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On circularly symmetric functions

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ABSTRACT

In this paper, we study the logarithmic coefficients of circularly symmetric functions. Also, we investigate the relative growth of successive coefficients of circularly symmetric functions. Furthermore, we obtain the sharp estimate for the order of $\|D_n| - |D_{n-1}\|$ by using the method of the logarithmic coefficients.

1. Introduction

Let S denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

which is analytic and univalent in the unit disk $\Delta = \{z \in C : |z| < 1\}$.

Let S^* denote the subset of S consisting of those functions $f(z)$ in S for which $f(\Delta)$ is starlike with respect to 0. It is well known (see [1] or [2]) that if $f(z)$ is analytic in Δ, then $f \in S^*$ if and only if $Re\frac{zf'(z)}{f(z)} > 0$, for all z in Δ. Finally, we let S_c denote the set of those functions $f(z)$ in S for which there exists a real number α and a function $g(z)$ in S^* such that

$$Re\frac{zf'(z)}{e^{i\alpha}g(z)} > 0, \quad z \in \Delta.$$

The elements of S_c are called close-to-convex functions. Clearly, $S^* \subset S_c$.

Associated with each $f(z)$ in S is a well-defined logarithmic function

$$\log\frac{f(z)}{z} = 2\sum_{k=1}^{\infty} \gamma_k z^k, \quad z \in \Delta. \tag{1.2}$$

The numbers γ_k are called the logarithmic coefficients of f. Thus the Koebe function $k(z) = z(1-z)^{-2}$ has logarithmic coefficients $\gamma_n = \frac{1}{n}$, $n \in N = \{1, 2, 3, \ldots\}$.

The inequality $|\gamma_n| \le \frac{1}{n}$ holds for functions $f(z)$ in S^*, but is false for the full class S, even in order of magnitude. Indeed, (see Theorem 8.4 on p. 242 of [1]) there exists a bounded function $f(z) \in S$ with logarithmic coefficients $\gamma_n \ne O(n^{-0.83})$.

In the paper [3], it is presented that the inequality $|\gamma_n| \le \frac{1}{n}$ is false for $n \ge 2$.

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Let D be a domain in C with $0 \in D$. We shall say that D is circularly symmetric if, for every R with $0 < R < +\infty$, $D \cap \{|z| = R\}$, is either empty, is the whole circle $|z| = R$, or is a single arc on $|z| = R$ which contains $z = R$ and is symmetric with respect to the real axis. Following [3,4], we shall denote by Y the class of those functions $f(z)$ in S which map Δ onto a circularly symmetric domain. The elements of Y will be called circularly symmetric functions.

The objective of the present paper is to study the logarithmic coefficients of circularly symmetric functions Y (except $f(z) \equiv z$). We obtain the inequality $|\gamma_n| \le An^{-1} \log n$ (A is an absolute constant) which holds for circularly symmetric functions Y. Furthermore, we investigate the relative growth of successive coefficients of circularly symmetric functions Y (except $f(z) \equiv z$), and obtain the sharp estimate of $\| D_n| - |D_{n-1}\|$ by using the method of the logarithmic coefficients.

Let A denote the absolute constant whose value varies in different places.

2. A necessary condition for a function to be in *Y*

First, we give the following lemmas.

Lemma 1 ([4]). *Let $f(z) \in Y$. Then*

$$Im\left\{\frac{zf'(z)}{f(z)}\right\} \ge 0, \quad Im\{z\} \ge 0;$$
$$Im\left\{\frac{zf'(z)}{f(z)}\right\} \le 0, \quad Im\{z\} \le 0.$$

Lemma 2 ([4]). *Let the function $f(z)$ be defined by (1.1). If $f(z) \in Y$, then $f(z) \equiv z$ or $a_2 > 0$.*

Theorem 1. *If $f(z) \in Y$, then*

$$Re\left\{(1-z^2)\frac{g(z)}{z}\right\} > 0,$$

where $g(z) = \frac{1}{a_2}\{\frac{zf'(z)}{f(z)} - 1\}$.

Proof. Suppose $z = re^{i\theta}$, $0 < r < 1$, $0 \le \theta \le \pi$, then

$$Re\left\{(r^2-z^2)\frac{g(z)}{z}\right\} = Re\{2r\sin\theta \cdot (-ig(z))\} = 2r\sin\theta Im\{g(re^{i\theta})\} = 2r\sin\theta \cdot \frac{1}{a_2} Im\left\{\frac{zf'(z)}{f(z)}\right\}.$$

By means of Lemmas 1 and 2, we obtain

$$Re\left\{(r^2-z^2)\frac{g(z)}{z}\right\} \ge 0. \tag{2.1}$$

When $\pi \le \theta \le 2\pi$, the inequality (2.1) also holds. So the inequality (2.1) also holds for $|z| < r$ from the maximum principle. Letting $r \to 1-0$ in (2.1), we obtain $Re\{(1-z^2)\frac{g(z)}{z}\} > 0$. □

3. Logarithmic coefficients of circularly symmetric functions *Y*

In order to obtain Theorem 2, we need the following lemmas.

Lemma 3. *Let $f(z) \in S$. Then, for $z = re^{i\theta}$, $0 < r < 1$,*

$$\left|\int_0^{2\pi} \frac{\partial}{\partial\theta}\left(\log\frac{f(z)}{z}\right) e^{-i\arg\frac{z}{1-z^2}} d\theta\right| \le A\log\frac{1}{1-r}. \tag{3.1}$$

Proof. Integration by parts gives

$$\left|\int_0^{2\pi} \frac{\partial}{\partial\theta}\left(\log\frac{f(z)}{z}\right) e^{-i\arg\frac{z}{1-z^2}} d\theta\right| = \left|\int_0^{2\pi} \log\frac{f(z)}{z} e^{-i\arg\frac{z}{1-z^2}} \frac{\partial}{\partial\theta}\left(\arg\frac{z}{1-z^2}\right) d\theta\right|. \tag{3.2}$$

Applying the distortion theorems of the univalent function, we have (see [5])

$$\left|\log\frac{f(z)}{z}\right| \le \log\left|\frac{f(z)}{z}\right| + \left|\arg\frac{f(z)}{z}\right| \le 3\log\frac{1}{1-r}. \tag{3.3}$$

Since function $\frac{z}{1-z^2} \in S^*$, we have (see [2])

$$\frac{\partial}{\partial\theta}\left(\arg\frac{z}{1-z^2}\right) > 0 \quad \text{and} \quad \int_0^{2\pi} \frac{\partial}{\partial\theta}\left(\arg\frac{z}{1-z^2}\right) \mathrm{d}\theta = 2\pi. \tag{3.4}$$

By applying (3.3) and (3.4), from (3.2) we obtain

$$\left|\int_0^{2\pi} \frac{\partial}{\partial\theta}\left(\log\frac{f(z)}{z}\right) \mathrm{e}^{-\mathrm{i}\arg\frac{z}{1-z^2}} \mathrm{d}\theta\right| \le A\log\frac{1}{1-r}. \quad \square$$

Lemma 4. *Let $f(z) \in S$. Then, for $z = r\mathrm{e}^{\mathrm{i}\theta}$, $0 < r < 1$, $n \ge 2$,*

$$\frac{1}{2\pi}\left|\int_0^{2\pi} \frac{zf'(z)}{f(z)} \mathrm{e}^{-2\mathrm{i}\arg\frac{z}{1-z^2}} \mathrm{e}^{\mathrm{i}n\theta} \mathrm{d}\theta\right| \le \frac{2}{n} + 2\sqrt{2}\left(\log\frac{1}{1-r^2}\right)^{\frac{1}{2}}. \tag{3.5}$$

Proof. Since $z(\log\frac{f(z)}{z})' = \frac{zf'(z)}{f(z)} - 1$, we have $\frac{zf'(z)}{f(z)} = 1 + \sum_{k=1}^{\infty} 2k\gamma_k z^k$. So integration by parts gives

$$\begin{aligned}
\frac{1}{2\pi}\left|\int_0^{2\pi} \frac{zf'(z)}{f(z)} \mathrm{e}^{-2\mathrm{i}\arg\frac{z}{1-z^2}} \mathrm{e}^{\mathrm{i}n\theta} \mathrm{d}\theta\right| &= \frac{1}{2\pi}\left|\int_0^{2\pi} \left(1 + \sum_{k=1}^{\infty} 2k\gamma_k z^k\right) \mathrm{e}^{-2\mathrm{i}\arg\frac{z}{1-z^2}} \mathrm{e}^{\mathrm{i}n\theta} \mathrm{d}\theta\right| \\
&= \frac{1}{\pi}\left|\int_0^{2\pi} \left[\frac{\mathrm{e}^{\mathrm{i}n\theta}}{n} + \sum_{k=1}^{\infty} \frac{2k\gamma_k}{n+k} r^k \mathrm{e}^{\mathrm{i}(n+k)\theta}\right] \mathrm{e}^{-2\mathrm{i}\arg\frac{z}{1-z^2}} \frac{\partial}{\partial\theta}\left(\arg\frac{z}{1-z^2}\right) \mathrm{d}\theta\right| \\
&\le \frac{1}{\pi}\int_0^{2\pi} \left|\frac{\mathrm{e}^{\mathrm{i}n\theta}}{n} + \sum_{k=1}^{\infty} \frac{2k\gamma_k}{n+k} r^k \mathrm{e}^{\mathrm{i}(n+k)\theta}\right| \frac{\partial}{\partial\theta}\left(\arg\frac{z}{1-z^2}\right) \mathrm{d}\theta.
\end{aligned} \tag{3.6}$$

It is well known that (see [6])

$$\sum_{k=1}^{\infty} k|\gamma_k|^2 r^{2k} \le \log\frac{1}{1-r^2}. \tag{3.7}$$

Applying the Schwarz inequality, we have

$$\begin{aligned}
\frac{1}{n} + \sum_{k=1}^{\infty} \frac{2k}{n+k}|\gamma_k| r^k &\le \frac{1}{n} + \left(\sum_{k=1}^{\infty} k|\gamma_k|^2 r^{2k} \sum_{k=1}^{\infty} \frac{4k}{n^2}\right)^{\frac{1}{2}} \\
&\le \frac{1}{n} + \left(\log\frac{1}{1-r^2}\right)^{\frac{1}{2}} \left(\sum_{k=1}^{\infty} \frac{4k}{n^2}\right)^{\frac{1}{2}} = \frac{1}{n} + \sqrt{2}\left(\log\frac{1}{1-r^2}\right)^{\frac{1}{2}}.
\end{aligned} \tag{3.8}$$

By applying (3.8), from (3.6) we obtain

$$\frac{1}{2\pi}\left|\int_0^{2\pi} \frac{zf'(z)}{f(z)} \mathrm{e}^{-2\mathrm{i}\arg\frac{z}{1-z^2}} \mathrm{e}^{\mathrm{i}n\theta} \mathrm{d}\theta\right| \le \frac{2}{n} + 2\sqrt{2}\left(\log\frac{1}{1-r^2}\right)^{\frac{1}{2}}. \quad \square$$

Theorem 2. *Let $f(z) \in Y$. Then, for $n \ge 2$,*

$$|\gamma_n| \le An^{-1}\log n, \tag{3.9}$$

where the exponent -1 is the best possible.

Proof. Write $p(z) = (1-z^2)\frac{g(z)}{z}$, where $g(z) = \frac{1}{a_2}\{\frac{zf'(z)}{f(z)} - 1\}$. Then $Rep(z) > 0$. Clearly,

$$p(z) = 2Rep(z) - \overline{p(z)}. \tag{3.10}$$

Since $z(\log\frac{f(z)}{z})' = \frac{zf'(z)}{f(z)} - 1$, we have

$$\frac{zf'(z)}{f(z)} = 1 + \sum_{k=1}^{\infty} 2k\gamma_k z^k.$$

Then, for $z = re^{i\theta}$, $0 < r < 1$, we have

$$2n\gamma_n = \frac{1}{2\pi i}\oint_{|z|=r}\frac{zf'(z)}{f(z)}z^{-n-1}dz = \frac{1}{2\pi}\int_0^{2\pi}\frac{zf'(z)}{f(z)}r^{-n}e^{-in\theta}d\theta. \tag{3.11}$$

By applying (3.10), from (3.11) we obtain

$$\begin{aligned}|2n\gamma_n r^n| &= \left|\frac{1}{2\pi}\int_0^{2\pi}\frac{zf'(z)}{f(z)}e^{-in\theta}d\theta\right| = \frac{1}{2\pi}\left|\int_0^{2\pi}p(z)\frac{a_2 z}{1-z^2}e^{-in\theta}d\theta + \int_0^{2\pi}e^{-in\theta}d\theta\right|\\ &\le \frac{a_2}{2\pi}\left|\int_0^{2\pi}2Rep(z)\frac{z}{1-z^2}e^{-in\theta}d\theta\right| + \frac{a_2}{2\pi}\left|\int_0^{2\pi}\overline{p(z)}\frac{z}{1-z^2}e^{-in\theta}d\theta\right| = I_1 + I_2.\end{aligned} \tag{3.12}$$

Now we estimate two terms I_1 and I_2.

$$\begin{aligned}I_1 &\le \frac{a_2}{\pi}\int_0^{2\pi}Rep(z)\left|\frac{z}{1-z^2}\right|d\theta = \frac{a_2}{\pi}Re\int_0^{2\pi}p(z)\left|\frac{z}{1-z^2}\right|d\theta\\ &= \frac{1}{\pi}Re\left[\int_0^{2\pi}\frac{zf'(z)}{f(z)}\frac{1-z^2}{z}\left|\frac{z}{1-z^2}\right|d\theta - \int_0^{2\pi}\frac{1-z^2}{z}\left|\frac{z}{1-z^2}\right|d\theta\right]\\ &= \frac{1}{\pi}Re\left[\int_0^{2\pi}\frac{zf'(z)}{f(z)}e^{-i\arg\frac{z}{1-z^2}}d\theta - \int_0^{2\pi}e^{-i\arg\frac{z}{1-z^2}}d\theta\right].\end{aligned} \tag{3.13}$$

It is clear that

$$\frac{zf'(z)}{f(z)} = \frac{1}{i}\frac{\partial}{\partial\theta}\left(\log\frac{f(z)}{z}\right) + 1. \tag{3.14}$$

By means of Lemma 3 and (3.14), from (3.13) we get

$$\begin{aligned}I_1 &\le \frac{1}{\pi}\left|\int_0^{2\pi}\frac{\partial}{\partial\theta}\left(\log\frac{f(z)}{z}\right)e^{-i\arg\frac{z}{1-z^2}}d\theta\right| + \frac{2}{\pi}\left|\int_0^{2\pi}e^{-i\arg\frac{z}{1-z^2}}d\theta\right|\\ &\le A\log\frac{1}{1-r} + 4.\end{aligned} \tag{3.15}$$

By means of Lemma 4, we obtain

$$\begin{aligned}I_2 &= \frac{1}{2\pi}\left|\int_0^{2\pi}\overline{\left(\frac{zf'(z)}{f(z)}\right)\left(\frac{1-z^2}{z}\right)}\frac{z}{1-z^2}e^{-in\theta}d\theta - \int_0^{2\pi}\overline{\left(\frac{1-z^2}{z}\right)}\frac{z}{1-z^2}e^{-in\theta}d\theta\right|\\ &= \frac{1}{2\pi}\left|\int_0^{2\pi}\overline{\left(\frac{zf'(z)}{f(z)}\right)}e^{2i\arg\frac{z}{1-z^2}}e^{-in\theta}d\theta - \int_0^{2\pi}e^{2i\arg\frac{z}{1-z^2}}e^{-in\theta}d\theta\right|\\ &\le \frac{1}{2\pi}\left|\int_0^{2\pi}\frac{zf'(z)}{f(z)}e^{-2i\arg\frac{z}{1-z^2}}e^{in\theta}d\theta\right| + 1 \le \frac{2}{n} + 2\sqrt{2}\left(\log\frac{1}{1-r^2}\right)^{\frac{1}{2}} + 1.\end{aligned} \tag{3.16}$$

Combining (3.15) and (3.16), from (3.12) we obtain

$$|\gamma_n| \le \frac{1}{n}(2r^n)^{-1}\left[A\log\frac{1}{1-r} + 4 + \frac{2}{n} + 2\sqrt{2}\left(\log\frac{1}{1-r^2}\right)^{\frac{1}{2}} + 1\right]. \tag{3.17}$$

Let $r = 1 - \frac{1}{n}$ in (3.17), we obtain, for $n = 2, 3, \ldots$

$$|\gamma_n| \le An^{-1}\log n. \tag{3.18}$$

Since the Koebe function $k(z) = z(1-z)^{-2} \in Y$ has logarithmic coefficients $\gamma_n = \frac{1}{n}$, the exponent -1 is the best possible. □

4. On the successive coefficients of circularly symmetric functions Y

If $f(z) \in S$, and taking into account (1.2), we obtain

$$\varphi(z) = \left[\frac{f(z)}{z}\right]^{\lambda} = 1 + \sum_{k=1}^{\infty}D_k(\lambda)z^k = \exp\left(\sum_{k=1}^{\infty}2\lambda\gamma_k z^k\right) \quad 0 < \lambda \le 1. \tag{4.1}$$

We form a function with an arbitrary parameter t, $|t| = 1$,

$$\psi(z) = (1 - tz)\varphi(z) = 1 + \sum_{k=1}^{\infty}(D_k - tD_{k-1})z^k, \tag{4.2}$$

and we shall start from its representation

$$\psi(z) = q(z)(1 - tz)^{1-\lambda}, \tag{4.3}$$

where

$$q(z) = (1 - tz)^{\lambda}\varphi(z) = \sum_{k=0}^{\infty} B_k(t)z^k, \tag{4.4}$$

$$\log q(z) = \sum_{k=1}^{\infty}\left(2\lambda\gamma_k - \frac{\lambda t^k}{k}\right)z^k = \sum_{k=1}^{\infty} A_k(t)z^k. \tag{4.5}$$

The relative growth of successive coefficients is a difficult and interesting problem in the univalent function area. The authors studied the relative growth of successive coefficients $\|D_n| - |D_{n-1}\|$ of $\varphi(z)$ (see [5,7–13]). In this paper, for the circularly symmetric function which is the subclass of the univalent function, we have resolved the problem and obtained a sharp estimate. In order to obtain our result (Theorem 3), we need the following lemma.

Lemma 5 *([7]). For coefficients B_k of function $q(z)$, for any $n \geq 1$ and for some t_n, $|t_n| = 1$, one has the inequality*

$$\sum_{k=0}^{n-1} |B_k| \leq An^{\lambda}. \tag{4.6}$$

Theorem 3. *Let $f(z) \in Y$, $\varphi(z) = [\frac{f(z)}{z}]^{\lambda} = 1 + \sum_{k=1}^{\infty} D_k(\lambda)z^k,\ 0 < \lambda \leq 1$. Then*

$$\|D_n| - |D_{n-1}\| \leq An^{\lambda-1}\log n.$$

Proof. From (4.2)–(4.4), we have

$$\sum_{k=0}^{\infty}(D_k - tD_{k-1})z^k = \sum_{k=0}^{\infty} B_k z^k\left(1 - \sum_{k=1}^{\infty}\beta_k t^k z^k\right) \quad (D_{-1}(\lambda) = 0, D_0(\lambda) = 1), \tag{4.7}$$

where

$$(1 - tz)^{1-\lambda} = 1 - \sum_{k=1}^{\infty}\beta_k t^k z^k.$$

It is known that for $k \geq 1$

$$\beta_k = \frac{1-\lambda}{k}\left(1 - \frac{1-\lambda}{1}\right)\left(1 - \frac{1-\lambda}{2}\right)\cdots\left(1 - \frac{1-\lambda}{k-1}\right). \tag{4.8}$$

So we have that $\beta_k > 0$ for $k \geq 1$ and $\{\beta_k\}$ is a monotonically decreasing sequence.

Comparing the coefficients of the same powers of z in (4.7) and taking absolute value, we obtain

$$|D_n - tD_{n-1}| \leq |B_n| + \beta_1|B_{n-1}| + \cdots + \beta_n.$$

We make the last inequality coarser

$$|D_n - tD_{n-1}| \leq \max_{n-m\leq w\leq n}|B_w|\left(\sum_{k=1}^{m}\beta_k + 1\right) + \beta_{m+1}\sum_{k=0}^{n-m-1}|B_k|, \quad 0 \leq m < n. \tag{4.9}$$

First we estimate $|B_w|$, $w = n - m, n - m + 1, \dots, n$. From (4.4) and (4.5) it follows that

$$wB_w = wA_w + (w-1)A_{w-1}B_1 + \cdots + 2A_2B_{w-2} + A_1B_{w-1},$$

where $wA_w = 2\lambda w\gamma_w - \lambda t^w$.

However, for circularly symmetric functions Y it is known (see Theorem 2) that

$$w|\gamma_w| \leq A\log w, \quad w = 2, 3, \dots.$$

Therefore

$$|wA_w| = |2\lambda w\gamma_w - \lambda t^w| \leq (2A\log w + 1)\lambda$$

and

$$w|B_w| \leq \max_{1\leq w\leq n} |wA_w| \sum_{k=0}^{n-1} |B_k| \leq A\lambda \log n \sum_{k=0}^{n-1} |B_k|.$$

Taking into account (4.6), we obtain for $t = t_n$, $|t_n| = 1$ and $w = 1, 2, \ldots, n$

$$|B_w| \leq \frac{A\lambda \log n}{w} n^{\lambda}. \tag{4.10}$$

Also by (4.6) we have for $0 \leq m \leq n-1$

$$\sum_{k=0}^{n-m-1} |B_k| \leq \sum_{k=0}^{n-1} |B_k| \leq An^{\lambda}. \tag{4.11}$$

Now we estimate β_k. It is known that

$$\sum_{k=1}^{n} \beta_k \leq 1 \tag{4.12}$$

and

$$\begin{aligned} \beta_k &= \frac{1-\lambda}{k}\left(1-\frac{1-\lambda}{1}\right)\left(1-\frac{1-\lambda}{2}\right)\cdots\left(1-\frac{1-\lambda}{k-1}\right) = \frac{1-\lambda}{k}\left(1+\frac{\lambda-1}{1}\right)\left(1+\frac{\lambda-1}{2}\right)\cdots\left(1+\frac{\lambda-1}{k-1}\right) \\ &\leq \frac{1-\lambda}{k}\exp(\lambda-1)\left(1+\frac{1}{2}+\cdots+\frac{1}{k-1}\right) \leq \frac{1-\lambda}{k}k^{\lambda-1} = \frac{1-\lambda}{k^{2-\lambda}}, \quad k \geq 1. \end{aligned} \tag{4.13}$$

We insert (4.10)–(4.13) into (4.9), assuming that $m = \frac{n}{2}$ if n is even and $m = \frac{n+1}{2}$ if n is odd. We carry out the computations for n even (for n odd it is similar). We have for $n \geq 1$

$$|D_n - tD_{n-1}| \leq \frac{A\lambda n^{\lambda}\log n}{\frac{n}{2}} + \frac{1-\lambda}{\left(\frac{n}{2}+1\right)^{2-\lambda}}An^{\lambda} \leq An^{\lambda-1}\log n.$$

So, we obtain

$$\|D_n| - |D_{n-1}\| \leq |D_n - tD_{n-1}| \leq An^{\lambda-1}\log n. \quad \square$$

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References

[1] P.L. Duren, Univalent Functions, Springer-Verlag, New York, 1983.
[2] Ch. Pommerenke, Univalent Functions, Vandenkoeck and Ruprecht, Göttingen, 1975.
[3] Girele Daniel, Logarithmic coefficients of univalent functions, Ann. Acad. Sci. Fenn. Math. 25 (2000) 337–350.
[4] J.A. Jenkins, On circularly symmetric functions, Proc. Amer. Math. Soc. 6 (1955) 620–624.
[5] G.M. Goluzin, Geometric Theorem of Functions of a Complex Variable, Moscow, 1952.
[6] I.M. Milin, Univalent Functions and Orthonormal Systems, Moscow, 1971 (in Russian).
[7] Q. Deng, The estimate of the difference of the moduli of adjacent coefficients of univalent functions, J. Jiangxi Normal Univ. (Natural Science Edition) 26 (2) (2002) 128–131.
[8] W.K. Hayman, On successive coefficients of univalent functions, J. Lond. Math. Soc. 38 (1963) 228–243.
[9] K.W. Lucas, On successive coefficients of areally mean p-valent functions, J. Lond. Math. Soc. 44 (1969) 631–642.
[10] H. Ke, On successive coefficients of univalent functions, Proc. Amer. Math. Soc. 95 (1986) 37–41.
[11] V.I. Milin, Adjacent coefficients of odd univalent functions, Sib. Math. J. 22 (1981) 283–290.
[12] Zhongqiu Ye, On the successive coefficients of close-to-convex functions, J. Math. Anal. Appl. 283 (2003) 689–695.
[13] Q. Deng, The estimate of the difference of the moduli of adjacent coefficients of Bazilevic functions, Acta Math. Sinica (Chin. Ser.) 49 (5) (2006) 1195–1200.