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Expected lengths and distribution functions for Young diagrams in the hook ☆

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Abstract

We consider β-Plancherel measures [J. Baik, E. Rains, The asymptotics of monotone subsequences of involutions, Duke Math. J. 109 (2001) 205–281] on subsets of partitions—and their asymptotics. These subsets are the Young diagrams contained in a (k,ℓ)-hook, and we calculate the asymptotics of the expected shape of these diagrams, relative to such measures. We also calculate the asymptotics of the distribution function of the lengths of the rows and the columns for these diagrams. This might be considered as the restriction to the (k,ℓ)-hook of the fundamental work of Baik, Deift and Johansson [J. Baik, P. Deift, K. Johansson, On the distribution of the length of the longest increasing subsequence of random permutations, J. Amer. Math. Soc. 12 (1999) 1119–1178]. The above asymptotics are given here by ratios of certain Selberg-type multi-integrals.

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1. Introduction

This paper studies the asymptotics of certain "β-Plancherel" measures on subsets of partitions. Let $Y_n = \{\lambda \mid \lambda \vdash n\}$ denote the partitions of n, and let f^λ denote the number of standard Young tableaux of shape $\lambda = (\lambda_1, \lambda_2, \ldots)$. For general references regarding partitions, Young diagrams

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and Young tableaux—see [13,20]. Let $0 < \beta \in \mathbb{R}$. Baik and Rains [5] consider the following "*β-Plancherel*" measure M_n^β on Y_n:

$$M_n^\beta(\lambda) := \frac{(f^\lambda)^\beta}{\sum_{\mu \vdash n} (f^\mu)^\beta}. \tag{1}$$

Indeed, M_n^2 is the so-called Plancherel measure on Y_n. We generalize to subsets $\Gamma_n \subseteq Y_n$, considering the subset of the partitions (i.e. diagrams) in the (k,ℓ) hook: Let $k, \ell \geqslant 0$ be integers and let $\Gamma_n = H(k,\ell;n)$ denote the following subset of Y_n:

$$H(k,\ell;n) = \{\lambda \vdash n \mid \lambda_{k+1} \leqslant \ell\}.$$

These subsets arise in the representation theory of Lie groups, algebras and superalgebras, see for example [6]. The measures $\rho_n^{(\beta;k,\ell)}$ below are the (k,ℓ)-hook restrictions of the above measures M_n^β.

Definition 1.1. Let $\lambda \in H(k,\ell;n)$ and $\beta > 0$, then

$$\rho^{(\beta;k,\ell)}(\lambda) = \rho_n^{(\beta;k,\ell)}(\lambda) := \frac{(f^\lambda)^\beta}{\sum_{\mu \in H(k,\ell;n)} (f^\mu)^\beta}.$$

1.1. Expected shape

Given $\Gamma_n \subseteq Y_n$, $n = 1, 2, \ldots$, and the probability measures $\rho = \{\rho_n\}_{n=1}^\infty$ on the Γ_ns, one studies the asymptotics of the expected value (i.e. average length) of the first row λ_1, denoted $\lambda_{1,E}$, and similarly for the second row $\lambda_{2,E}$, etc. Similarly for the columns. Explicitly, when $\Gamma_n = H(k,\ell;n)$, expected values are given by the following definition.

Definition 1.2. If $\lambda \vdash n$, we write $\lambda = (\lambda_{1,n}, \lambda_{2,n}, \ldots)$. Also, λ' is the conjugate partition of λ. Let $1 \leqslant p \leqslant k$ and $1 \leqslant q \leqslant \ell$. The expected value of the pth row is $E(\lambda_p) = \lambda_{p,E}^{(\beta;k,\ell)}(n)$, where

$$\lambda_{p,E}^{(\beta;k,\ell)}(n) = \frac{\sum_{\lambda \in H(k,\ell;n)} \lambda_{p,n} \cdot (f^\lambda)^\beta}{\sum_{\lambda \in H(k,\ell;n)} (f^\lambda)^\beta} \quad \text{and} \quad \lambda_E^{(\beta;k,\ell)}(n) = \bigl(\lambda_{1,E}^{(\beta;k,\ell)}(n), \lambda_{2,E}^{(\beta;k,\ell)}(n), \ldots\bigr).$$

Similarly for the expected qth column

$$\lambda_{q,E}^{\prime(\beta;k,\ell)}(n) = \frac{\sum_{\lambda \in H(k,\ell;n)} \lambda'_{q,n} \cdot (f^\lambda)^\beta}{\sum_{\lambda \in H(k,\ell;n)} (f^\lambda)^\beta} \quad \text{and} \quad \lambda_E^{\prime(\beta;k,\ell)}(n) = \bigl(\lambda_{1,E}^{\prime(\beta;k,\ell)}(n), \lambda_{2,E}^{\prime(\beta;k,\ell)}(n), \ldots\bigr).$$

Of course, one can replace $H(k,\ell;n)$ in the above definition by other subsets $\Gamma_n \subseteq Y_n$.

The case $\Gamma_n = Y_n$ and $\beta = 2$ (Plancherel) has a long history. Let

$$w(n) = \frac{\sum_{\lambda \vdash n} \lambda_{1,n} \cdot (f^\lambda)^2}{\sum_{\lambda \vdash n} (f^\lambda)^2} = \frac{\sum_{\lambda \vdash n} \lambda_{1,n} \cdot (f^\lambda)^2}{n!}$$

be the expected value of the first row—for the Plancherel measure M_n^2. Hammersley [10] showed that the limit $c = \lim_{n\to\infty} w(n)/\sqrt{n}$ exists. Vershik and Kerov [24] proved that $c = 2$ (independently, Logan and Shepp [12] proved that $c \leqslant 2$). Vershik and Kerov—and Logan and Shepp—also determined the asymptotics of the expected shape λ in this case.

Recently, in a major breakthrough paper, Baik, Deift and Johansson [3] determined the distribution function of the asymptotics of the first row, relating it to the *Tracy–Widom distribution* [22], see also [5,23]. The distribution function for the second row is given, by these same authors, in [4]. The distribution functions for the general rows are given in [8,11,16]; see also [7] for the analogue results for colored permutations. The above results also establish deep connections with the theory of *random matrices* [15]. For detailed reviews of these results—see [1,21].

The main objective of the present paper is to compute the asymptotics of the above expected values (i.e. shapes) $\lambda_E^{(\beta;k,\ell)}(n)$, as well as the corresponding distribution functions. The first term approximation is relatively simple, as we show that for each $1 \leqslant p \leqslant k$ and $1 \leqslant q \leqslant \ell$

$$\lambda_{p,E}^{(\beta;k,\ell)}(n),\ \lambda_{q,E}^{\prime(\beta;k,\ell)}(n) \simeq \frac{n}{k+\ell},$$

see Theorem 4.1. Second-term approximations of $\lambda_{p,E}^{(\beta;k,\ell)}(n)$ are introduced and studied in Sections 5, 6, and they have different values for different rows and for different columns. These second-term approximations are given as ratios of certain Selberg-type integrals, see Theorems 5.3 and 6.3 below.

1.2. Distribution functions

In Section 7 we introduce and study the asymptotics of the distribution functions $\lambda_p^{(\beta;k,\ell)}(n,z)$, $\lambda_q^{\prime(\beta;k,\ell)}(n,z)$ for the lengths of the rows and the columns in $H(k,\ell;n)$—with respect to the above measures. We are able to calculate, asymptotically, a first-term approximation of these functions, but only a conjecture is given, about the second-term approximations. That first-term approximation is

$$\lambda_p^{(\beta;k,\ell)}(n,z),\ \lambda_q^{\prime(\beta;k,\ell)}(n,z) \simeq \frac{n}{k+\ell}\cdot r_{(k,\ell),\beta}(z),$$

where $r_{(k,\ell),\beta}(z)$ is given by Eq. (20), see Theorem 7.3.

1.3. Comparison with maximal shape

Given a subset of partitions $\Gamma_n \subseteq Y_n$, one looks for $\lambda \in \Gamma_n$ with maximal degree f^λ. Call it *maximal shape* (with respect to Γ_n) and denote it by $\lambda_{\max}$. In Sections 8, 9 and 10 the expected shapes for $\beta = 1, 2$ are compared with the maximal shape. When $\Gamma_n = Y_n$, the asymptotics of $\lambda_{\max}$ was calculated by Vershik and Kerov [24,25], and by Logan and Shepp [12]. In particular, they proved that asymptotically, the *expected* shape for $\beta = 2$ and the *maximal* shape are the same, and that shape is given by the two axes and by the curve

$$y = 1 + \left(\frac{2}{\pi}\right)\left[x\cdot\sqrt{1-x^2} - \arccos x\right]. \tag{2}$$

A comparison with the case $\Gamma_n = H(k,\ell;n)$ is intriguing.

When Γ_n is the k-strip $\Gamma_n = H(k,0;n)$, the asymptotics of λ with maximal f^λ was calculated in [2], and is given by the curve

$$y = \left(\frac{2}{\pi}\right)\left[x \cdot \arcsin x + \sqrt{1-x^2}\,\right]. \tag{3}$$

When $\Gamma_n = H(k,\ell;n)$, the maximal λ was given in [18]. These results are reviewed in Section 8.2. Consider for example the 'strip' case $\ell = 0$, and denote the maximal λ by $\lambda_{\max}^{(k,0)}$. Comparing it with $\lambda_E^{(2;k,0)}$, the 'Plancherel' expected λ in $H(k,0;n)$, we show that these asymptotic shapes are *not* equal—even in their first raw. Nevertheless, *numerically* $\lambda_{\max}^{(k,0)}$ and $\lambda_E^{(2;k,0)}$ are remarkably close, at least in the few special cases we check below, see Section 8.

Also, the asymptotics of $\lambda_{\max}$ for $\Gamma_n = Y_n$ is *not* the limit case of $\lambda_{\max}^{(k,0)}$ as $k \to \infty$, but the similarity between (2) and (3) is intriguing. It should be interesting to see if the ratios of the Selberg-type integrals, which give the expected shapes and the distribution functions for $\Gamma_n = H(k,\ell;n)$, are in any way related to the Tracy–Widom distributions [22,23], which give the (Plancherel and the 'involution') distribution functions in Y_n.

1.4. RSK

In the case of $\rho_n^{(1;k,\ell)}$ and $\rho_n^{(2;k,\ell)}$, the RSK correspondence provides an interesting interpretation of the above asymptotics. The RSK (Robinson–Schensted–Knuth) correspondence $\sigma \leftrightarrow (P_\lambda, Q_\lambda)$ corresponds $\sigma \in S_n$ with a pair of standard Young tableaux of shape λ [20]. In the Plancherel case $\beta = 2$ it relates the above expected values of the first row to the statistics of the longest increasing (and decreasing) subsequences in permutation. For example, when $\sigma \leftrightarrow (P_\lambda, Q_\lambda)$, λ_1 is the length of a longest increasing subsequence in σ, while λ_1' is the length of a longest decreasing subsequence in σ. By C. Green's theorem [9] there are similar interpretations for λ_2, λ_3, etc. For a detailed account of the RSK see [20]. Thus the results in [3] etc. can also be stated in terms of longest increasing subsequences in permutations.

It is well known that σ is an involution iff $\sigma \leftrightarrow (P_\lambda, P_\lambda)$. The analogue Probability theory of longest increasing subsequences in involutions in S_n is done in [5].

Denote by $S_{k,\ell;n} \subseteq S_n$ the subset of the permutations $\sigma \in S_n$ such that under the RSK correspondence $\sigma \leftrightarrow (P_\lambda, Q_\lambda)$, we have $\lambda \in H(k,\ell;n)$. For example, $S_{k,0;n}$ is the subset of those permutations in S_n where any descending subsequence has length $\leqslant k$. Thus, $\lambda_{1,E}^{(1;k,\ell)}$ is the expected value of the longest increasing subsequence in the involutions in $S_{k,\ell;n}$.

2. Selberg-type integrals

As mentioned above, the main results in this paper involve Selberg-type integrals, hence we briefly review these type of multi-integrals. In [19] A. Selberg proved the following formula:

$$\int_0^1 \cdots \int_0^1 (u_1 \cdots u_n)^{x-1} \cdot \left[(1-u_1)\cdots(1-u_n)\right]^{y-1} \cdot \prod_{1\leqslant i<j\leqslant n} |u_i - u_j|^{2z} du_1 \cdots du_n$$
$$= \prod_{k=1}^{n} \frac{\Gamma(1+kz)\cdot\Gamma(x+(k-1)z)\cdot\Gamma(y+(k-1)z)}{\Gamma(1+z)\cdot\Gamma(x+y+(n+k-2)z)}.$$

Various integral formulas can be deduced from Selber's integral, see for example [14] for the Macdonald–Mehta integrals. For example Mehta's integral formula (which was a conjecture for some time)

$$\int_{\mathbb{R}^k} e^{-(1/2)(\sum x_i^2)} \prod_{1\leqslant i<j\leqslant k} (x_i - x_j)^{2z} dx_1 \cdots dx_k = \left(\sqrt{2\pi}\right)^k \cdot \prod_{j=1}^{k} \frac{\Gamma(1+jz)}{\Gamma(1+z)} \tag{4}$$

can be deduced from Selberg's formula, see [15] for details. We call these and related integrals "Selberg-type integrals." A connection between the RSK and these integrals, as well as with *random matrices*, appears in [6,17]. Since the formulas from [6,17] are needed later, we record it here, together with certain additional asymptotics and integrals that are also needed below. Let

$$\Omega_k = \left\{(x_1, \ldots, x_k) \in \mathbb{R}^k \mid x_1 \geqslant x_2 \geqslant \cdots \geqslant x_k \text{ and } x_1 + \cdots + x_k = 0\right\},$$

and more generally,

$$\Omega_{(k,\ell)} = \left\{(x_1, \ldots, x_k, y_1, \ldots, y_\ell) \mid x_1 \geqslant \cdots \geqslant x_k;\ y_1 \geqslant \cdots \geqslant y_\ell;\ \sum x_i + \sum y_j = 0\right\}.$$

Theorem 2.1 *(Theorem* 2.10 *in [17]). Let* $\gamma_k = (1/\sqrt{2\pi})^{k-1} \cdot k^{k^2/2}$ *and* $D_k(x) = \prod_{1\leqslant i<j\leqslant k}(x_i - x_j)$*, then*

$$\sum_{\lambda \in H(k,0;n)} \left(f^\lambda\right)^\beta \simeq \left[\gamma_k \cdot \left(\frac{1}{n}\right)^{(k-1)(k+2)/4} \cdot k^n\right]^\beta \cdot \left(\sqrt{n}\right)^{k-1} \cdot I(k,0,\beta),$$

where

$$I(k,0,\beta) = \int_{\Omega_k} \left[D_k(x) \cdot e^{-\frac{k}{2}(\sum x_i^2)}\right]^\beta d^{(k-1)}x.$$

Here

$$D_k(x) = \prod_{1\leqslant i<j\leqslant k} (x_i - x_j).$$

Note that by the symmetry of the (absolute value of the) above integrand, Ω_k is transformed in [17] into $\mathbb{R}^k$, and the corresponding integral is then evaluated by (4). In most of the integrals below there is no such symmetry, hence no such simplification of the domain of integration. Therefore, at the moment, we do not have explicit evaluations of these integrals.

Given $\lambda = (\lambda_{1,n}, \lambda_{2,n}, \ldots) \vdash n$, write $\lambda_{p,n} = n/k + c_{p,n} \cdot \sqrt{n}$ and denote $c_{p,n} = c_{p,n}(\lambda)$.

Theorem 2.2.

$$\sum_{\lambda \in H(k,0;n)} c_{p,n}(\lambda) \cdot \left(f^\lambda\right)^\beta \simeq \left[\gamma_k \cdot \left(\frac{1}{n}\right)^{(k-1)(k+2)/4} \cdot k^n\right]^\beta \cdot \left(\sqrt{n}\right)^{k-1} \cdot I^*(k,0,\beta), \tag{5}$$

where

$$I^*(k,0,\beta)=\int_{\Omega_k} x_p\cdot\Big[D_k(x)\cdot e^{-\frac{k}{2}(\sum x_i^2)}\Big]^{\beta} d^{(k-1)}x.$$

Proof. We sketch the proof. Let $\lambda=(\lambda_1,\ldots,\lambda_k)$ and write $\lambda_j=n/k+c_j\sqrt{n}$. Fix some $a,\delta>0$ and consider partitions $\lambda\vdash n$ with all $|c_j|<a$ and $c_i-c_j\geqslant\delta$ if $i<j$. By approximation arguments similar to those in Lemma 1.1 of [17], (with $d_\lambda=f^\lambda$ and $c_p=c_{p,n}$) it follows here that when n goes to infinity,

$$c_{p,n}\cdot\big(f^\lambda\big)^\beta\simeq g(\lambda)\cdot h(n), \tag{6}$$

where

$$g(\lambda)=c_p\cdot\big[D_k(c_1,\ldots,c_k)\cdot e^{(-k/2)c^2}\big]^\beta \quad\text{and}\quad h(n)=\big[\gamma_k\cdot n^{-(k-1)(k+2)/4}\cdot k^n\big]^\beta.$$

Here $\gamma_k=(1/\sqrt{2\pi}\,)^{k-1}\cdot k^{k^2/2}$, $D_k(c_1,\ldots,c_k)=\prod_{i<j}(c_i-c_j)$ and $e^{(-k/2)c^2}=e^{(-k/2)(c_1^2+\cdots+c_k^2)}$. The approximation (6) takes care of the summands where λ is in the vicinity of the $k\times(n/k)$ rectangle. For the summands where λ is not in the vicinity of the $k\times(n/k)$ rectangle, by approximation arguments similar to those in Proposition 1.6. of [17], deduce the following upper bound: As n goes to infinity,

$$c_p\cdot f^\lambda\leqslant b\cdot c_p\cdot D_\delta(c)\cdot e^{(-k^3/3)c^2}\cdot n^{-(k-1)(k+2)/4}\cdot k^n. \tag{7}$$

Here δ is a fixed number between 0 and 1/2, and

$$D_\delta(c)=\prod_{\substack{1\leqslant i<j\leqslant k\\ \delta\leqslant c_i-c_j}}(c_i-c_j).$$

Let l.h.s. denote the left-hand side of (5): l.h.s. $=\sum_{\lambda\in H(k,0;n)}c_{p,n}(\lambda)\cdot(f^\lambda)^\beta$. As in [17], the estimates (6) and (7) imply that the sum l.h.s. is dominated by the summands in the vicinity of the $k\times(n/k)$ rectangle. Namely, l.h.s. can be approximated by sums over the subsets of partitions $\Lambda(k,n,a,\delta)\subseteq H(k,0;n)$, where $\lambda\in\Lambda(k,n,a,\delta)$ if $\lambda=(\lambda_1,\ldots,\lambda_k)$, $\lambda_j=\frac{n}{k}+c_j\sqrt{n}$, all $|c_j|<a$ and $c_i-c_j\geqslant\delta$ when $i<j$. Then l.h.s. is approximated by first letting $n\to\infty$, then letting $a\to\infty$ and $\delta\to0$. By (6) this implies that as n goes to infinity,

$$\text{l.h.s.}\simeq\Bigg(\lim_{\substack{a\to\infty\\ \delta\to0}}\sum_{\lambda\in\Lambda(k,n,a,\delta)}g(\lambda)\Bigg)\cdot h(n).$$

Finally, in the first factor approximate summation by integration: this yields the integral $I^*(k,0,\beta)$ and completes the proof. □

The (k,ℓ)-hook analogue of Theorem 2.1 is Theorem 7.18 in [6] which we now quote.

Theorem 2.3 *(Theorem 7.18 in [6]). Let*

$$\gamma_{k,\ell} = \left(1/\sqrt{2\pi}\right)^{k+\ell-1} \cdot (k+\ell)^{(k^2+\ell^2)/2} \cdot (1/2)^{k\ell},$$

then

$$\sum_{\lambda \in H(k,\ell;n)} \left(f^\lambda\right)^\beta \simeq \left[\gamma_{k,\ell} \cdot \left(\frac{1}{n}\right)^{(k(k+1)+\ell(\ell+1)-2)/4} \cdot (k+\ell)^n\right]^\beta \cdot \left(\sqrt{n}\right)^{k+\ell-1} \cdot I(k,\ell,\beta),$$

where

$$I(k,\ell,\beta) = \int\limits_{\Omega_{k,\ell}} \left[D_k(x) \cdot D_\ell(y) \cdot e^{-\frac{k+\ell}{2}(\sum x_i^2 + \sum y_j^2)}\right]^\beta d^{(k+\ell-1)}(x;y).$$

Because of the symmetry in the x's and the symmetry in the y's in the integrand, the integral $I(k,\ell,\beta)$ can be reduced to the integral in (4) and thus can be evaluated, see Proposition 7.20 of [6].

We shall also need

Theorem 2.4. *Let* $1 \leqslant p \leqslant k$*, let* $\lambda = (\lambda_{1,n}, \lambda_{2,n}, \ldots) \in H(k,\ell;n)$ *and define* $c_{p,n}(\lambda)$ *via*: $\lambda_{p,n} = n/(k+\ell) + c_{p,n}(\lambda) \cdot \sqrt{n}$. *Similarly for* $1 \leqslant q \leqslant \ell$ *and* $c'_{q,n}(\lambda) := c_{q,n}(\lambda')$. *Then, as n goes to infinity,*

$$\sum_{\lambda \in H(k,\ell;n)} c_{p,n}(\lambda) \cdot \left(f^\lambda\right)^\beta \simeq$$

$$\left[\gamma_{k,\ell} \cdot \left(\frac{1}{n}\right)^{(k(k+1)+\ell(\ell+1)-2)/4} \cdot (k+\ell)^n\right]^\beta \cdot \left(\sqrt{n}\right)^{k+\ell-1} \cdot I^*(k,\ell,\beta), \tag{8}$$

where

$$I^*(k,\ell,\beta) = \int\limits_{\Omega_{k,\ell}} x_p \cdot \left[D_k(x) \cdot D_\ell(y) \cdot e^{-\frac{k+\ell}{2}(\sum x_i^2 + \sum y_j^2)}\right]^\beta d^{(k+\ell-1)}(x;y).$$

Similarly for the sum

$$\sum_{\lambda \in H(k,\ell;n)} c'_{q,n}(\lambda) \cdot \left(f^\lambda\right)^\beta,$$

with the corresponding integral

$$I'^*(k,\ell,\beta) = \int\limits_{\Omega_{k,\ell}} y_q \cdot \left[D_k(x) \cdot D_\ell(y) \cdot e^{-\frac{k+\ell}{2}(\sum x_i^2 + \sum y_j^2)}\right]^\beta d^{(k+\ell-1)}(x;y).$$

Proof. Denote

$$\text{l.h.s.} = \sum_{\lambda \in H(k,\ell;n)} c_{p,n}(\lambda) \cdot (f^\lambda)^\beta \quad \text{and} \quad \text{l.h.s.}' = \sum_{\lambda \in H'(k,\ell;n)} c_{p,n}(\lambda) \cdot (f^\lambda)^\beta$$

where $H'(k,\ell;n)$ are those partitions in $H(k,\ell;n)$ which contain the $k \times \ell$ rectangle: $H'(k,\ell;n) = \{\lambda \in H(k,\ell;n) \mid \lambda + k \geqslant \ell\}$. We show that the asymptotics of l.h.s.′ equals the right-hand side of (8), and this clearly will imply the proof.

For $\lambda \in H'(k,\ell;n)$ denote $\nu = \nu(\lambda) = (\nu_1, \ldots, \nu_k)$ where $\nu_i = \lambda_i - \ell$; similarly $\mu = \mu(\lambda) = (\mu_1, \ldots, \mu_\ell)$ is given by $\mu'_j = \lambda'_j - k$. Similar to the proof of Theorem 2.2, the sum l.h.s.′ is dominated by the summands with $\nu(\lambda)$ and $\mu(\lambda)$ close to being the appropriate rectangles. For such $\lambda \in H'(k,\ell;n)$ with $\nu(\lambda)$ and $\mu(\lambda)$ and with $R = R_{k,\ell}$ the $k \times \ell$ rectangle, using the hook formula one proves a decomposition formula as follows: (see (7.14.1) of [6]),

$$f^\lambda = \frac{n!}{|\nu|! \cdot |\mu|!} \cdot \frac{1}{\prod_{z \in R} h(z)} \cdot f^\nu \cdot f^\mu,$$

thus

$$c_p \cdot (f^\lambda)^\beta = \left(\frac{n!}{|\nu|! \cdot |\mu|!} \cdot \frac{1}{\prod_{z \in R} h(z)} \right)^\beta \cdot c_p \big(f^\nu\big)^\beta \cdot \big(f^\mu\big)^\beta.$$

The approximation of the term

$$\left(\frac{n!}{|\nu|! \cdot |\mu|!} \cdot \frac{1}{\prod_{z \in R} h(z)} \right)^\beta$$

is given by 7.15 in [6]. The asymptotics of the summation of the factors $c_p(f^\nu)^\beta \cdot (f^\mu)^\beta$ is done, essentially, by a combination of the arguments that proved Theorems 2.1 and 2.2. Together, this implies that

$$\text{l.h.s.}' \simeq \left[\gamma_{k,\ell} \cdot \left(\frac{1}{n}\right)^{(k(k+1)+\ell(\ell+1)-2)/4} \cdot (k+\ell)^n \right]^\beta \cdot \big(\sqrt{n}\big)^{k+\ell-1} \cdot I^*(k,\ell,\beta). \tag{9}$$

Finally note that because of the asymptotics (9), the summation on the complement partitions $\lambda \in H(k,\ell,n) - H'(k,\ell,n)$ will be exponentially smaller that the summation on $\lambda \in H'(k,\ell,n)$, and this completes the proof. □

We shall also need

Theorem 2.5. *Let $z > 0$, denote*

$$H(k,\ell;n,z) = \left\{ \lambda \in H(k,\ell;n) \mid \lambda_{1,n}, \lambda'_{1,n} \leqslant \frac{n}{k+\ell} + z\sqrt{n} \right\}$$

and let

$$\Omega_{(k,\ell),z} = \left\{ (x_1, \ldots, x_k; y_1, \ldots, y_\ell) \in \Omega_{(k,\ell)} \mid x_1, y_1 \leqslant z \right\}.$$

Then

$$\frac{\sum_{\lambda\in H(k,\ell;n,z)}(f^\lambda)^\beta}{\sum_{\lambda\in H(k,\ell;n)}(f^\lambda)^\beta}\simeq\frac{\int_{\Omega_{(k,\ell),z}}[D_k(x)\cdot D_\ell(y)\cdot e^{-\frac{k+\ell}{2}(\sum x_i^2+\sum y_j^2)}]^\beta d^{(k+\ell-1)}(x;y)}{\int_{\Omega_{(k,\ell)}}[D_k(x)\cdot D_\ell(y)\cdot e^{-\frac{k+\ell}{2}(\sum x_i^2+\sum y_j^2)}]^\beta d^{(k+\ell-1)}(x;y)}.$$

Proof. Since the asymptotics of the denominator is given in Theorem 2.3, we only need to compute the asymptotics of the numerator. Comparing with the asymptotics of the denominator, here we have the extra constraints $\lambda_{1,n}$, $\lambda'_{1,n}\leqslant n/(k+\ell)+z\sqrt{n}$, namely, $c_1, c'_1\leqslant z$. In the asymptotics, this constraints only affect the domain of integration, making it $\Omega_{(k,\ell),z}$ instead of $\Omega_{(k,\ell)}$. Thus, in the asymptotics of the above ratio, all the terms except the integrals cancel, and the proof follows. □

We call all the above "*Selberg-type integrals*," and remark that the expected values and distribution functions discussed in Sections 1.1 and 1.2 are given below as ratio of such integrals.

3. The main results

3.1. The expected values

We study the expected values of the row and of the column lengths in $\Gamma_n=H(k,\ell;n)$ with respect to the measures $\rho_n^{(\beta;k,\ell)}$ introduced in Definition 1.1. The first term asymptotics is given by

Theorem 3.1 *(See Theorem* 4.1*). Let $\beta>0$ and $\Gamma_n=H(k,\ell;n)$. For each $1\leqslant p\leqslant k$,*

$$\lambda_{p,E}^{(\beta;k,\ell)}(n)\simeq\frac{n}{k+\ell},\quad \textit{namely}\quad \lim_{n\to\infty}\big(\lambda_{p,E}^{(\beta;k,\ell)}(n)\big)\Big/\Big(\frac{n}{k+\ell}\Big)=1.$$

Similarly, for each $1\leqslant q\leqslant\ell$,

$$\lambda_{q,E}^{\prime(\beta;k,\ell)}(n)\simeq\frac{n}{k+\ell}.$$

The second term approximations are given as follows. Define $c_{p,E}^{(\beta;k,\ell)}(n)$ and $c_{p,E}^{(\beta;k,\ell)}$ via

$$\lambda_{p,E}^{(\beta;k,\ell)}(n)=\frac{n}{k+\ell}+c_{p,E}^{(\beta;k,\ell)}(n)\cdot\sqrt{n},\quad\text{and}\quad c_{p,E}^{(\beta;k,\ell)}=\lim_{n\to\infty}c_{p,E}^{(\beta;k,\ell)}(n).$$

Similarly for the columns. Then

Theorem 3.2 *(see Theorem* 6.3*). Let $1\leqslant p\leqslant k$, then the limit $c_{p,E}^{(\beta;k,\ell)}$ exists, and is given as follows*:

$$c_{p,E}^{(\beta;k,\ell)}=\frac{\int_{\Omega_k}x_p\cdot[D_k(x_1,\dots,x_k)\cdot e^{-\frac{k+\ell}{2}(x_1^2+\cdots+x_k^2)}]^\beta d^{(k-1)}(x)}{\int_{\Omega_k}[D_k(x_1,\dots,x_k)\cdot e^{-\frac{k+\ell}{2}(x_1^2+\cdots+x_k^2)}]^\beta d^{(k-1)}(x)}.\tag{10}$$

Similarly for the columns.

Equation (10) (or (15)) is a consequence of the seemingly more symmetric Eq. (17).

In Sections 8, 9 and 10 the expected shapes for $\beta = 1, 2$ are compared with the maximal shape $\lambda_{\max}$ in few special cases. As mentioned in Section 1.3, this shows a different behavior in the hook case $\Gamma_n = H(k, \ell; n)$ compared with the general case $\Gamma_n = Y_n$.

3.2. The distribution function for the first row

In Section 7 we study the distribution functions of the length on the rows and the columns. We calculate the first term approximations and conjecture the second term approximations. Given $0 < z \in \mathbb{R}$, denote

$$H(k, \ell; n, z) = \left\{\lambda \in H(k, \ell; n) \mid \lambda_{1,n}, \lambda'_{1,n} \leqslant \frac{n}{k+\ell} + z\sqrt{n}\right\}.$$

Let $1 \leqslant p \leqslant k,\ 1 \leqslant q \leqslant \ell$. The distribution of the length of the pth row as a function of z is defined as

$$\lambda_p^{(\beta;k,\ell)}(n, z) = \frac{\sum_{\lambda \in H(k,\ell;n,z)} \lambda_{p,n} \cdot (f^\lambda)^\beta}{\sum_{\lambda \in H(k,\ell;n)} (f^\lambda)^\beta},$$

and similarly for the columns. Recall $\Omega_{(k,\ell)}$ from Section 2 and denote

$$\Omega_{(k,\ell),z} = \left\{(x_1, \ldots, x_k; y_1, \ldots, y_\ell) \in \Omega_{(k,\ell)} \mid x_1, y_1 \leqslant z\right\}.$$

Then

Theorem 3.3 *(see Theorem* 7.3*). Let* $\beta, z > 0$ *and denote*

$$r_{(k,\ell),\beta}(z) = \frac{\int_{\Omega_{(k,\ell),z}} [D_k(x) \cdot D_\ell(y) \cdot e^{-\frac{k+\ell}{2}(\sum x_i^2 + \sum y_j^2)}]^\beta d^{(k+\ell-1)}(x; y)}{\int_{\Omega_{(k,\ell)}} [D_k(x) \cdot D_\ell(y) \cdot e^{-\frac{k+\ell}{2}(\sum x_i^2 + \sum y_j^2)}]^\beta d^{(k+\ell-1)}(x; y)}. \tag{11}$$

Then

$$\lambda_p^{(\beta;k,\ell)}(n, z), \lambda_q'^{(\beta;k,\ell)}(n, z) \simeq \frac{n}{k+\ell} \cdot r_{(k,\ell),\beta}(z).$$

In Section 7.2 we make some conjectures about the second term approximations of $\lambda_p^{(\beta;k,\ell)}(n, z)$ and $\lambda_q'^{(\beta;k,\ell)}(n, z)$, both in terms of ratios of Selberg-type integrals. In the last three sections (Sections 7–9), we calculate some special cases and include also some computer calculations.

Part I. Expected shape, the β-Plancherel probability

4. First term approximation

Recall the notation $\simeq$: Let a_n, b_n be two sequences of, say, real numbers, and assume $b_n \neq 0$ if n is large enough. Then $a_n \simeq b_n$ if $\lim_{n\to\infty} a_n/b_n = 1$. Extend $\simeq$ to vectors as follows: $(a_{1,n}, \ldots, a_{r,n}) \simeq (b_{1,n}, \ldots, b_{r,n})$ iff $a_{i,n} \simeq b_{i,n}$ for $i = 1, \ldots, r$.

Following the techniques and arguments in Section 7 of [6] (see also [17]), we show below the following first approximation of the expected shape $\lambda_E = \lambda_E^{(\beta;k,\ell)}$.

Theorem 4.1. *Let $\beta > 0$ and $\Gamma_n = H(k,\ell;n)$ and let $\lambda_{p,E}^{(\beta;k,\ell)}(n)$ and $\lambda_{q,E}^{\prime(\beta;k,\ell)}(n)$ be given by Definition* 1.2. *For each $1 \leqslant p \leqslant k$,*

$$\lambda_{p,E}^{(\beta;k,\ell)}(n) \simeq \frac{n}{k+\ell}, \quad \text{namely} \quad \lim_{n\to\infty} \bigl(\lambda_{p,E}^{(\beta;k,\ell)}(n)\bigr) \Big/ \Bigl(\frac{n}{k+\ell}\Bigr) = 1.$$

Similarly, for each $1 \leqslant q \leqslant \ell$,

$$\lambda_{q,E}^{\prime(\beta;k,\ell)}(n) \simeq \frac{n}{k+\ell}.$$

Proof. We only sketch the proof for the expected row length in the case $\ell = 0$, thus showing that

$$\lambda_{p,E}^{(\beta;k,0)}(n) \simeq \frac{n}{k}.$$

The main point is that since $\beta > 0$, both sums in the numerator and the denominator of Definition 1.2 are dominated by the summands corresponding to the partitions λ, such that $\lambda_i = n/k + c_i \cdot \sqrt{n}$ and with c_is in a bounded interval. In other words, let $a > 0$ and denote

$$H_a(k,0;n) = \Bigl\{\lambda \in H(k,0;n) \mid \lambda_i = \frac{n}{k} + c_i \cdot \sqrt{n}, \text{ where } |c_i| \leqslant a,\ i = 1,2,\ldots,k\Bigr\}.$$

Then

$$\lim_{n\to\infty} \lambda_{p,E}^{(\beta;k,0)}(n) = \lim_{n\to\infty} \frac{\sum_{\lambda\in H(k,0;n)} \lambda_{p,n} \cdot (f^\lambda)^\beta}{\sum_{\lambda\in H(k,0;n)} (f^\lambda)^\beta} = \lim_{a\to\infty} \Biggl[\lim_{n\to\infty} \frac{\sum_{\lambda\in H_a(k,0;n)} \lambda_{p,n} \cdot (f^\lambda)^\beta}{\sum_{\lambda\in H_a(k,0;n)} (f^\lambda)^\beta}\Biggr].$$

Writing $\lambda_{p,n} = n/k + c_{p,n} \cdot \sqrt{n}$, the expression in the brackets equals

$$\frac{\sum_{\lambda\in H_a(k,0;n)} (n/k)(f^\lambda)^\beta}{\sum_{\lambda\in H_a(k,0;n)} (f^\lambda)^\beta} + \frac{\sum_{\lambda\in H_a(k,0;n)} c_{p,n}\sqrt{n}(f^\lambda)^\beta}{\sum_{\lambda\in H_a(k,0;n)} (f^\lambda)^\beta}.$$

The first summand equals n/k while the absolute value of the second summand is bounded by $a\sqrt{n}$ since all $|c_p(n)| \leqslant a$. Since $n/k + a\sqrt{n} \simeq n/k$, it follows that

$$\lambda_{p,E}^{(\beta;k,0)}(n) \simeq \frac{n}{k},$$

which completes the proof in the case $\ell = 0$. $\square$

5. Second term approximation, the 'strip' case ($\ell = 0$)

Because of Theorem 4.1, we look for a more subtle approximation of $\lambda_{p,E}^{(\beta;k,\ell)}(n)$, namely, we look for the expected deviation—of the form $c \cdot \sqrt{n}$—from $n/(k+\ell)$. This leads us to introduce the asymptotic expected value $c_{p,E}^{(\beta;k,\ell)}$ below. We begin with the 'strip' case $\ell = 0$. The general (k, ℓ)-hook case is given in the next section.

Definition 5.1. Let $\Gamma_n = H(k, 0; n)$ and let $1 \leqslant p \leqslant k$, with $\lambda_{p,E}^{(\beta;k,0)}(n)$ given by Definition 1.2. Define $c_{p,E}^{(\beta;k,0)}(n)$ via the equation

$$\lambda_{p,E}^{(\beta;k,0)}(n) = \frac{n}{k} + c_{p,E}^{(\beta;k,0)}(n) \cdot \sqrt{n}, \quad \text{and} \quad c_{p,E}^{(\beta;k,0)} = \lim_{n\to\infty} c_{p,E}^{(\beta;k,0)}(n).$$

Thus, when n goes to infinity,

$$\lambda_{p,E}^{(\beta;k,0)}(n) \simeq \frac{n}{k} + c_{p,E}^{(\beta;k,0)} \cdot \sqrt{n}.$$

Remark 5.2. It is not obvious that the limit $c_{p,E}^{(\beta;k,0)} = \lim_{n\to\infty} c_{p,E}^{(\beta;k,0)}(n)$ exists. However, Theorem 5.3 asserts that in fact, this limit does exist.

Our aim is to calculate $c_{p,E}^{(\beta;k,0)}$, thus calculating "the second term" in the approximation of the expected value of the pth row-length $\lambda_{p,E}^{(\beta;k,0)}$. Definitions 1.2 and 5.1 obviously imply the equation

$$c_{p,E}^{(\beta;k,0)}(n) = \left(\frac{\sum_{\lambda\in H(k,0;n)} \lambda_{p,n}(f^\lambda)^\beta}{\sum_{\lambda\in H(k,0;n)}(f^\lambda)^\beta} - \frac{n}{k}\right)\frac{1}{\sqrt{n}}. \tag{12}$$

If $k = 1$, Eq. (12) implies that for any $\beta > 0$, $c_{1,E}^{(\beta;1,0)} = 0$.

Theorem 5.3. *Let $k \geqslant 2$. The limit $c_{p,E}^{(\beta;k,0)} = \lim_{n\to\infty} c_{p,E}^{(\beta;k,0)}(n)$ exists, and is given by*

$$c_{p,E}^{(\beta;k,0)} = \frac{\int_{\Omega_k} x_p \cdot [D_k(x_1,\ldots,x_k)\cdot e^{-\frac{k}{2}(x_1^2+\cdots+x_k^2)}]^\beta d^{(k-1)}x}{\int_{\Omega_k}[D_k(x_1,\ldots,x_k)\cdot e^{-\frac{k}{2}(x_1^2+\cdots+x_k^2)}]^\beta d^{(k-1)}x}. \tag{13}$$

Proof. In Eq. (12) write $\lambda_{i,n} = \frac{n}{k} + c_{i,n}\cdot\sqrt{n}$, and consider λs with c_i bounded in some interval: $|c_i| \leqslant a$ for some $a > 0$. By an argument similar to the proof of Theorem 4.1, it follows that as $n \to \infty$, $c_{p,E}^{(\beta;k,0)}(n)$ is approximated by the ratio

$$\frac{\sum_{\lambda\in H_a(k,0;n)} c_{p,n}(f^\lambda)^\beta}{\sum_{\lambda\in H_a(k,0;n)}(f^\lambda)^\beta}. \tag{14}$$

The proof now follows by applying Theorem 2.1 to the denominator, Theorem 2.2 to the numerator, then canceling equal terms. □

Remark 5.4. Note that the denominator of (13) is actually a "Selberg"—or a Macdonald–Mehta—integral, which can be evaluated for any β. For example, let $\beta = 2$. By comparing (F.2.10) with (F.4.5.2) of [17], deduce that

$$\int_{\Omega_k} \Big[D_k(x_1, \ldots, x_k) \cdot e^{-\frac{k}{2}(x_1^2+\cdots+x_k^2)} \Big]^2 d^{(k-1)}x$$

$$= \big(\sqrt{2\pi}\big)^{k-1} \cdot \left(\frac{1}{\sqrt{2}}\right)^{k^2-1} \cdot \left(\frac{1}{\sqrt{k}}\right)^{k^2} \cdot 1! \cdot 2! \cdots (k-1)!.$$

Similarly, by comparing (F.2.10) with (F.4.5.1) of [17], deduce that when $\beta = 1$,

$$\int_{\Omega_k} \Big[D_k(x_1, \ldots, x_k) \cdot e^{-\frac{k}{2}(x_1^2+\cdots+x_k^2)} \Big] d^{(k-1)}x$$

$$= \left(\frac{1}{\sqrt{k}}\right)^{k(k+1)} \cdot \frac{1}{k!} \cdot \big(\sqrt{2}\big)^{3k-1} \cdot \frac{1}{\sqrt{\pi}} \cdot \prod_{j=1}^{k} \Gamma\left(1 + \frac{1}{2} j\right).$$

For explicit values, recall that $\Gamma(3/2) = \pi/2$, that $\Gamma(1) = 1$, and that $\Gamma(z+1) = z\Gamma(z)$.

6. Second term approximation, the (k, ℓ)-hook case

We turn now to the general (k, ℓ)-hook case.

Definition 6.1. Let $\beta > 0$ and let $1 \leqslant p \leqslant k$ and $1 \leqslant q \leqslant \ell$, with $\lambda_{p,E}^{(\beta;k,\ell)}(n)$ given by Definition 1.2. Define $c_{p,E}^{(\beta;k,\ell)}$ via the equation

$$\lambda_{p,E}^{(\beta;k,\ell)}(n) = \frac{n}{k+\ell} + c_{p,E}^{(\beta;k,\ell)}(n) \cdot \sqrt{n}, \quad \text{and} \quad c_{p,E}^{(\beta;k,\ell)} = \lim_{n\to\infty} c_{p,E}^{(\beta;k,\ell)}(n).$$

Similarly for the columns:

$$\lambda_{q,E}^{\prime(\beta;k,\ell)}(n) = \frac{n}{k+\ell} + c_{q,E}^{\prime(\beta;k,\ell)}(n) \cdot \sqrt{n}, \quad \text{and} \quad c_{q,E}^{\prime(\beta;k,\ell)} = \lim_{n\to\infty} c_{q,E}^{\prime(\beta;k,\ell)}(n).$$

The existence of the limits $c_{p,E}^{(\beta;k,\ell)}$ and $c_{q,E}^{\prime(\beta;k,\ell)}$ is asserted by Theorem 6.3 below.
Definitions 1.2 and 6.1 obviously imply

Remark 6.2.

$$c_{p,E}^{(\beta;k,\ell)}(n) = \left(\frac{\sum_{\lambda\in H(k,\ell;n)} \lambda_{p,n} (f^\lambda)^\beta}{\sum_{\lambda\in H(k,\ell;n)} (f^\lambda)^\beta} - \frac{n}{k+\ell} \right) \cdot \frac{1}{\sqrt{n}},$$

and

$$c_{q,E}^{\prime(\beta;k,\ell)}(n) = \left(\frac{\sum_{\lambda\in H(k,\ell;n)} \lambda'_{q,n} (f^\lambda)^\beta}{\sum_{\lambda\in H(k,\ell;n)} (f^\lambda)^\beta} - \frac{n}{k+\ell} \right) \cdot \frac{1}{\sqrt{n}}.$$

Theorem 6.3. *Let $1 \leqslant p \leqslant k$, $1 \leqslant q \leqslant \ell$, then the limits $c_{p,E}^{(\beta;k,\ell)}$ and $c_{q,E}^{\prime(\beta;k,\ell)}$ exist, and are given as follows.*

$$c_{p,E}^{(\beta;k,\ell)} = \frac{\int_{\Omega_k} x_p \cdot [D_k(x_1,\ldots,x_k) \cdot e^{-\frac{k+\ell}{2}(x_1^2+\cdots+x_k^2)}]^\beta d^{(k-1)}(x)}{\int_{\Omega_k} [D_k(x_1,\ldots,x_k) \cdot e^{-\frac{k+\ell}{2}(x_1^2+\cdots+x_k^2)}]^\beta d^{(k-1)}(x)}, \tag{15}$$

and

$$c_{q,E}^{\prime(\beta;k,\ell)} = \frac{\int_{\Omega_\ell} y_q \cdot [D_\ell(y_1,\ldots,y_\ell) \cdot e^{-\frac{k+\ell}{2}(y_1^2+\cdots+y_\ell^2)}]^\beta d^{(\ell-1)}(y)}{\int_{\Omega_\ell} [D_\ell(y_1,\ldots,y_\ell) \cdot e^{-\frac{k+\ell}{2}(y_1^2+\cdots+y_\ell^2)}]^\beta d^{(\ell-1)}(y)}. \tag{16}$$

Thus, as n goes to infinity,

$$\lambda_{p,E}^{(\beta;k,\ell)}(n) \simeq \frac{n}{k+\ell} + c_{p,E}^{(\beta;k,\ell)} \cdot \sqrt{n} \quad \textit{and} \quad \lambda_{q,E}^{\prime(\beta;k,\ell)}(n) \simeq \frac{n}{k+\ell} + c_{q,E}^{\prime(\beta;k,\ell)} \cdot \sqrt{n}.$$

Proof. We prove for $c_{p,E}^{(\beta;k,\ell)}$—in two steps.

Step 1. We claim that the limit $c_{p,E}^{(\beta;k,\ell)}$ exists, and is given by

$$c_{p,E}^{(\beta;k,\ell)} = \frac{\int_{\Omega_{(k,\ell)}} x_p \cdot [D_k(x) \cdot D_\ell(y) \cdot e^{-\frac{k+\ell}{2}(\sum x_i^2 + \sum y_j^2)}]^\beta d^{(k+\ell-1)}(x;y)}{\int_{\Omega_{(k,\ell)}} [D_k(x) \cdot D_\ell(y) \cdot e^{-\frac{k+\ell}{2}(\sum x_i^2 + \sum y_j^2)}]^\beta d^{(k+\ell-1)}(x;y)}. \tag{17}$$

The proof of Eq. (17) is essentially the same as that of Eq. (13). Its starting point is Remark 6.2 (instead of Eq. (12)). Here we write, for $1 \leqslant i \leqslant k$ and for $1 \leqslant j \leqslant \ell$,

$$\lambda_{i,n} = \frac{n}{k+\ell} + c_{i,n} \cdot \sqrt{n} \quad \text{and} \quad \lambda'_{j,n} = \frac{n}{k+\ell} + c'_{j,n} \cdot \sqrt{n},$$

and consider λs with c_i and c'_j bounded in some interval. Now follow a 'hook'-generalization of the proof of Theorem 5.3, applying Theorems 2.3 and 2.4, and complete the proof of Eq. (17).

Step 2. We now transform (17) into (15). Let $I_p^{(\beta;k,\ell)}$ denote the numerator of (17):

$$I_p^{(\beta;k,\ell)} = \int\limits_{\Omega_{(k,\ell)}} x_p \big[D_k(x) \cdot D_\ell(y) \cdot e^{-((k+\ell)/2)(\sum x_i^2 + \sum y_j^2)}\big]^\beta d^{(k+\ell-1)}(x;y).$$

Setting $\sum x_i = u$ we have $\sum y_j = -u$, and $I_p^{(\beta;k,\ell)} = \int_{-\infty}^{\infty} K(u)L(-u)\,du$, where

$$K(u) = \int\limits_{M_k(x,u)} x_p \big[D_k(x) \cdot e^{-((k+\ell)/2)(\sum x_i^2)}\big]^\beta d^{(k-1)}x$$

and

$$L(-u) = \int\limits_{M_\ell(y,-u)} \left[D_\ell(y) \cdot e^{-((k+\ell)/2)(\sum y_j^2)}\right]^\beta d^{(k-1)}x.$$

Here $M_k(x,u) = \{(x_1,\dots,x_k) \mid x_1 \geqslant \cdots \geqslant x_k \text{ and } \sum x_i = u\}$ and similarly, $M_\ell(y,-u) = \{(y_1,\dots,y_\ell) \mid y_1 \geqslant \cdots \geqslant y_\ell \text{ and } \sum y_j = -u\}$.

To evaluate $I_p^{(\beta;k,\ell)}$, proceed as follows. In $K(u)$ and $L(-u)$ substitute $x'_i = x_i - (u/k)$, and $y'_j = y_j + (u/\ell)$. The Jacobians are $= 1$, $x_t = x'_t + (u/k)$; $D_k(x') = D_k(x)$; $D_\ell(y') = D_\ell(y)$; $\sum x_i^2 = \sum x_i'^2 + (u^2/k)$ and $\sum y_j^2 = \sum y_j'^2 + (u^2/\ell)$. Replacing x'_i by x_i and y'_j by y_j, it follows that

$$I_p^{(\beta;k,\ell)} = J_1(x) \cdot J_3(y) \cdot A(u) + J_2(x) \cdot J_3(y) \cdot B(u).$$

Here

$$A(u) = \int_{-\infty}^{\infty} e^{-\frac{(k+\ell)^2 u^2 \beta}{2k\ell}}\, du, \qquad B(u) = \int_{-\infty}^{\infty} \frac{u}{k} \cdot e^{-\frac{(k+\ell)^2 u^2 \beta}{2k\ell}}\, du,$$

$$J_1(x) = \int\limits_{M_k(x,0)} x_p \cdot \left[D_k(x) \cdot e^{-\frac{k+\ell}{2}(\sum x_i^2)}\right]^\beta d^{(k-1)}x, \quad \bigl(J_1(x) = 1 \text{ if } k = 1\bigr)$$

$$J_2(x) = \int\limits_{M_k(x,0)} \left[D_k(x) \cdot e^{-\frac{k+\ell}{2}(\sum x_i^2)}\right]^\beta d^{(k-1)}x,$$

and

$$J_3(y) = \int\limits_{M_\ell(y,0)} \left[D_\ell(x) \cdot e^{-\frac{k+\ell}{2}(\sum y_j^2)}\right]^\beta d^{(\ell-1)}x, \quad \bigl(J_3(y) = 1 \text{ if } \ell = 1\bigr)$$

$(M_r(x,0) = \{(x_1,\dots,x_r) \mid x_1 \geqslant \cdots \geqslant x_r \text{ and } x_1 + \cdots + x_r = 0\}$, etc.). Since, trivially, $B(u) = 0$, deduce that $I_p^{(\beta;k,\ell)} = J_1(x) \cdot J_3(y) \cdot A(u)$.

Let $\bar{I}^{(\beta;k,\ell)}$ denote the denominator in Eq. (17). By exactly the same arguments it follows that $\bar{I}^{(\beta;k,\ell)} = J_2(x) \cdot J_3(y) \cdot A(u)$. By (17)

$$c_{p,E}^{(\beta;k,\ell)} = \frac{I_p^{(\beta;k,\ell)}}{\bar{I}^{(\beta;k,\ell)}} = \frac{J_1(x)}{J_2(x)},$$

which is the right-hand side of Eq. (15). This completes the proof. □

Remark 6.4. The denominator-integral in (17) is

$$\bar{I}^{(\beta;k,\ell)} = \int\limits_{\Omega_{(k,\ell)}} \Big[D_k(x_1,\dots,x_k)\cdot D_\ell(y_1,\dots,y_\ell)\cdot e^{-\frac{k+\ell}{2}(x_1^2+\cdots+x_k^2+y_1^2+\cdots+y_\ell^2)}\Big]^\beta d^{(k+\ell-1)}(x;y),$$

and is calculated explicitly in [6, Section 7] (here $\beta = 2z$):

$$I^*(k,\ell) = I(k,\ell,2) = \frac{1}{k!\ell!}\sqrt{2\pi^{k+\ell-1}}\cdot\sqrt{\frac{\beta}{2\pi}}\cdot\left(\frac{1}{\beta(k+\ell)}\right)^{\frac{1}{2}[(k(k-1)+\ell(\ell-1))(\beta/2)+k+\ell]}$$
$$\times\frac{\prod_{i=1}^k\Gamma(i\beta/2+1)\cdot\prod_{j=1}^\ell\Gamma(j\beta/2+1)}{\Gamma(\beta/2+1)},$$

where Γ is the *Gamma* function ($\Gamma(n+1)=n!$).

Theorems 5.3 and 6.3 imply

Corollary 6.5. *Let* $1\leqslant p\leqslant k$ *and* $1\leqslant q\leqslant\ell$. *Then*

$$c_{p,E}^{(\beta;k,\ell)} = \sqrt{\frac{k}{k+\ell}}\cdot c_{p,E}^{(\beta;k,0)}, \quad \textit{and similarly} \quad c_{q,E}^{\prime(\beta;k,\ell)} = \sqrt{\frac{\ell}{k+\ell}}\cdot c_{q,E}^{\prime(\beta;0,\ell)}.$$

Proof. Let $\alpha = k/(k+\ell)$ and in Eq. (15) substitute $x=\sqrt{\alpha}\cdot v$. By routine calculations, this substitution transforms the ration of integrals (15) into the ratio in (13)—multiplied by the factor $\sqrt{\alpha}$, which completes the proof. □

7. The distribution functions

7.1. First term approximation

Definition 7.1. Let $z>0$. Denote

$$H(k,\ell;n,z) = \Big\{\lambda\in H(k,\ell;n)\mid \lambda_{1,n},\lambda'_{1,n}\leqslant\frac{n}{k+\ell}+z\sqrt{n}\Big\}.$$

Let $\beta>0$, $1\leqslant p\leqslant k$ and $1\leqslant q\leqslant\ell$. The distribution of the length of the pth row as a function of z is defined as

$$\lambda_p^{(\beta;k,\ell)}(n,z) = \frac{\sum_{\lambda\in H(k,\ell;n,z)}\lambda_{p,n}\cdot(f^\lambda)^\beta}{\sum_{\lambda\in H(k,\ell;n)}(f^\lambda)^\beta}. \tag{18}$$

Similarly, the distribution of the length of the qth column as a function of z is defined as

$$\lambda_q^{\prime(\beta;k,\ell)}(n,z) = \frac{\sum_{\lambda\in H(k,\ell;n,z)}\lambda'_{q,n}\cdot(f^\lambda)^\beta}{\sum_{\lambda\in H(k,\ell;n)}(f^\lambda)^\beta}. \tag{19}$$

Let $\lambda \in H(k,\ell;n,z)$ with n large, then necessarily $\lambda_1,\dots,\lambda_k \geqslant \ell$ and $\lambda'_1,\dots,\lambda'_\ell \geqslant k$ (in fact, Lemma 7.2 proves a much stronger property) so that $\lambda_1,\dots,\lambda_k+\lambda'_1,\dots,\lambda'_\ell = n+k\ell$. Write

$$\lambda_p = \frac{n+k\ell}{k+l} + c_p\cdot\sqrt{n}, \quad p=1,\dots,k, \quad \text{and} \quad \lambda'_q = \frac{n+k\ell}{k+l} + c'_q\cdot\sqrt{n}, \quad q=1,\dots,\ell,$$

and notice that $\sum c_p + \sum c'_q = 0$.

Lemma 7.2. *With the above notations* (*and n large*),

$$-\frac{\ell+p-1}{k-p+1}\cdot z \leqslant c_p \leqslant z, \quad p=1,\dots,k, \quad \textit{and} \quad -\frac{k+q-1}{\ell-q+1}\cdot z \leqslant c'_q \leqslant z, \quad q=1,\dots,\ell.$$

Proof. Clearly, all $c_p, c'_q \leqslant z$. Assume for example that

$$c_p < -\frac{\ell+p-1}{k-p+1}\cdot z, \quad \text{hence also} \quad c_k, c_{k-1},\dots,c_p < -\frac{\ell+p-1}{k-p+1}\cdot z.$$

Thus

$$0 = c_1+\cdots+c_k+c'_1+\cdots+c'_\ell < -(k-p+1)\cdot\frac{\ell+p-1}{k-p+1}\cdot z + c_1+\cdots+c_{p-1}$$
$$+c'_1+\cdots+c'_\ell \leqslant -(\ell+p-1)\cdot z + \ell+p-1)\cdot z = 0,$$

a contradiction. Similarly for c'_q. This proves the lemma. □

Recall that

$$\Omega_{(k,\ell)} = \left\{(x_1,\dots,x_k,y_1,\dots,y_\ell) \mid x_1 \geqslant \cdots \geqslant x_k;\ y_1 \geqslant \cdots \geqslant y_\ell;\ \sum x_i + \sum y_j = 0\right\}$$

and

$$\Omega_{(k,\ell),z} = \left\{(x_1,\dots,x_k;y_1,\dots,y_\ell) \in \Omega_{(k,\ell)} \mid x_1, y_1 \leqslant z\right\}.$$

For example, $\Omega_{(2,0),z} = \{(x,-x) \mid 0 \leqslant x \leqslant z\}$.

Theorem 7.3. *Let* $\beta, z > 0$ *and denote*

$$r_{(k,\ell),\beta}(z) = \frac{\int_{\Omega_{(k,\ell),z}} [D_k(x)\cdot D_\ell(y)\cdot e^{-\frac{k+\ell}{2}(\sum x_i^2+\sum y_j^2)}]^\beta d^{(k+\ell-1)}(x;y)}{\int_{\Omega_{(k,\ell)}} [D_k(x)\cdot D_\ell(y)\cdot e^{-\frac{k+\ell}{2}(\sum x_i^2+\sum y_j^2)}]^\beta d^{(k+\ell-1)}(x;y)}. \tag{20}$$

Let $1 \leqslant p \leqslant k$, $1 \leqslant q \leqslant \ell$ *and let* $n \to \infty$, *then*

$$\lambda_p^{(\beta;k,\ell)}(n,z), \lambda_q'^{(\beta;k,\ell)}(n,z) \simeq \frac{n}{k+\ell}\cdot r_{(k,\ell),\beta}(z).$$

In other words,

$$\lim_{n\to\infty}\frac{k+\ell}{n}\cdot\lambda_p^{(\beta;k,\ell)}(n,z)=\lim_{n\to\infty}\frac{k+\ell}{n}\cdot\frac{\sum_{\lambda\in H(k,\ell;n,z)}\lambda_{p,n}\cdot(f^\lambda)^\beta}{\sum_{\lambda\in H(k,\ell;n)}(f^\lambda)^\beta}$$
$$=\frac{\int_{\Omega_{(k,\ell),z}}[D_k(x)\cdot D_\ell(y)\cdot e^{-\frac{k+\ell}{2}(\sum x_i^2+\sum y_j^2)}]^\beta d^{(k+\ell-1)}(x;y)}{\int_{\Omega_{(k,\ell)}}[D_k(x)\cdot D_\ell(y)\cdot e^{-\frac{k+\ell}{2}(\sum x_i^2+\sum y_j^2)}]^\beta d^{(k+\ell-1)}(x;y)}. \tag{21}$$

Similarly for

$$\lim_{n\to\infty}\frac{k+\ell}{n}\cdot\lambda_q'^{(\beta;k,\ell)}(n,z).$$

Proof. is similar to the proof of Theorem 4.1: Let $\lambda_{p,n}=n/(k+\ell)+c_{p,n}\cdot\sqrt{n}$, then

$$\frac{k+\ell}{n}\cdot\lambda_p^{(\beta;k,\ell)}(n,z)=\frac{\sum_{\lambda\in H(k,\ell;n,z)}(f^\lambda)^\beta}{\sum_{\lambda\in H(k,\ell;n)}(f^\lambda)^\beta}+\frac{k+\ell}{\sqrt{n}}\cdot\left(\frac{\sum_{\lambda\in H(k,\ell;n,z)}c_{p,n}\cdot(f^\lambda)^\beta}{\sum_{\lambda\in H(k,\ell;n)}(f^\lambda)^\beta}\right).$$

By Theorem 2.5, the first summand approaches $r_{(k,\ell),\beta}(z)$ as n goes to infinity, and by Lemma 7.2, $|c_{p,n}|\leqslant b\cdot z$ for an appropriate constant $b>0$, therefore the second summand obviously goes to zero as n goes to infinity. □

Theorem 7.3 is a first-term approximation of the distribution function.

7.2. Conjectures about the second term approximation

Let

$$s_{(k,\ell),\beta}(n,z)=\frac{\sum_{\lambda\in H(k,\ell;n,z)}(f^\lambda)^\beta}{\sum_{\lambda\in H(k,\ell;n)}(f^\lambda)^\beta}. \tag{22}$$

As in the proof of Theorem 7.3, $\lim_{n\to\infty}s_{(k\ell),\beta}(n,z)=r_{(k,\ell),\beta}(z)$. Numerical evidence suggest the following (vague) conjecture.

Conjecture 7.4. *For all $k,\ell\geqslant 0$ and $\beta,z>0$, as n goes to infinity the expression*

$$\frac{\sqrt{n}}{k+\ell}\cdot\left[s_{(k,\ell),\beta}(n,z)-r_{(k,\ell),\beta}(z)\right] \tag{23}$$
$$=\frac{\sqrt{n}}{k+\ell}\cdot\left[\frac{\sum_{\lambda\in H(k,\ell;n,z)}(f^\lambda)^\beta}{\sum_{\lambda\in H(k,\ell;n)}(f^\lambda)^\beta}-\frac{\int_{\Omega_{(k,\ell),z}}[D_k(x)\cdot D_\ell(y)\cdot e^{-\frac{k+\ell}{2}(\sum x_i^2+\sum y_j^2)}]^\beta d^{(k+\ell-1)}(x;y)}{\int_{\Omega_{(k,\ell)}}[D_k(x)\cdot D_\ell(y)\cdot e^{-\frac{k+\ell}{2}(\sum x_i^2+\sum y_j^2)}]^\beta d^{(k+\ell-1)}(x)}\right]$$

oscillates in some symmetric bounded interval centered at zero. We denote that interval as $(-L(k,\ell,\beta,z),L(k,\ell,\beta,z))$, where (we conjecture that) $-L(k,\ell,\beta,z)$ and $L(k,\ell,\beta,z)$ are the respective infimum and supremum of the values in Eq. (23).

Definition 7.5. Let $\beta,\ z > 0$, $1 \leqslant p \leqslant k$ and $1 \leqslant q \leqslant \ell$. Then $\lambda_p^{(\beta;k,\ell)}(n,z)$ is given by Eq. (18). Similarly for the columns $\lambda_q'^{(\beta;k,\ell)}(n,z)$. Now define $c_p^{(\beta;k,\ell)}(n,z)$ via the equation

$$\lambda_p^{(\beta;k,\ell)}(n,z) = \frac{n}{k+\ell} \cdot r_{(k,\ell),\beta}(z) + c_p^{(\beta;k,\ell)}(n,z)\sqrt{n}.$$

Similarly for $c_q'^{(\beta;k,\ell)}(n,z)$. We would like to understand the behavior of $c_p^{(\beta;k,\ell)}(n,z)$ and $c_q'^{(\beta;k,\ell)}(n,z)$ as n goes to infinity. We consider $c_p^{(\beta;k,\ell)}(n,z)$.

Note that Definition 7.1, Theorem 7.3 and Definition 7.5 imply

Proposition 7.6.

$$c_p^{(\beta;k,\ell)}(n,z) = \left(\frac{\sum_{\lambda \in H(k,\ell;n,z)} \lambda_{p,n}(f^\lambda)^\beta}{\sum_{\lambda \in H(k,\ell;n)}(f^\lambda)^\beta} - \frac{n}{k+\ell} \cdot r_{(k,\ell),\beta}(z)\right) \cdot \frac{1}{\sqrt{n}}. \tag{24}$$

Conjecture 7.7. *Recall the interval* $(-L(k,\ell,\beta,z), L(k,\ell,\beta,z))$ *from Conjecture* 7.4. *As n goes to infinity,* $c_p^{(\beta;k,\ell)}(n,z)$ *oscillates in the interval* $(-L(k,\ell,\beta,z), L(k,\ell,\beta,z)) + s_p^{(k,\ell),\beta}(z)$, *where*

$$s_p^{(k,\ell),\beta}(z) = \frac{\int_{\Omega_{(k,\ell),z}} x_p \cdot [D_k(x) \cdot D_\ell(y) \cdot e^{-\frac{k+\ell}{2}(\sum x_i^2 + \sum y_j^2)}]^\beta d^{(k+\ell-1)}(x)}{\int_{\Omega_{(k,\ell)}} [D_k(x) \cdot D_\ell(y) \cdot e^{-\frac{k+\ell}{2}(\sum x_i^2 + \sum y_j^2)}]^\beta d^{(k+\ell-1)}(x)}. \tag{25}$$

Proof. Is based on Conjecture 7.4 as follows. In (24) write $\lambda_{p,n} = n/(k+\ell) + c_{p,n} \cdot \sqrt{n}$, then $c_p^{(\beta;k,\ell)}(n,z) = A(n) + B(n)$, where

$$A(n) = \frac{\sqrt{n}}{k+\ell} \cdot \left(\frac{\sum_{\lambda \in H(k,\ell;n,z)}(f^\lambda)^\beta}{\sum_{\lambda \in H(k,\ell;n)}(f^\lambda)^\beta} - r_{(k,\ell),\beta}(z)\right)$$

and

$$B(n) = \frac{\sum_{\lambda \in H(k,\ell;n,z)} c_{p,n} \cdot (f^\lambda)^\beta}{\sum_{\lambda \in H(k,\ell;n)}(f^\lambda)^\beta}.$$

By arguments similar to those in previous proofs, $\lim_{n\to\infty} B(n) = s_{(k,\ell),\beta}(z)$, and by Conjecture 7.4, $A(n)$ oscillates in the interval $(-L(k,\ell,\beta,z), L(k,\ell,\beta,z))$. $\square$

Part II. Some special cases

8. $\beta = 2$, comparison of expected and maximal shapes

8.1. Expected shape

Recall the RSK bijection $\sigma \leftrightarrow (P_\lambda, Q_\lambda)$ and the subsets

$$S_{k,\ell;n} = \{\sigma \in S_n \mid \text{under the RSK } \sigma \longleftrightarrow (P_\lambda, Q_\lambda),\ \lambda \in H(k,\ell;n)\}$$

from Section 1.4. For example if $\ell = 0$, it follows from well-known properties of the RSK that $S_{k,0;n}$ are the permutations in S_n with longest decreasing subsequence having length $\leqslant k$. In general, if $\sigma \in S_n$ is of shape $\lambda = (\lambda_1, \lambda_2, \ldots)$, then λ_1 is the length of a maximal increasing subsequence in σ. Thus, for example, $\lambda_{1,E}^{(2;k,\ell)}$ (see Definition 1.2) is the expected length of the longest increasing subsequences in $S_{k,\ell;n}$.

Consider first the case $\ell = 0$. Since the case $k = 1$ is trivial, assume $k \geqslant 2$ (and $\ell = 0$). In that case, by Theorem 5.3, the expected shape in $S_{k,0;n}$ is $\lambda_E^{(2;k,0)} \simeq (\frac{n}{k} + c_1 \cdot \sqrt{n}, n/k + c_2 \cdot \sqrt{n}, \ldots, n/k + c_k \cdot \sqrt{n})$, where for $1 \leqslant p \leqslant k$,

$$c_p = c_{p,E}^{(2;k,0)} = \frac{\int_{\Omega_k} x_p \cdot [D_k(x_1, \ldots, x_k) \cdot e^{-\frac{k}{2}(x_1^2 + \cdots + x_k^2)}]^2 d^{(k-1)}x}{\int_{\Omega_k} [D_k(x_1, \ldots, x_k) \cdot e^{-\frac{k}{2}(x_1^2 + \cdots + x_k^2)}]^2 d^{(k-1)}x}. \tag{26}$$

In general, the expected shape of the permutations in $S_{k,\ell;n}$ is given by Definition 1.2, and the asymptotic shape as $n \to \infty$ is given by Theorem 6.3, both with $\beta = 2$.

8.2. Maximal f^λ

Given a subset of partitions $\Gamma_n \subseteq Y_n$, one looks for $\lambda = \lambda_{\max} \in \Gamma_n$ with maximal degree f^λ, see Section 1.3. In the case $\Gamma_n = H(k, 0; n)$, the asymptotics of $\lambda_{\max}$ is given in [2], which we briefly describe here. The analogue result from [18], for the case $\Gamma_n = H(k, \ell; n)$, is also described below.

Let $H_k(x)$ denote the kth Hermit polynomial. It is defined via the equation

$$\frac{d^k}{dx^k}\left(e^{-x^2}\right) = (-1^k) H_k(x) e^{-x^2}.$$

Thus $H_0(x) = 1$, $H_1(x) = 2x$, $H_2(x) = 4x^2 - 2$, $H_3(x) = 4x(2x^2 - 3)$, $H_4(x) = 16x^4 - 48x^2 + 12$, etc. $H_k(x)$ is of degree k and its roots are real and distinct, denoted

$$x_1^{(k)} < x_2^{(k)} < \cdots < x_k^{(k)}.$$

Also, $x_1^{(k)} + x_2^{(k)} + \cdots + x_k^{(k)} = 0$. The following theorem is proved in [2]:

Theorem 8.1. *As $n \to \infty$, the maximum* $\max\{f^\lambda \mid \lambda \in H(k, 0; n)\}$ *occurs when*

$$\lambda \simeq \lambda_{\max}^{(k,0)} = \left(\frac{n}{k} + x_k^{(k)} \sqrt{\frac{k}{n}}, \ldots, \frac{n}{k} + x_1^{(k)} \sqrt{\frac{k}{n}}\right).$$

The analogue result for $\Gamma_n = H(k, \ell; n)$ is given in [18]:

Theorem 8.2. *Let $\lambda = \lambda_{\max}^{(k,\ell)} \in H(k, \ell; n)$ maximize f^λ in $H(k, \ell; n)$ and write $\lambda_{\max}^{(k,\ell)} = (\lambda_1, \ldots, \lambda_k, \ldots)$ and $\lambda_{\max}^{\prime(k,\ell)} = (\lambda'_1, \ldots, \lambda'_\ell, \ldots)$, and assume $n \to \infty$, then*

$$(\lambda_1, \ldots, \lambda_k) \simeq \left(\frac{n}{k+\ell} + x_k^{(k)} \sqrt{\frac{n}{k+\ell}}, \ldots, \frac{n}{k+\ell} + x_1^{(k)} \sqrt{\frac{n}{k+\ell}}\right)$$

and

$$\left(\lambda'_1,\dots,\lambda'_\ell\right)\simeq\left(\frac{n}{k+\ell}+x_\ell^{(\ell)}\sqrt{\frac{n}{k+\ell}},\dots,\frac{n}{k+\ell}+x_1^{(\ell)}\sqrt{\frac{n}{k+\ell}}\right).$$

Here $x_1^{(k)}<\cdots<x_k^{(k)}$ *are the roots of the kth Hermit polynomial and similarly for the* $x_j^{(\ell)}$*s.*

8.3. Examples of some $(k,0)$ cases

Let $\Gamma_n=H(k,0;n)$ and $\beta=2$ ($\leftrightarrow_{RSK} S_{k,0;n}$) and compare the expected shape $\lambda_E^{(2;k,0)}$ with the maximizing shape $\lambda_{\max}^{(k,0)}$. We begin with

The case $k=2$. Here the expected shape is given by $\lambda_E^{(2;2,0)}\simeq(n/2+c_{1,E}^{(2;2,0)}\sqrt{n},n/2+c_{2,E}^{(2;2,0)}\sqrt{n})$, where $c_{2,E}^{(2;2,0)}=-c_{1,E}^{(2;2,0)}$, and by Eq. (26), $c_{1,E}^{(2;2,0)}=I_1/I_2$ is the ratio of the following integrals:

$$I_1=\int_0^\infty x\left[2x\cdot e^{-2x^2}\right]^2dx=\frac{1}{8},\quad\text{and}\quad I_2=\int_0^\infty\left[2x\cdot e^{-2x^2}\right]^2dx=\frac{\sqrt{\pi}}{8}.$$

Thus $c_{1,E}^{(2;2,0)}=\frac{1}{\sqrt{\pi}}$, hence

$$\lambda_E^{(2;2,0)}\simeq\left(\frac{n}{2}+\frac{1}{\sqrt{\pi}}\cdot\sqrt{n},\frac{n}{2}-\frac{1}{\sqrt{\pi}}\cdot\sqrt{n}\right)=\left(\frac{n}{2}+0.56419\sqrt{n},\frac{n}{2}-0.56419\sqrt{n}\right).\qquad(27)$$

Compare $\lambda_E^{(2;2,0)}$ with $\lambda_{\max}^{(2,0)}$: Here $x_2^{(2,0)}=1/\sqrt{2}$, $x_1^{(2,0)}=-1/\sqrt{2}$, so

$$\lambda_{\max}^{(2,0)}=\left(\frac{n}{2}+\frac{1}{2}\sqrt{n},\frac{n}{2}-\frac{1}{2}\sqrt{n}\right)=\left(\frac{n}{2}+0.5\sqrt{n},\frac{n}{2}-0.5\sqrt{n}\right).$$

Note: Working with Eq. (12) and with 'Mathematica' we calculated $c_{1,E}^{(2;2,0)}(n)$. For $n=100$, 200, 300, 400, 500, 600, 700, 800, 900, the corresponding values of $c_{1,E}^{(2;2,0)}(n)$ are: 0.517699, 0.530593, 0.536496, 0.54007, 0.542533, 0.544364, 0.545795, 0.546952 and 0.547915, agreeing with Theorem 5.3.

The case $k=3$. Here $\lambda_E^{(2;3,0)}\simeq(n/3+c_{1,E}^{(2;3,0)}\sqrt{n},n/3+c_{2,E}^{(2;3,0)}\sqrt{n},n/3+c_{3,E}^{(2;3,0)}\sqrt{n})$, so we calculate the cs. By Eq. (26), $c_{1,E}^{(2;3,0)}=J_1/J_2$ is the ratio of the following integrals:

$$J_1=\int_{\Omega_3}x_1\left[D_3(x)e^{-\frac{3}{2}(x_1^2+x_2^2+x_3^2)}\right]^2d^{(2)}x\quad\text{and}\quad J_2=\int_{\Omega_3}\left[D_3(x)e^{-\frac{3}{2}(x_1^2+x_2^2+x_3^2)}\right]^2d^{(2)}x.$$

By Remark 5.4 with $\beta=2$ and $k=3$, $J_2=\pi/324\sqrt{3}$. We calculate the numerator J_1. The domain Ω_3 of integration is defined by: $x_1\geqslant x_2\geqslant x_3$ and $x_3=-(x_1+x_2)$, so $x_1\geqslant x_2\geqslant$

$-(x_1+x_2)$. When $x_2 \leqslant 0$, that condition is equivalent to $x_2 \geqslant -x_1/2$. When $x_2 \geqslant 0$, that condition is equivalent to $x_2 \leqslant x_1$. It follows that

$$J_1 = \int_0^{\infty}\left[\int_{-x_1/2}^{x_1}\left(x_1\left[(x_1-x_2)(2x_1+x_2)(x_1+2x_2)e^{-3(x_1^2+x_2^2+x_1x_2)}\right]^2\right)dx_2\right]dx_1.$$

After some routine calculations (‘Mathematica’ was used here) we obtain $J_1 = \sqrt{\pi}/(288\sqrt{2})$. It follows that when $k=3$,

$$\lim_{n\to\infty} c_{1,E}^{(2;3,0)}(n) = c_{1,E}^{(2;3,0)} = \frac{9\sqrt{3}}{8\sqrt{2\pi}} = 0.777362\ldots. \tag{28}$$

By similar calculations it follows that $c_{2,E}^{(2;3,0)} = 0$, hence $c_{3,E}^{(2;3,0)} = -c_{1,E}^{(2;3,0)}$. Thus

$$\lambda_E^{(2;3,0)} \simeq \left(\frac{n}{3} + \frac{9\sqrt{3}}{8\sqrt{2\pi}}\sqrt{n}, \frac{n}{3}, \frac{n}{3} - \frac{9\sqrt{3}}{8\sqrt{2\pi}}\sqrt{n}\right) = \left(\frac{n}{3} + 0.777362\sqrt{n}, \frac{n}{3}, \frac{n}{3} - 0.777362\sqrt{n}\right).$$

Compare now with $\lambda_{\max}^{(3,0)}$. Since $H_3(x) = 4x(2x^2-3)$, $x_3^{(3,0)} = \sqrt{3}/\sqrt{2}$, $x_2^{(3,0)} = 0$ and $x_1^{(3,0)} = -\sqrt{3}/\sqrt{2}$. Thus

$$\lambda_{\max}^{(3,0)} \simeq \left(\frac{n}{3} + \frac{1}{\sqrt{2}}\sqrt{n}, \frac{n}{3}, \frac{n}{3} - \frac{1}{\sqrt{2}}\sqrt{n}\right) = \left(\frac{n}{3} + 0.707107\sqrt{n}, \frac{n}{3}, \frac{n}{3} - 0.707107\sqrt{n}\right).$$

Note: For $n = 200, 300, 400, 500, 700, 850$ and 1100, ‘Mathematica’ and Eq. (12) give the following corresponding values of $c_{1,E}^{(2;3,0)}(n)$: 0.719084, 0.729014, 0.735096, 0.739317, 0.74494, 0.747816 and 0.751261, in accordance with Theorem 5.3.

9. Examples of some (k,ℓ)-hook cases, $\beta = 2$

9.1. *The* $(1,1)$ *case*

By Eq. (15) $c_{1,E}^{(2;1,1)} = 0$. (Alternatively, calculate $c_{1,E}^{(\beta;1,1)}$ by Eq. (17). Here $\Omega_{1,1} = \{(x,y) \mid x+y=0\}$, so $y=-x$ and $-\infty \leqslant x \leqslant \infty$. Thus $c_{1,E}^{(\beta;1,1)} = I_1(\beta)/I_2(\beta)$ where $I_1(\beta) = \int_{-\infty}^{\infty} x[e^{-2x^2}]^\beta\,dx = 0$.) Similarly $c_{1,E}'^{(2;1,1)} = 0$ by Eq. (16). It follows that (for any $\beta > 0$)

$$\lambda_E^{(\beta;1,1)} \simeq \left(\frac{n}{2}, 1^{n/2}\right).$$

9.2. *The* $(2,1)$ *case*

By Corollary 6.5

$$\lim_{n\to\infty} c_{1,E}^{(2;2,1)}(n) = c_{1,E}^{(2;2,1)} = \sqrt{\frac{2}{3}} \cdot c_{1,E}^{(2;2,0)} = \sqrt{\frac{2}{3}} \cdot \frac{1}{\sqrt{\pi}} = 0.460659. \tag{29}$$

Since $c_{2,E}^{(2;2,0)} = -c_{1,E}^{(2;2,0)}$, deduce that

$$c_{2,E}^{(2;2,1)} = -\sqrt{\frac{2}{3}} \cdot \frac{1}{\sqrt{\pi}} = -0.460659. \tag{30}$$

Since the sum of the coordinates in $\lambda_E^{(2;2,1)}(n)$ is n, it follows that $c_{1,E}^{\prime(2;2,1)} = 0$ (this can also be deduced directly from Theorem 6.3). Therefore

$$\lambda_E^{(2;2,1)} \simeq \left(\frac{n}{3} + 0.460659\sqrt{n},\ \frac{n}{3} - 0.460659\sqrt{n},\ 1^{n/3}\right).$$

Note: For $n = 100$, 200, 300, 400, 700 and 1100, Remark (6.2) and 'Mathematica' give the following values of $c_{1,E}^{(2;2,1)}(n)$: 0.45423, 0.45571, 0.456475, 0.456962, 0.457777 and 0.458317, agreeing with Theorem 6.3.

10. Expected shape, $\beta = 1$

10.1. The general case

Since $\sigma \in S_n$ is an involution iff the RSK yields $\sigma \leftrightarrow (P_\lambda, P_\lambda)$, therefore $\lambda_E^{(1;k,\ell)}(n)$ is the *expected shape of the involutions in* $S_{k,\ell;n}$. Note that when $\Gamma_n = Y_n$ (and $\beta = 1$) Baik and Rains [5] showed that the expected length of the first row (or of the longest increasing subsequence in involutions in S_n) is again $2\sqrt{n}$, i.e. the same as the first row of $\lambda_{\max}$, see [5] (1.5). As we show below, this is *not* the case when $\Gamma_n = H(k, \ell; n)$.

Denote $\tilde{\lambda}^{(k,\ell)}(n) = \lambda_E^{(1;k,\ell)}(n)$. We summarize:

(1) Let $1 \leqslant p \leqslant k$, $1 \leqslant q \leqslant \ell$, then

$$\tilde{\lambda}_{p,n}^{(k,\ell)} = \frac{\sum_{\lambda \in H(k,\ell;n)} \lambda_{p,n} \cdot f^\lambda}{\sum_{\lambda \in H(k,\ell;n)} f^\lambda} \quad \text{and} \quad \tilde{\lambda}_{q,n}^{\prime(k,\ell)} = \frac{\sum_{\lambda \in H(k,\ell;n)} \lambda'_{q,n} \cdot f^\lambda}{\sum_{\lambda \in H(k,\ell;n)} f^\lambda}.$$

(2) Define $\tilde{c}_{p,n}^{(k,\ell)}$ via

$$\tilde{\lambda}_{p,n}^{(k,\ell)} = \frac{n}{k+\ell} + \tilde{c}_{j,n}^{(k,\ell)} \sqrt{n}, \quad \text{and} \quad \tilde{c}_p^{(k,\ell)} = \lim_{n \to \infty} \tilde{c}_{p,n}^{(k,\ell)}.$$

Similarly for $\tilde{c}_{q,n}^{\prime(k,\ell)}$ and $\tilde{c}_q^{\prime(k,\ell)}$. Thus, when $n \to \infty$,

$$\tilde{\lambda}_{p,n}^{(k,\ell)} \simeq \frac{n}{k+\ell} + \tilde{c}_p^{(k,\ell)} \cdot \sqrt{n} \quad \text{and} \quad \tilde{\lambda}_{q,n}^{\prime(k,\ell)} \simeq \frac{n}{k+\ell} + \tilde{c}_q^{\prime(k,\ell)} \cdot \sqrt{n}.$$

(3) (Theorem 6.3, $\beta = 1$) Let $1 \leqslant p \leqslant k$, $1 \leqslant q \leqslant \ell$, then the limits $\tilde{c}_p^{(k,\ell)}$ and $\tilde{c}_q^{\prime(k,\ell)}$ exist, and are given as follows.

$$\tilde{c}_p^{(k,\ell)} = \frac{\int_{\Omega_k} x_p \cdot D_k(x_1, \ldots, x_k) \cdot e^{-\frac{k+\ell}{2}(x_1^2 + \cdots + x_k^2)}\, d^{(k-1)}(x)}{\int_{\Omega_k} D_k(x_1, \ldots, x_k) \cdot e^{-\frac{k+\ell}{2}(x_1^2 + \cdots + x_k^2)}\, d^{(k-1)}(x)}. \tag{31}$$

Similarly for the expected columns:

$$\tilde{c}_q'^{\,(k,\ell)} = \frac{\int_{\Omega_\ell} y_q \cdot D_\ell(y_1,\ldots,y_\ell)\cdot e^{-\frac{k+\ell}{2}(y_1^2+\cdots+y_\ell^2)}\, d^{(\ell-1)}(y)}{\int_{\Omega_\ell} D_\ell(y_1,\ldots,y_\ell)\cdot e^{-\frac{k+\ell}{2}(y_1^2+\cdots+y_\ell^2)}\, d^{(\ell-1)}(y)}. \tag{32}$$

(4) (Corollary 6.5, $\beta = 1$) We have

$$\tilde{c}_p^{(k,\ell)} = \sqrt{\frac{k}{k+\ell}}\cdot \tilde{c}_p^{(k,0)}, \quad \text{and similarly} \quad \tilde{c}_q'^{\,(k,\ell)} = \sqrt{\frac{\ell}{k+\ell}}\cdot \tilde{c}_q'^{(0,\ell)}.$$

10.2. Examples for involutions in $S_{k,0;n}$

(i.e. $\beta = 1$ and $\ell = 0$).

When $k = 2$, we obtain $\int_0^\infty 2x\cdot e^{-2x^2} = 1/2$ and $\int_0^\infty x2x\cdot e^{-2x^2} = \sqrt{\pi}/(4\sqrt{2})$. Thus $\tilde{c}_1^{(2,0)} = \sqrt{\pi}/(2\sqrt{2})$ and

$$\begin{aligned}\tilde{\lambda}^{(2,0)} &= \left(\tilde{\lambda}_1^{(2,0)}, \tilde{\lambda}_2^{(2,0)}\right) \simeq \left(\frac{n}{2} + \frac{\sqrt{\pi}}{2\sqrt{2}}\cdot\sqrt{n}, \frac{n}{2} - \frac{\sqrt{\pi}}{2\sqrt{2}}\cdot\sqrt{n}\right) \\ &= \left(\frac{n}{2} + 0.626657\sqrt{n}, \frac{n}{2} - 0.626657\sqrt{n}\right).\end{aligned} \tag{33}$$

Note: For $n = 200$, 400, 600, 800, 1000, 1200 and 1400, Remark 6.2 and 'Mathematica' give the following values of $\tilde{c}_1^{(2,0)}(n)$: 0.592086, 0.602049, 0.606506, 0.609175, 0.611002, 0.612354 and 0.613406, agreeing with Eq. (31).

When $k = 3$ we have

By Eq. (31), $c_{1,E}^{(1;3,0)} = \tilde{J}_1/\tilde{J}_2$ is the ratio of the following integrals:

$$\tilde{J}_1 = \int_{\Omega_3} x_1 D_3(x) e^{-\frac{3}{2}(x_1^2+x_2^2+x_3^2)} d^{(2)}x \quad \text{and} \quad \tilde{J}_2 = \int_{\Omega_3} D_3(x) e^{-\frac{3}{2}(x_1^2+x_2^2+x_3^2)} d^{(2)}x.$$

By Remark 5.4 ($\beta = 1$, $k = 3$), $\tilde{J}_2 = \sqrt{\pi}/27$. We calculate the numerator $\tilde{J}_1$. Similar to the evaluation of J_1 in Section 8.3, here

$$\tilde{J}_1 = \int_0^\infty \left[\int_{-x_1/2}^{x_1} \left(x_1(x_1-x_2)(2x_1+x_2)(x_1+2x_2)e^{-3(x_1^2+x_2^2+x_1x_2)}\right) dx_2\right] dx_1 = 1/18.$$

It follows that when $k = 3$, $\tilde{c}_1^{(3,0)} = c_{1,E}^{(1;3,0)} = 3/(2\sqrt{\pi}) = 0.846284\ldots$. Again, $\tilde{c}_2^{(3,0)} = 0$ so $\tilde{c}_3^{(3,0)} = -\tilde{c}_1^{(3,0)}$. Thus, when $k = 3$,

$$\begin{aligned}\tilde{\lambda}^{(3,0)} = \lambda_E^{(1;3,0)} &= \left(\frac{n}{3} + \frac{3}{2\sqrt{\pi}}\sqrt{n},\ \frac{n}{3},\ \frac{n}{3} - \frac{3}{2\sqrt{2\pi}}\sqrt{n}\right) \\ &= \left(\frac{n}{3} + 0.846284\sqrt{n},\ \frac{n}{3},\ \frac{n}{3} - 0.846284\sqrt{n}\right).\end{aligned} \tag{34}$$

Note: For $n = 200$, 400, 600, 800, 1000, and 1200, Remark 6.2 and 'Mathematica' give the following values of $\tilde{c}_1^{(3,0)}(n) = c_{1,E}^{(1;3,0)}$: 0.789051, 0.80486, 0.812115, 0.816513, 0.819547 and 0.821802, as predicted by Eq. (31).

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References

[1] D.J. Aldous, P. Diaconis, Longest increasing subsequences: from patience sorting to the Baik–Deift–Johansson theorem, Bull. Amer. Math. Soc. (N.S.) 36 (1999) 413–432.
[2] R. Askey, A. Regev, Maximal degrees for Young diagrams in a strip, European J. Combin. 5 (1982) 189–191.
[3] J. Baik, P. Deift, K. Johansson, On the distribution of the length of the longest increasing subsequence of random permutations, J. Amer. Math. Soc. 12 (1999) 1119–1178.
[4] J. Baik, P. Daift, K. Johansson, On the distribution of the length of the second row of a Young diagram under Plancherel measure, Geom. Funct. Anal. 10 (6) (2000) 1606–1607.
[5] J. Baik, E. Rains, The asymptotics of monotone subsequences of involutions, Duke Math. J. 109 (2001) 205–281.
[6] A. Berele, A. Regev, Hook Young diagrams with applications to combinatorics and to representations of Lie superalgebras, Adv. Math. 64 (1987) 118–175.
[7] A. Borodin, Longest increasing subsequence of random colored permutations, Electron. J. Combin. 6 (1999), Research Paper 13, 12 pp.
[8] A. Borodin, A. Okounkov, G. Olshanski, Asymptotics of Plancherel measures for symmetric groups, J. Amer. Math. Soc. 13 (2000) 481–515.
[9] C. Green, An extension of Schensted theorem, Adv. Math. 14 (1974) 254–265.
[10] J.M. Hammersley, in: Proc. Sixth Berkely Symposium on Mathematical Statistics and Probability, vol. 1, Univ. of California Press, Berkeley/Los Angeles, 1972, pp. 345–394.
[11] K. Johansson, Discrete orthogonal polynomial ensembles and Plancherel measure, Ann. of Math. 153 (2001) 259–296.
[12] B.F. Logan, L.A. Shepp, A variational problem for random Young tableaux, Adv. Math. 26 (1977) 206–222.
[13] I.G. Macdonald, Symmetric Functions and Hall Polynomials, second ed., Oxford Univ. Press, 1995.
[14] I.G. Macdonald, Some conjectures for root systems, SIAM J. Math. Anal. 13 (1982) 988–1007.
[15] M.L. Mehta, Random Matrices, second ed., Academic Press, Boston, 1991.
[16] A. Okounkov, Random matrices and random permutations, Int. Math. Res. Not. 20 (2000) 1043–1095.
[17] A. Regev, Asymptotic values for degrees associated with strips of Young diagrams, Adv. Math. 41 (2) (1981) 115–136.
[18] A. Regev, Maximal degrees for Young diagrams in the (k, ℓ) hook, European J. Combin. 19 (1998) 721–726.
[19] A. Selberg, Bemerkninger om et multiplet integral, Nordisk Mat. Tidskr. 26 (1944) 71–78.
[20] R. Stanley, Enumerative Combinatorics, vol. 2, Cambridge Univ. Press, Cambridge, 1991.
[21] R. Stanley, Recent progress in algebraic combinatorics, Mathematical Challenges of the 21st Century, Los Angeles, CA, 2000, Bull. Amer. Math. Soc. (N.S.) 40 (2003) 55–68 (electronic).
[22] C.A. Tracy, H. Widom, Level spacing distributions and the Bessel kernel, Comm. Math. Phys. 161 (2) (1994) 289–309.
[23] C.A. Tracy, H. Widom, On orthogonal and symplectic matrix ensembles, Comm. Math. Phys. 177 (3) (1996) 727–754.
[24] A.M. Vershik, S.V. Kerov, Asymptotics of the Plancherel measure of the symmetric group and the limit of Young tableaux, Dokl. Akad. Nauk 233 (1977) 1024–1027, English translation in Soviet Math. Dokl. 18 (1977) 527–531.
[25] A.M. Vershik, S.V. Kerov, Asymptotics of maximal and typical dimensions of irreducible representations of the symmetric group, Funktsional. Anal. i Prilozhen. 19 (1981) 25–36.