1, n-1}$ and $\alpha \equiv \alpha_{2, k}$ and the subscript indices on A and \tilde{A} have been suppressed for clarity. In (16) one of the shifted $s_{\hat{1}2\dots k}$ poles has been substituted by an unshifted pole $s_{12\dots k}$ from calculating the single-pole residues. Noting that $s_{\hat{1}2\dots n-1} = s_{\hat{P}k+1\dots n-1}$, and using $\mathcal{S}[\gamma\sigma | \alpha\beta]_{p_1} = \mathcal{S}[\sigma | \alpha]_{p_1} \times \mathcal{S}[\gamma | \beta]_{\hat{P}}$, we can rewrite the above expression

$$\frac{1}{s_{12\dots k}} \sum_h \left[\left(\sum_{\sigma, \alpha} \frac{\tilde{A}(-\hat{P}^h, \sigma, \hat{1}) \mathcal{S}[\sigma | \alpha]_{p_1} A(\hat{1}, \alpha, -\hat{P}^h)}{s_{\hat{1}2\dots k}} \right) \left(\sum_{\gamma, \beta} \frac{\tilde{A}(\hat{n}, \gamma, \hat{P}^{-h}) \mathcal{S}[\gamma | \beta]_{\hat{P}} A(\hat{P}^{-h}, \beta, \hat{n})}{s_{\hat{P}k+1\dots(n-1)}} \right) \right] \\ + \frac{1}{s_{12\dots k}} \sum_h \left[\left(\sum_{\sigma, \alpha} \frac{\tilde{A}(-\hat{P}^h, \sigma, \hat{1}) \mathcal{S}[\sigma | \alpha]_{p_1} A(\hat{1}, \alpha, -\hat{P}^h)}{s_{\hat{1}2\dots k}} \right) \left(\sum_{\gamma, \beta} \frac{\tilde{A}(\hat{n}, \gamma, \hat{P}^{-h}) \mathcal{S}[\gamma | \beta]_{\hat{P}} A(\hat{P}^h, \beta, \hat{n})}{s_{\hat{P}k+1\dots(n-1)}} \right) \right]. \quad (17)$$

We now want to argue that line one and two of (17) will always be zero separately. We start by giving the arguments mostly in words, and afterward give another, more mathematical, proof. In what follows when we talk about the set α we will actually mean the above α set plus leg 1, and when talking about the β set we mean the above β set plus leg n . There are several different cases.

1. All flipped legs lie within the α set in (17). In this case the first term in line one is zero by the induction principle, and (at least) the second term in line two is zero (since it will be a single-leg flip relation).
2. There are flipped legs in both the α and β set in (17), but the legs in the α set still satisfy the rule. In this case the first term in line one is again zero because of the induction assumption. In line two we can have two kind of situations:
 - (a) The difference in the number of negative and positive helicity-flipped legs in the α set is greater than 1, so the addition of one flipped leg (*i.e.* $-\widehat{P}$) does not ruin the zero.
 - (b) The difference in the number of negative and positive helicity-flipped legs in the α set is equal to 1. Here the first term of line two will only be zero for one of the $h = \pm$ values. But since the \widehat{P} leg in the amplitude containing the β set always has the opposite helicity of the $-\widehat{P}$ leg in the amplitude containing the α set, the relation with the β set must be zero in the case where the relation with the α set is not necessarily zero. If this was not so, we would have chosen $n_+ = n_-$ which is against our starting assumption.
3. There are flipped legs in both the α and β sets in (17), but there is an equal number of negative and positive helicity-flipped legs in the α set. However, since $n_+ \neq n_-$ the remaining flipped legs in the β set must now satisfy our rule, and therefore make the second term of line one vanish. Since the α set contains an equal number of flipped legs, adding one flipped leg, either of positive or negative helicity, will give us an unequal number, and the first term of line two therefore vanishes.
4. All flipped legs lie in the β set in (17). These contributions are zero by arguments similar to those in 1).

The above result can also be stated in a more mathematical language. In Eq. (17), we see that the counting of n_+ , n_- in line one schematically can be written as

$$(n_+, n_-) \rightarrow (n_+^\alpha, n_-^\alpha) \times (n_+^\beta, n_-^\beta), \quad (18)$$

while line two can be written as

$$(n_+, n_-) \rightarrow (n_+^\alpha + 1, n_-^\alpha) \times (n_+^\beta, n_-^\beta + 1) + (n_+^\alpha, n_-^\alpha + 1) \times (n_+^\beta + 1, n_-^\beta). \quad (19)$$

For line one to be nonzero, we need to have

$$n_+^\alpha = n_-^\alpha, \quad \text{and} \quad n_+^\beta = n_-^\beta \implies n_+ = n_-, \quad (20)$$

which is not true. For line two to be nonzero, we need to have

$$n_+^\alpha + 1 = n_-^\alpha, \quad \text{and} \quad n_+^\beta = n_-^\beta + 1 \implies n_+ = n_-, \quad (21)$$

or

$$n_+^\alpha = n_-^\alpha + 1, \quad \text{and} \quad n_+^\beta + 1 = n_-^\beta \implies n_+ = n_-, \quad (22)$$

which is again not true from our start assumption.

This concludes the induction proof of (12).

5. Identities at one-loop level

So far our discussion has been restricted to tree level. Identities among tree level amplitudes can clearly have consequences for loop amplitudes through generalized unitarity [16], and in particular quadruple-cut constructions [17]. Indeed, we find a series of new identities also at one-loop level of gauge theories. To illustrate this, we first remind the reader of some of the one-loop results derived in Ref. [18]. In that paper expressions for $\mathcal{N} = 8$ supergravity box-coefficients were linked directly to products of box coefficients of $\mathcal{N} = 4$ super Yang–Mills. Box coefficients are multiplying a class of integral functions known as scalar box integrals. Such integrals have four propagator lines which are integrated over. As was shown in Ref. [18], one can derive the following set of relations valid at six-point scattering

$$\begin{aligned} c_{N=8}^{(ab)c(de)f} &= 0, \\ c_{N=8}^{(ab)(cd)ef} &= 2s_{ab}s_{cd} \times \left(\sum c_i^{(ab)(cd)ef} \times c_i^{(ba)(dc)ef} \right), \\ c_{N=8}^{(abc)def} &= 2s_{ab}s_{cl_c} \sum (c_i^{(abc)def} c_i^{(bac)def} + c_i^{(abc)def} c_i^{(bca)def} + c_i^{(abc)def} c_i^{(cba)def}) \\ &\quad + 2s_{ac}s_{bl_c} \sum (c_i^{(acb)def} c_i^{(cab)def} + c_i^{(acb)def} c_i^{(cba)def} + c_i^{(acb)def} c_i^{(bca)def}), \end{aligned} \quad (23)$$

for all choices of helicities. We refer to Ref. [18] for the precise definition of the box coefficients $c_{N=8}$ and c_i that enter the above expression.

Using our rule for vanishing identities between gauge theory amplitudes this leads to results such as (the superscript \pm on leg c denotes that helicity on that leg has been chosen oppositely)

$$\begin{aligned} 0 &= 2s_{ab}s_{cl_c} \sum (c_i^{(abc^+)def} c_i^{(bac^-)def} + c_i^{(abc^+)def} c_i^{(bc^-a)def} + c_i^{(abc^+)def} c_i^{(c^-ba)def}) \\ &\quad + 2s_{ac}s_{bl_c} \sum (c_i^{(ac^+b)def} c_i^{(c^-ab)def} + c_i^{(ac^+b)def} c_i^{(c^-ba)def} + c_i^{(ac^+b)def} c_i^{(bc^-a)def}). \end{aligned} \quad (24)$$

From this one can directly link coefficients of different boxes with opposite helicity configurations. This provides a link at one-loop level for our gauge theory identities. One readily checks that they indeed are satisfied. We could of course consider flipping more legs and also higher n -point functions in a similar fashion. This as well as multi-loop considerations are beyond the scope of this Letter and we leave it for future work.

6. On the origin of the identities

As we have seen above, when $n_+ = n_- = 0$ our formulae simply express the standard KLT-relations, i.e. $X_n^{(0,0)}$ give the gravity tree-level amplitude, but when $n_+ \neq n_-$ we get $X_n^{(n_+, n_-)} = 0$, the new gauge theory identities. It is interesting to consider what happens if $n_+ = n_- \neq 0$. In that case we do not get identities among gauge theory amplitudes. Since our focus in this Letter is just on these gauge theory identities, our discussion around this case will be brief.

The origin of the non-vanishing of $X_n^{(n_+, n_-)}$ in the cases where $n_+ = n_- \neq 0$ can be illustrated by a rewriting of the 4-point case with $n_+ = n_- = 1$. We have,

$$-s_{12}A_4(1^-, 2^-, 3^+, 4^+)\tilde{A}_4(3^-, 4^+, 2^+, 1^-) = -s_{12}A_4(1^-, 2_s, 3_s, 4^+)\tilde{A}_4(3_s, 4^+, 2_s, 1^-), \quad (25)$$

where the s subscript denotes a scalar particle, and where we have used supersymmetric Ward identities to write the purely gluonic amplitudes in terms of gluon/scalar amplitudes. We see that the $n_+ = n_- = 1$ case is nothing but a KLT-relation involving scalars, and therefore obviously should not vanish. Doing the same in the case of $n_+ = n_- = 2$ we obtain 4-point scalar amplitudes. One could of course also think of more exotic relations involving for example flipped fermion spins (see e.g. Ref. [19] for different types of mixed matter relations for which spin flips could be considered). In [20] different relations are considered between operators of $\mathcal{N} = 8$ and $\mathcal{N} = 4$ it would also be interesting to think of flips of spins in such a context.

Because of the simple form of supersymmetric Ward identities for MHV amplitudes, the above rewriting can easily be extended to higher-point relations. It is straightforward to check that it holds in the 5-point case as well. It thus becomes clear that the new gauge theory identities arise when there are no matching amplitudes in the (super)gravity sector: These are the “forbidden” combinations that provide constraints on the gauge theory sectors alone. An understanding of this helicity selection rule at the string theory level could be most interesting.

7. Conclusions

In this Letter we have discussed some rather intriguing identities among Yang–Mills amplitudes. They were found using inspiration from the field theory limit of the well-known KLT-relations after flipping one or more helicities between the two gauge theory amplitudes on the right-hand side. We have found it useful to present our results using a generalized and more symmetric version of the KLT-relations which we have uncovered in the process of this investigation. Although we have only considered gluon scattering amplitudes in this Letter, extensions certainly exist for amplitudes involving external fermions and scalars. We have also considered the impact of these identities on loop amplitudes. Relations between box coefficients of one-loop amplitudes in $\mathcal{N} = 4$ super Yang–Mills theory follow in this way. Further progress in this direction is likely and it would clearly be interesting to investigate more identities at one and possibly multi-loop level. It would also be an interesting task to understand better the origin of these identities. It seems quite likely that string theory also here holds the clue. This seems a promising avenue for future studies.

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