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On the extraconnectivity of graphs ☆

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Abstract

Given a simple connected graph G, let $\kappa(n)$ $[\lambda(n)]$ be the minimum cardinality of a set of vertices [edges], if any, whose deletion disconnects G and every remaining component has more than n vertices. For instance, the usual connectivity and the superconnectivity of G correspond to $\kappa(0)$ and $\kappa(1)$, respectively. This paper gives sufficient conditions, relating the diameter of G with its girth, to assure optimum values of these conditional connectivities.

1. Introduction

The standard graph theoretic terms not defined in this paper can be found in [3].

A simple connected graph $G = (V, E)$ with diameter D is said to be *l-geodetic* if l is the maximum integer, $1 \leqslant l \leqslant D$, such that for any $x, y \in V(G)$ there exists at most one $x \leftrightarrow y$ path of length less than or equal to l. If $l = D$, the graph G is called *strongly geodetic*, see [2,8]. Notice that if G has girth g, then G is l-geodetic for $l = \lfloor (g-1)/2 \rfloor$. Reciprocally, if G is l-geodetic, then its girth g is either $2l+1$ or $2l+2$.

A sufficient condition for an l-geodetic graph to have maximum connectivity [edge-connectivity] can be formulated in terms of l and D, see [4,9,10].

Theorem 1.1. *Let G be an l-geodetic graph with minimum degree δ, diameter D, connectivity κ and edge-connectivity λ. Then*

$$\kappa = \delta \quad \text{if } D \leqslant 2l - 1,$$
$$\lambda = \delta \quad \text{if } D \leqslant 2l.$$

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Suppose that $G \neq K_{\delta+1}$ is a maximally connected graph with minimum degree δ, i.e. $\kappa = \delta$. If $x \in V(G)$ is a vertex of degree δ, then the set of vertices adjacent to x, $\Gamma(x)$, is a *trivial* minimum order disconnecting set of vertices. It is said that G is *super-κ* if every disconnecting set of vertices of cardinality δ is trivial, see [1]. Analogously, G is said to be super-λ if all its minimum edge-disconnecting sets are trivial.

Let us define a non-trivial set of vertices or edges as a vertex or edge set that does not contain a trivial disconnecting one. The authors and Escudero have proved in [6] that if $G = (V,E)$ is l-geodetic with minimum degree $\delta > 2$ and diameter $D \leqslant 2l - 2$, and $F \subset V$, $|F| \leqslant 2\delta - 3$, is non-trivial, then $G - F$ is connected. Analogously, if $D \leqslant 2l - 1$ and $A \subset E$, $|A| \leqslant 2\delta - 3$, is non-trivial, then $G - A$ is connected. Thus, G is super-κ if $D \leqslant 2l - 2$ and G is super-λ if $D \leqslant 2l - 1$. To reformulate these results, let us define $\kappa(1)$ as the minimum cardinality of a non-trivial set of vertices F, if any, such that $G - F$ is not connected. Define $\lambda(1)$ in a similar way. Then, $\kappa(1)$ and $\lambda(1)$ measure the superconnectivity and edge-superconnectivity of G. Hence, from the above results, we have:

Theorem 1.2. *Let G be an l-geodetic graph with minimum degree $\delta > 2$ and diameter D. Then,*

$$\kappa(1) \geqslant 2\delta - 2 \text{ if } D \leqslant 2l - 2,$$
$$\lambda(1) \geqslant 2\delta - 2 \text{ if } D \leqslant 2l - 1.$$

If we have no further information about the structure of G, then Theorem 1.2 is best possible in the following sense. Suppose that G contains an edge with endvertices x and y of degree δ and such that $\Gamma(x) \cap \Gamma(y) = \emptyset$. The set $F = \Gamma(x) \cup \Gamma(y) - \{x, y\}$ could be an example of non-trivial disconnecting set with $2\delta - 2$ vertices. Thus, for such a graph G, $\kappa(1) \leqslant 2\delta - 2$ and, by the results given in Theorem 1.2, $D \leqslant 2l - 2$ is a sufficient condition for $\kappa(1) = 2\delta - 2$. The edge case can be discussed similarly.

2. The connectivities $\kappa(n)$ and $\lambda(n)$

If H is a subgraph of G, let $N(H)$ denote the set $\bigcup_{u \in V(H)} \Gamma(u) - V(H)$.

Given a graph $G = (V,E)$ and a fixed integer $n \geqslant 0$, let us say that $F \subset V(G)$ is *non-trivial* if F does not contain a set $N(H)$ for any subgraph $H \subset G$ with k vertices, $1 \leqslant k \leqslant n$ (for $n = 0$, any $F \subset V$ is non-trivial). Now, generalizing the definition of $\kappa(1)$ given in Section 1, let us define the *conditional connectivity* $\kappa(n)$ as the minimum cardinality of a non-trivial disconnecting set. In what follows it is supposed that, for the graphs considered, such $\kappa(n)$ exists. The conditional edge-connectivity $\lambda(n)$ can be defined in an analogous way.

Given a graph G and a graph-theoretic property $\mathscr{P}$, Harary [7] defined the conditional connectivity $\kappa(G;\mathscr{P})$ $[\lambda(G;\mathscr{P})]$ as the minimum cardinality of a set of vertices [edges], if any, whose deletion disconnects the graph and every remaining component has property $\mathscr{P}$. From this point of view, $\kappa(n) \equiv \kappa(G;\mathscr{P}_n)$ $[\lambda(n) \equiv \lambda(G;\mathscr{P}_n)]$ where $\mathscr{P}_n$ is the property of having more than n vertices.

When G is not a complete graph, then $\kappa(0)$ $[\lambda(0)]$ corresponds to the connectivity κ $[\lambda]$. So, $\kappa(0)\leqslant\delta$ $[\lambda(0)\leqslant\delta]$ and, by Theorem 1.1, $D\leqslant 2l-1$ $[D\leqslant 2l]$ is a sufficient condition for G to be maximally connected, i.e. $\kappa=\delta$ $[\lambda=\delta]$. For $n=1$, $\kappa(1)$ $[\lambda(1)]$ measures the superconnectivity [edge-superconnectivity] of G and Theorem 1.2 gives a sufficient condition to have optimum superconnectivity [edge-superconnectivity].

If $n>1$, let us say that $\kappa(n)$ and $\lambda(n)$ measure the n-extraconnectivity of G. Suppose that a tree T_{n+1}, with $n+1$ vertices each of degree δ in G, is a subgraph of G. If $F=N(T_{n+1})$, then T_{n+1} is a component of $G-F$. Moreover, if $G-F$ is not connected and each other component has at least $n+1$ vertices, then it is clear that $\kappa(n)\leqslant|F|\leqslant(n+1)\delta-2n$. In the following section, a sufficient condition for $\kappa(n)$ $[\lambda(n)]$ to be optimum, i.e. $\kappa(n)\geqslant(n+1)\delta-2n$ $[\lambda(n)\geqslant(n+1)\delta-2n]$, is derived. This condition relates the parameters l and D. To derive it we always assume that $\delta>2$.

3. Maximally extraconnected graphs with large girth

In what follows, $n\geqslant 0$ denotes an even integer, G an l-geodetic graph with parameter $l>\frac{1}{2}n$ and $F\subset V(G)$, $|F|<(n+1)\delta-2n$, stands for a non-trivial set of vertices. Given a component C of $G-F$, the set of vertices in C at maximum distance from F is denoted $Z(C)$, i.e. $Z(C)=\{z\in V(C): d(z,F)=r\}$, where $r=\max_{x\in V(C)} d(x,F)$.

Proposition 3.1. *Any $z\in Z(C)$ is in a path P_z of $G-F$ of length at least $\frac{1}{2}n+1$.*

Proof. The case $n=0$ being trivial (as $|F|<\delta$ in that case), assume $n\geqslant 2$. If C contains a cycle, then its length is at least $n+3$ because $l>\frac{1}{2}n$ and the result clearly holds. Now suppose that C is a tree. Condition $l>\frac{1}{2}n$ implies that $N(u)\neq N(v)$ for any pair of vertices of C, u,v, such that $d(u,v)\leqslant n$. Hence, C must have diameter greater than n; otherwise $|N(C)|=|F|\geqslant(n+1)\delta-2n$. Then, component C contains at least one $u\leftrightarrow v$ shortest path of length at least $n+1$. Consequently, for any $z\in Z(C)$ there exists in $G-F$ either a $z\leftrightarrow u$ or a $z\leftrightarrow v$ path of length greater than $\frac{1}{2}n$. $\square$

Note that, in fact, Proposition 3.1 holds for any $z\in V(C)$.

To prove our main theorem, we need to take into account a tree T, considered as a subgraph of C, of one of the following types:

Type I: T is simply a path of length $n\geqslant 0$,

$$w_0, w_1, \ldots, w_{n/2-1}, w_{n/2}, w_{n/2+1}, \ldots, w_{n-1}, w_n$$

such that $d(w_i,F)=d(w_{n-i},F)=r-i$, $0\leqslant i\leqslant\frac{1}{2}n$.

Type II: Let $n\geqslant 2$. The structure of T is as shown in Fig. 1. More precisely, given $z\in Z(C)$, consider a path P_z as described in Proposition 3.1 and take a subpath P'_z, of length $\frac{1}{2}n$, that contains z. The tree T has order n and is obtained by attaching an

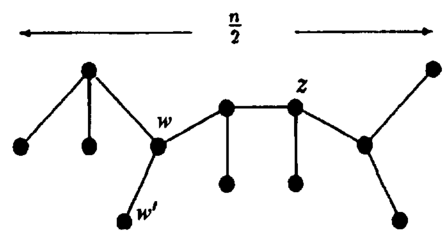

Fig. 1. Tree of type II.

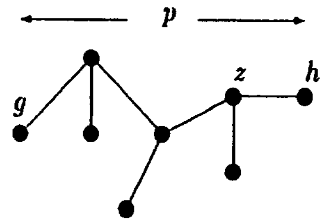

Fig. 2. Tree T'.

edge ww' to each internal vertex w of P_z'. Note that if every internal vertex w of P_z' satisfies $d(w,F) > 1$, then C contains a tree T of type II (as $\delta > 2$). Moreover, if $n > 2$ let z and P_z' be such that z is not an endvertex of P_z'.

Type III: Again let n be at least 2. If $d(u,F) = 1$ for some vertex u in the path P_z that contains z, then it could happen that component C does not contain a tree of type II. In this case, let us consider in C a tree T' with structure as shown in Fig. 2. As in the preceding case, T' is obtained by joining an edge to each internal vertex of a path that contains a vertex $z \in Z(C)$, but now this path P has length $p < \frac{1}{2}n$. The endvertices of P, g and h, satisfy $d(g,F) = d(h,F) = 1$ and $d(w,F) > 1$ for every internal vertex w of P. The order of T' is $2p$. Now, let T be a tree of order n that contains T'. As C has more than n vertices, the existence of such a tree T is assured.

Let T be a tree contained in C such that T contains a vertex $z \in Z(C)$. For every vertex u of T consider a path $P_u = u_0, u_1, \ldots, u_{s-1}, u_s$, $s \geqslant 1$, $u_0 = u$, $u_1 \notin V(T)$, such that $d(u_i,F) > d(u_{i-1},F)$, $1 \leqslant i \leqslant s$, and $d(v,F) \leqslant d(u_s,F)$ for every $v \neq u_{s-1}$ adjacent to u_s (if such a path does not exist, let $s = 0$ and consider the trivial path $P_u = u$). Define $N^*(u)$ as the set of vertices adjacent to u_s that are different from u_{s-1} (if $s = 0$, then define $N^*(u)$ as $\Gamma(u) - V(T)$). Given $u, v \in V(T)$, let $p_T(u,v)$ denote the $u \leftrightarrow v$ path in T. Besides, given a path P in the graph G, $|P|$ will denote its length.

Lemma 3.1. *Let T be a tree of type* I, II *or* III. *For any pair u, v of different vertices of T, the length of the path*

$$u_s, u_{s-1}, \ldots, u_1, p_T(u,v), v_1, \ldots, v_{s'-1}, v_{s'} \tag{1}$$

is at most n. Moreover, $N^*(u) \cap N^*(v) = \emptyset$.

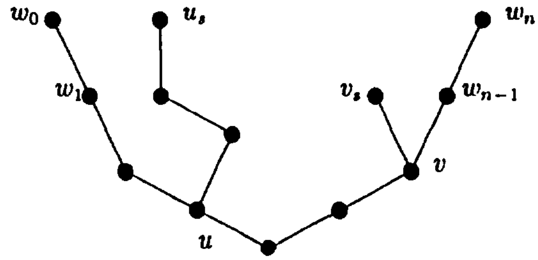

Fig. 3. Tree of type I with the paths P_u and P_v.

Proof. According to the type of T, consider the following cases:

Type I: By the structure of the path T, if $u = w_i$, $0 \leqslant i \leqslant n$, we have

$$|u_s \leftrightarrow u| \leqslant \begin{cases} i, & 0 \leqslant i \leqslant \frac{1}{2}n, \\ n - i, & \frac{1}{2}n < i \leqslant n. \end{cases}$$

Moreover, if $u = w_i$ and $v = w_j$, $0 \leqslant i < j \leqslant n$, then $|p_T(u,v)| = j - i$. Therefore, the length $|u_s \leftrightarrow u| + |p_T(u,v)| + |v \leftrightarrow v_{s'}|$ of the path given in (1) is bounded by

$$\begin{aligned} & i + (j-i) + j = 2j \leqslant n, \quad 0 \leqslant i < j \leqslant \tfrac{1}{2}n, \\ & i + (j-i) + (n-j) = n, \quad 0 \leqslant i \leqslant \tfrac{1}{2}n,\ \tfrac{1}{2}n < j \leqslant n, \\ & (n-i) + (j-i) + (n-j) = 2(n-i) < n, \quad \tfrac{1}{2}n < i < j \leqslant n. \end{aligned}$$

See Fig. 3.

Type II: First, suppose that $p_T(u,z)$ and $p_T(z,v)$ have a common subpath of length $k > 0$, and assume $|p_T(u,z)| \geqslant |p_T(z,v)|$. Clearly, the length of the path $u_s, u_{s-1}, \ldots, u_1$, u is at most $r - d(u,F)$. Analogously, the length of $v, v_1, \ldots, v_{s'-1}, v_{s'}$ is at most $r - d(v,F)$. Moreover, $|p_T(u,z)| \geqslant r - d(u,F)$, $|p_T(z,v)] \geqslant r - d(v,F)$ and $|p_T(z,v)| \leqslant k + 1$. Thus, the length of (1) is upper bounded by

$$\begin{aligned} & (r - d(u,F)) + |p_T(u,z)| + |p_T(z,v)| - 2k + (r - d(v,F)) \\ & \leqslant 2(|p_T(u,z)| + |p_T(z,v)| - k) \leqslant 2(|p_T(u,z)| + 1) \leqslant n. \end{aligned} \tag{2}$$

If $p_T(u,z)$ and $p_T(z,v)$ are edge disjoint paths, clearly $|p_T(u,z)| + |p_T(z,v)| = |p_T(u,v)| \leq \frac{1}{2}n$ and, reasoning as in Eq. (2), we find that the length of (1) is bounded by

$$\begin{aligned} & (r - d(u,F)) + |p_T(u,z)| + |p_T(z,v)| + (r - d(v,F)) \\ & \leqslant 2(|p_T(u,z)| + |p_T(z,v)|) \leqslant n. \end{aligned}$$

Type III: The length of the path given in (1) is now bounded by

$$|u_s \leftrightarrow u| + |p_T(u,v)| + |v \leftrightarrow v_s|.$$

But $|u_s \leftrightarrow u|$ and $|v \leftrightarrow v_s|$ are at most $r - 1$ and $p \geqslant 2(r-1)$. Besides, we clearly have $|p_T(u,v)| \leqslant (n - 2p) + p$ since in the worst case $p_T(u,v)$ contains vertices of T'

which has diameter p. Thus, the length of (1) is bounded by

$$2(r-1)+p+(n-2p)=n-p+2r-2\leqslant n.$$

Note that if $p=0$ (and so $r=1$), the above bound is in fact $n-1$ since in this case $|p_T(u,v)|\leqslant n-1$.

These results imply that all the vertices in (1) must be different and that $N^*(u)\cap N^*(v)=\emptyset$, otherwise $g(G)\leqslant n+2$ contradicting $l>\frac{1}{2}n$. □

Given a tree T contained in C such that T contains a vertex $z\in Z(C)$, let $N^*(T)$ be the set $\bigcup_{u\in V(T)}N^*(u)$. Moreover, if T is of type I, II or III, then $|N^*(T)|\geqslant|N(T)|$. Besides, given $x\in N^*(T)$ let f_x denote a vertex in F such that $d(x,f_x)=d(x,F)$.

Lemma 3.2. *Let $n\geqslant 0$ be an even integer and let G be an l-geodetic graph, $l>\frac{1}{2}n$. If $F\subset V(G)$, $|F|<(n+1)\delta-2n$, is non-trivial, then in any component of $G-F$ there exists a vertex z such that $d(z,F)\geqslant l-\frac{1}{2}n$.*

Proof. The proof goes through the following argument: in any given component C of $G-F$ a tree T' of order $n+1$ containing a vertex $z\in Z(C)$, and such that $|N^*(T')|\geqslant|N(T')|\geqslant(n+1)\delta-2n>|F|$ can be found . Thus, we have $f_x=f_y=f$ for some $x,y\in N^*(T')$, $x\neq y$. Vertices x and y are adjacent to u_s and $v_{s'}$, respectively, endvertices of the paths $P_u=u,u_1,\ldots,u_s$ and $P_v=v,v_1,\ldots,v_{s'}$ for some u and v in $V(T')$. By the construction of P_u and P_v , it is clear that a cycle containing u_s, $v_{s'}$ and f is formed by considering the closed walk

$$f\leftrightarrow x,u_s,\ldots,u_1,p_{T'}(u,v),v_1\ldots,v_{s'},y\leftrightarrow f, \tag{3}$$

where $f\leftrightarrow x$ and $y\leftrightarrow f$ are shortest paths. As we will see, tree T' is in general a tree obtained by adding a vertex to a tree T of type II or type III. In any case, Lemma 3.1 will assure that the length of (3) is at most $d(x,F)+d(y,F)+n+2$. Thus, $d(x,F)+d(y,F)+n+2\geqslant g(G)\geqslant 2l+1$. It follows that either x or y is at distance at least $l-\frac{1}{2}n$ from F, as claimed. Moreover, for any $z\in Z(C)$, $r=d(z,F)\geqslant l-\frac{1}{2}n$.

Certainly any component C of $G-F$ contains a tree of type II or type III. Just begin at a vertex z of $Z(C)$, form two paths towards F and stop if either the length of the combined path (with z as a middle vertex) has length $\frac{1}{2}n$ or if the endvertices have distance 1 to F. Now construct from this combined path a tree of type II or type III as described above. However, to handle some particular values of n and $r=d(z,F)$ it is useful to consider trees of type I. Note that the above reasoning proves the lemma when C contains a tree T of type I ($T'=T$ in this case).

(a) Suppose that C contains a tree T of type II ($n\geqslant 2$). Suppose also that for a certain vertex $w\in V(T)$, the path $P_w=w,w_1,\ldots,w_s$ has length $s>0$. In this case, let T' be the tree obtained by joining to T the edge ww_1, and consider now $N^*(T')$. Reasoning as in the proof of Lemma 3.1, we conclude that for any pair u,v of different vertices of T', $N^*(u)\cap N^*(v)=\emptyset$ and the length of path (1) is at most n. Since T' has order $n+1$, we have $|N^*(T')|\geqslant(n+1)\delta-2n>|F|$ and $f_x=f_y=f$ for some

$x, y \in N^*(T')$. So, a cycle of length at most $d(x,F) + d(y,F) + n + 2$ is found from the closed walk (3). It follows that either x or y is at distance at least $l - \frac{1}{2}n$ from F.

If T is such that for every $w \in V(T)$ the length of the corresponding path P_w is 0, let $v \notin V(T)$ be a vertex adjacent to an endvertex u of P'_z, the path of length $\frac{1}{2}n$ that contains vertex z (by Proposition 3.1 and the definition of T, P'_z is a subpath of a path P_z of length $\frac{1}{2}n + 1$). Let T' be the tree obtained by joining to T the edge vu. Now, consider the path $P_v = v_0, v_1, \ldots, v_s$, defined with respect to T'. The length of the path

$$v_s, v_{s-1}, \ldots, v_1, v, p_T(u,w) \tag{4}$$

is bounded by $(r - d(v,F)) + 1 + |p_T(u,w)|$, for any $w \in V(T)$. Let us consider the following subcases:

(a.1) If $n > 2$, since z is not an endvertex of the path P'_z, we have $r - d(v,F) \leqslant |p_{T'}(v,z)| \leqslant \frac{1}{2}n$ and the length of (4) is at most $\frac{1}{2}n + \frac{1}{2}n + 1 = n + 1$. If the length of (4) is precisely $n + 1$, then $d(v_s,F) = r$ and the path $p_T(z,u), v, v_1, \ldots, v_s$ is a tree of type I contained in C. So, in this case the lemma holds. On the other hand, if the length of (4) is bounded by n, consider $N^*(T')$. Again we have $|N^*(T')| > |F|$ and, reasoning as before, a vertex x such that $d(x,F) \geqslant l - \frac{1}{2}n$ can be found in C.

(a.2) In the case $n = 2$, if $r = d(z,F) = 1$, then the path $P_v = v$ is trivial and the length of (4) is at most 2. Else, when $r > 1$, take as v a vertex adjacent to z and reason as in case (a.1). In particular, if $d(v,F) = r - 1$, consider the tree of type I formed by z, v, v_1, where v, v_1 is P_v.

(b) Now let us consider a tree T of type III contained in the given component C ($n \geqslant 2$). If P_w is non-trivial for at least one vertex w in $V(T)$, the lemma is proved as in case (a). If the length of P_w is 0 for every $w \in V(T)$, then join to T an edge uv for some $v \notin V(T)$ adjacent to $u \in V(T)$. If $p > 0$, reasoning as in the proof of Lemma 3.1, we obtain that, for any $w \in V(T)$, the length of the path $v_s, \ldots, v_1, v, p_T(u,w)$ is now bounded by

$$(r - 1) + 1 + p + (n - 2p) = r - p + n \leqslant n,$$

because $p \geqslant 2(r - 1)$. If $p = 0$ (and $r = 1$), then $v_s = v$ and the length of $v, p_T(u,w)$ is again at most n.

Now, reasoning as in case (a), the vertex claimed by the lemma is found. □

When n is an odd integer, apply Lemma 3.2 to $n' = n + 1$ to obtain the following corollary.

Corollary 3.1. *Let n be an odd positive integer and let G be an l-geodetic graph, $l > \frac{1}{2}(n+1)$. If $F \subset V(G)$, $|F| < (n+1)\delta - 2n$, is non-trivial, then in any component of $G - F$ there exists a vertex z such that $d(z,F) \geqslant l - \frac{1}{2}(n + 1)$.*

A sufficient condition for $\kappa(n)$ to be optimum is given in the following theorem.

Theorem 3.1. *Let G be an l-geodetic graph with diameter D. Then, $\kappa(n) \geqslant (n+1)\delta - 2n$ if*
(a) n is even and $D \leqslant 2l - n - 1$; or
(b) n is odd and $D \leqslant 2l - n - 2$.

Proof. Let $F \subset V(G)$, $|F| < (n+1)\delta - 2n$, be a non-trivial vertex set. Let us consider the case when n is even. We will show that, if $D < 2l - n$, then $G - F$ is connected, that is, between any pair of vertices $x, y \in V(G)$ there is in G an $x \leftrightarrow y$ path that contains no vertex of F. Since $l \leqslant D$, condition $D < 2l - n$ implies $n < l$.

According to Lemma 3.2, in $G - F$ there exist $x \leftrightarrow x'$ and $y \leftrightarrow y'$ paths such that $d(x', F)$ and $d(y', F)$ are at least $l - \frac{1}{2}n$. Therefore, an $x' \leftrightarrow y'$ path of length at most $D < 2(l - \frac{1}{2}n)$ avoids F.

The case n odd is proved analogously from Corollary 3.1. □

In what follows the edge version of Theorem 3.1 is considered. We only give a sketch of the proof since it essentially goes along the same ideas used before.

Theorem 3.2. *Let G be an l-geodetic graph with diameter D. Then, $\lambda(n) \geqslant (n+1)\delta - 2n$ if*
(a) n is even and $D \leqslant 2l - n$; or
(b) n is odd and $D \leqslant 2l - n - 1$.

Suppose that $G - A$ is not connected and let A be minimal so that each component C of $G - A$ is an induced subgraph. Now, let F denote the set of endvertices of the edges of A belonging to C. As in the vertex case, the existence of a vertex $z \in C$ such that $d(z, F) \geqslant l - \frac{1}{2}n$ can be assured. The proof is based again on the existence in C of a tree T' of order $n + 1$, obtained from a tree T of type I, II or III, which satisfies Lemma 3.1. However, the distance $d(u, F)$, $u \in V(T')$, can now be equal to zero. From the extension of T', formed by attaching a path P_u to each vertex $u \in V(T')$, the existence from the closed walk (3) of a cycle containing z is obtained. The main difference from the vertex case is the following: if $u \in F$ and the path P_u of the extension of T' is trivial ($s = 0$), then define $N^*(u) = \{u\}$. Moreover, for each edge of A incident to such a vertex u consider a trivial path $u = f_u$.

The results given by Theorems 3.1 and 3.2 for $n = 2$ were previously obtained by the authors [5]. Besides, Theorems 3.1 and 3.2 prove, when n is even, the conjecture also stated in [5] that, for all n, $D \leqslant 2l - n - 1$ [$D \leqslant 2l - n$] suffices to assure $\kappa(n) \geqslant (n + 1)\delta - 2n$ [$\lambda(n) \geqslant (n + 1)\delta - 2n$].

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References

[1] F. Boesch and R. Tindell, Circulants and their connectivities, J. Graph Theory, 8 (1984) 487–499.
[2] J. Bosák, A. Kotzig and S. Znám, Strongly geodetic graphs, J. Combin. Theory 5 (1968) 170–176.
[3] G. Chartrand and L. Lesniak, Graphs and Digraphs (Wadsworth, Monterrey, 1986).
[4] J. Fàbrega and M.A. Fiol, Maximally connected digraphs, J. Graph Theory 13 (1989) 657–668.
[5] J. Fàbrega and M.A. Fiol, Extraconnectivity of graphs with large girth, Discrete Math. 127 (1994) 163–170.
[6] M.A. Fiol, J. Fàbrega and M. Escudero, Short paths and connectivity in graphs and digraphs, Ars Combin. 29B (1990) 17–31.
[7] F. Harary, Conditional connectivity, Networks 13 (1983) 347–357.
[8] J. Plesnik and S. Znám, Strongly geodetic directed graphs, Acta Fac. Rerum Natur. Univ. Comenian. Math. Publ. 29 (1974) 29–34.
[9] T. Soneoka, H. Nakada and M. Imase, Sufficient conditions for dense graphs to be maximally connected, in: Proc. ISCAS85 (IEEE Press, New York, 1985) 811–814.
[10] T. Soneoka, H. Nakada, M. Imase and C. Peyrat, Sufficient conditions for maximally connected dense graphs, Discrete Mathematics 63 (1987) 53–66.