Analysis of a predator–prey model with Holling II functional response concerning impulsive control strategy

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Abstract

According to biological and chemical control strategy for pest control, we investigate the dynamic behavior of a Holling II functional response predator–prey system concerning impulsive control strategy-periodic releasing natural enemies and spraying pesticide at different fixed times. By using Floquet theorem and small amplitude perturbation method, we prove that there exists a stable pest-eradication periodic solution when the impulsive period is less than some critical value. Further, the condition for the permanence of the system is also given. Numerical results show that the system we consider can take on various kinds of periodic fluctuations and several types of attractor coexistence and is dominated by periodic, quasiperiodic and chaotic solutions, which implies that the presence of pulses makes the dynamic behavior more complex. Finally, we conclude that our impulsive control strategy is more effective than the classical one if we take chemical control efficiently.

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Keywords: Holling II predator–prey model; Impulsive control strategy; Extinction; Permanence; Bifurcation

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1. Introduction

Pest outbreaks often cause serious ecological and economic problems. There are many ways to beat agricultural pests. Biological control is the reduction in pest populations from the actions of other living organisms, often called natural enemies or beneficial species (see [3,5,6]). Virtually all pests have some natural enemies, and the key to successful pest control is to identify the pest and its natural enemy, releasing the beneficial insect early when pest levels are low. One of the first successful cases of biological control in greenhouse was the use of parasitoid *Encarsia formosa* against the greenhouse whitefly *Trialeurodes vaporariorum* on tomatoes and cucumbers (see [25]). Another important method for pest control is chemical control. Pesticides are useful because they quickly kill a significant portion of a pest population and they sometimes provide the only feasible method for preventing economic loss. However, pesticide pollution is also recognized as a major health hazard to human beings and to natural enemies. Therefore, it is important to understand the life cycle of pest so that the pesticide can be applied when the pest is at its most vulnerable—the aim is to achieve maximum effect at minimum levels of pesticide.

Recently, in order to consider the consequences of spraying pesticide and introducing additional predators into a natural pest–predator system, many authors have suggested impulsive differential equations (see [2,11]) to investigate the dynamics of pest control model. Impulsive equations are found in almost every domain of applied science and have been studied in many investigations [1,7,9,10,12–24] in population dynamical systems. Ref. [15] developed Holling II functional response predator–prey system by periodic impulsive immigration of natural enemies. They gave the conditions for extinction of pest and permanence of the system and mainly studied the influence on the inherent oscillation caused by the impulsive perturbations. In Ref. [16], they presented and analyzed the pest–predator model under insecticides used impulsively and focused on the effects of the fraction of population which died due to the pesticide and the pulse period on the survive of the pest and predator. However, wherever possible, different pest control techniques should work together rather than against each other.

In Refs. [13,14], we constructed two kinds of predator–prey impulsive equations to model the process of periodic releasing natural enemies and spraying pesticide at fixed time, respectively. One is a classical Lotka–Volterra predator–prey impulsive system, which corresponding continuous system has a globally asymptotically stable positive equilibrium if it exists. The other is a predator–prey impulsive system with Holling I functional response, which corresponding continuous system may have a stable positive equilibrium and a stable limit cycle at the same time. In Ref. [14], from a biological point of view, we analyzed the dynamics of the system from two cases: general case (taking integrated pest management (IPM), that is, taking biological control and chemical control together) and special case (only choosing chemical control) and compared the validity of the IPM strategy with the classical methods (only biological control or chemical control). However, in Ref. [14], we ignored the side effects of pesticide on natural enemies and assumed the time of spraying pesticide and releasing natural enemies is the same. It is unreasonable. In Ref. [13], considering the effects of pesticide on natural enemies, we constructed a predator–prey impulsive system with Holling I functional response to model the process of periodic biological and chemical control at different fixed time. Since the unforced continuous predator–prey system with Holling I functional response has non-unique dynamics, that is, the solution of such system with different initial values either tends to a locally stable positive equilibrium or to a stable limit cycle, so in Ref. [13], we emphatically investigated the effects of the impulsive perturbations on the unforced continuous system and concluded that such impulsive system has different dynamic behaviors with
different range of initial values when the impulsive perturbations are small, and the solutions of such system are controlled by periodic, quasiperiodic and chaotic solutions.

In this paper, according to biological and chemical control strategy for pest control, we construct a Holling II functional response predator–prey impulsive system by periodic releasing natural enemies and spraying pesticide at different fixed times in Section 2. The dynamic behavior of its corresponding continuous system is different from the classical Lotka–Volterra predator–prey system and Holling I functional response predator–prey system, it only exists a stable limit cycle when its unique positive equilibrium loses its stability. We are interested to know whether this predator–prey impulsive model can lead to qualitatively different dynamics. On the one hand, we analyze the dynamic behavior of this system in Section 3. By using Floquet theorem and small amplitude perturbation method, we show that there exists a stable pest-eradication periodic solution when the period of impulsive effect is less than some critical value. The condition for the permanence of the system is also given. On the other hand, we investigate the effects of the impulsive perturbations on the stable limit cycle of the unforced Holling II predator–prey system in Section 4. Numerical simulations show that the impulsive perturbations destroy the stable limit cycle of the unforced system and make the dynamic behavior of the impulsive system we consider more complex. A brief discussion of our results is given in the last section and we conclude that our impulsive control strategy is more effective than the classical one (only the biological control or the chemical control) if we take chemical control efficiently. Since we are interested in the biological points, the proofs of Lemma 3.2, Theorems 3.1 and 3.2 will be given in Appendix.

2. Model formulation

Based on experiments, Holling [8] suggested three different kinds of functional responses for different kinds of species to model the phenomena of predation, which made the standard Lotka–Volterra system more realistic. So the basic model we consider is based on the following predator–prey model where the prey has logistic growth with no predators and the predator has Holling II functional response:

\[
\begin{align*}
\frac{dx_1(t)}{dt} & = rx_1(t) \left(1 - \frac{x_1(t)}{K}\right) - b \frac{x_1(t)x_2(t)}{\beta + x_1(t)}, \\
\frac{dx_2(t)}{dt} & = c \frac{x_1(t)x_2(t)}{\beta + x_1(t)} - dx_2(t),
\end{align*}
\]  

(2.1)

where \(r, K, b, c, \beta, d\) are positive constants. \(x_1(t)\) and \(x_2(t)\) denote the density of prey species and that of predator species, respectively. We assume that the prey is a dangerous pest, and that the predator was introduced to suppress its density.

We also assume that the prey (pest) with no predators (natural enemies) grows logistically to its environmental capacity \(K\), with an intrinsic birth rate constant \(r\). The per capita rate at which predator \(x_2\) captures prey \(x_1\) is represented by the term \(b x_1(t)/(\beta + x_1(t))\). It has the important feature of levelling off as the size of the prey population becomes large to maximum of \(b\). Since the capture rate is assumed to be proportional to the per capita growth rate of predator \(x_2\), \(c\) represents the maximum growth rate for predator \(x_2\). The constant \(\beta\) is the prey population size at which the growth rate of predator \(x_2\) is half its maximum and \(d\) is the death rate of predator \(x_2\).

For system (2.1), from Chen and Jing [4] we have the following results.
**Lemma 2.1.** System (2.1) has a unique limit cycle which is stable if \( \tilde{a}_2 + 2\tilde{a}_3 > 0 \). Otherwise, if \( \tilde{a}_2 + 2\tilde{a}_3 \leq 0 \), the system has no limit cycle, and further if \( \tilde{a}_1 + \tilde{a}_2 + \tilde{a}_3 > 0 \), then the system has a unique positive equilibrium which is a stable node or focus, where \( \tilde{a}_1 = a_1/d, \tilde{a}_2 = a_2/\delta, \tilde{a}_3 = d\alpha_3/\delta^2 \) and \( a_1 = r, a_2 = r/\beta - r/K, a_3 = -r/\beta K, \delta = (c - d)/\beta \).

Moreover, in system (2.1), it can be seen that (0, 0) is a saddle point and there exists no pest-eradication non-negative equilibrium, so in this case the pest cannot be extinct, and the classical approach of this kind in pest control is not effective. Now we will develop system (2.1) by introducing periodic spraying pesticides and releasing predators at different fixed moment, respectively. That is, we consider the following

de impulsive differential equation:

\[
\begin{align*}
\frac{dx_1}{dt} &= r x_1(t) \left( 1 - \frac{x_1(t)}{K} \right) - b x_1(t) x_2(t) \frac{x_1(t)}{\beta + x_1(t)}, \\
\frac{dx_2}{dt} &= c x_1(t) x_2(t) \frac{x_1(t)}{\beta + x_1(t)} - dx_2(t),
\end{align*}
\]

where \( x_1(t) = \lim_{s \to t^+} x_i(s), x_2(t) = \lim_{s \to t^+} x_i(s), 0 \leq p_1 < 1, \) \((0 \leq p_2 < 1)\) represents the fraction of pests (or predators) which die due to the pesticide at \( t = (n + k - 1)T, \) \( \mu > 0 \) is the release amount of predator at \( t = nT, n \in Z_+ \) and \( Z_+ = \{1, 2, \ldots\} \), \( T \) is the period of the impulsive effect. That is, we can use a combination of biological (periodic releasing natural enemies) and chemical (spraying pesticide) tactics to suppress the pest to the low level.

### 3. Qualitative analysis for model (2.2)

Firstly, we give some notations, definitions and lemmas which will be useful for our main results.

Let \( R_+ = [0, \infty), R_+^2 = \{x \in R^2 : x > 0\} \). Denote \( f = (f_1, f_2) \) the map defined by the right-hand side of system (2.2). The solution of system (2.2), denoted by \( x(t) = (x_1(t), x_2(t)) : R_+ \to R_+^2 \), is piecewise continuous, and it is continuous on \( ((n - 1)T, (n + k - 1)T) \) and \( ((n + k - 1)T, nT) \), \( x((n + k - 1)T^+) = \lim_{t \to (n + k - 1)T^+} x(t) \) and \( x(nT^+) = \lim_{t \to nT^+} x(t) \) exist. Obviously, the global existence and uniqueness of solutions of system (2.2) are guaranteed by the smoothness of \( f \) (see [11]). The following lemmas are obvious.

**Lemma 3.1.** Suppose \( x(t) \) is a solution of system (2.2) with \( x(0^+ \geq 0), \) then \( x(t) \geq 0 \) for \( t \geq 0 \).

**Lemma 3.2.** There exists a constant \( M > 0 \) such that \( x_i(t) \leq M, i = 1, 2 \) for each solution \( x(t) \) of (2.2) with \( t \) large enough.
We give some basic properties of the following subsystem of (2.2):

\[
\frac{dx_2(t)}{dt} = -dx_2(t), \quad t \neq (n + k - 1)T, \quad t \neq nT,
\]

\[
x_2(t^+) = (1 - p_2)x_2(t), \quad t = (n + k - 1)T,
\]

\[
x_2(t^+) = x_2(t) + \mu, \quad t = nT,
\]

\[
x_{20} = x_2(0^+).
\] (3.1)

System (3.1) is a periodically forced linear system; it is easy to obtain that

\[
\tilde{x}_2(t) = \begin{cases} 
\frac{\mu \exp(-d[t - (n - 1)T])}{1 - (1 - p_2)\exp(-dT)}, & (n - 1)T < t \leq (n + k - 1)T, \\
\frac{\mu(1 - p_2)\exp(-d[t - (n - 1)T])}{1 - (1 - p_2)\exp(-dT)}, & (n + k - 1)T < t \leq nT,
\end{cases}
\] (3.2)

\[(\tilde{x}_2(0^+) = \tilde{x}_2(nT^+) = \mu/(1 - (1 - p_2)\exp(-dT)), \tilde{x}_2(kT^+) = (\mu(1 - p_2)\exp(-dkT))/(1 - (1 - p_2)\exp(-dT)))\] is a positive periodic solution of system (3.1). Since the solution of system (3.1) is

\[
\begin{cases} 
(1 - p_2)^n - 1 \left(x_2(0^+) - \frac{\mu}{1 - (1 - p_2)\exp(-dT)}\right)\exp(-dt) + \tilde{x}_2(t), & (n - 1)T < t \leq (n + k - 1)T, \\
(1 - p_2)^n \left(x_2(0^+) - \frac{\mu}{1 - (1 - p_2)\exp(-dT)}\right)\exp(-dt) + \tilde{x}_2(t), & (n + k - 1)T < t \leq nT,
\end{cases}
\]

we have:

**Lemma 3.3.** System (3.1) has a positive periodic solution \(\tilde{x}_2(t)\) and for every solution \(x_2(t)\) of system (3.1) we have \(x_2(t) \to \tilde{x}_2(t)\) as \(t \to \infty\).

Therefore, system (2.2) has a pest-eradication periodic solution \((0, \tilde{x}_2(t))\).

Now we give the conditions which assure the locally asymptotical stability of the pest-eradication periodic solution \((0, \tilde{x}_2(t))\).

**Theorem 3.1.** Let \((x_1(t), x_2(t))\) be any solution of (2.2); then \((0, \tilde{x}_2(t))\) is locally asymptotically stable provided

\[
rT - \frac{b\mu[1 - p_2\exp(-dT) - (1 - p_2)\exp(-dT)]}{\beta d[1 - (1 - p_2)\exp(-dT)]} < \ln \frac{1}{1 - p_1}.
\] (3.3)

Now we investigate the permanence of system (2.2). Before stating our theorem, we give the following definition.
Fig. 1. Dynamical behavior of the pest–predator system (2.2) with $d = 0.2$, $r = 8$, $b = 1$, $β = 1$, $c = 0.95$, $K = 1.6$, $p_1 = 0.1$, $p_2 = 0.01$, $μ = 5.5$, $T = 3.3$, $k = 0.5$, $x_1(0) = 1.5$, $x_2(0) = 1.5$: (a) time-series of the pest population and (b) time-series of the predator population.

**Definition 3.2.** For all solutions $x_i(t)$ with initial values $x_i(0) > 0$ in system (2.2), if there are constants $m, M > 0$ (independent of initial value) and a finite time $T_0$, $m \leq x_i(t) \leq M$ ($i = 1, 2$) holds for $t \geq T_0$. Then system (2.2) is said to be permanent. Here $T_0$ may depend on the initial values $x_i(0)$.

**Theorem 3.2.** System (2.2) is permanent provided

$$rT - \frac{bμ[1 - p_2 \exp(-dkT) - (1 - p_2) \exp(-dT)]}{βd[1 - (1 - p_2) \exp(-dT)]} > \ln \frac{1}{1 - p_1}$$

(3.4)

holds true.

**Remark 3.1.** Let

$$f(T) = rT - \frac{bμ[1 - p_2 \exp(-dkT) - (1 - p_2) \exp(-dT)]}{βd[1 - (1 - p_2) \exp(-dT)]} - \ln \frac{1}{1 - p_1}.$$

Since $f(0) = -\ln 1/(1 - p_1) < 0$, $f(T) \to \infty$ as $T \to \infty$, and $f''(T) > 0$, so $f(T) = 0$ has a unique positive root, denoted by $T_{\text{max}}$.

From Theorem 3.1, we know that the pest-eradication periodic solution $(0, \bar{x}_2(t))$ is asymptotically stable when $T < T_{\text{max}}$. A typical pest-eradication periodic solution of system (2.2) is shown in Fig. 1, where we observe how the variable $x_2(t)$ oscillates in a stable cycle. In contrast, the pest $x_1(t)$ rapidly decreases to zero and $T_{\text{max}} \approx 3.4$. If the period of pulses $T$ is larger than $T_{\text{max}}$, the pest-eradication solution becomes unstable and undergoes a transcritical bifurcation, then the pest and predator can coexist on a stable limit cycle when $T > T_{\text{max}}$ and is close to $T_{\text{max}}$ (see Fig. 2) and system (2.2) can be permanent which follows from Theorem 3.2, but as $T$ increasing, this limit cycle may lose its stability and system (2.2) exhibits a wide variety of dynamic behavior. In the following section, we will analyze this in detail.
4. Impulsive perturbation analysis

Let \( d = 0.2, r = 8, b = 1, \beta = 1, c = 0.95, K = 1.6 \); then \( \bar{a}_2 + 2\bar{a}_3 \approx 0.445 > 0 \), from Lemma 2.1, we know system (2.1) only has a unique stable cycle (see Fig. 3). Now, we investigate the effect of impulsive perturbations on the unforced system (2.1). In Fig. 4, we have displayed bifurcation diagram for the pest population as \( T \) increasing from 3.4 to 7.17 with \( p_1 = 0.1, p_2 = 0.01, \mu = 5.5, k = 0.5 \). The resulting bifurcation diagram clearly shows that system (2.2) has rich dynamics including periodic oscillation, period-doubling bifurcation, period-halving bifurcation, chaos and non-unique dynamics.

The \( T \)-period solution of the forced system (2.2) is still stable for \( 3.4 < T < 4.14 \). When \( T > 4.14 \), it becomes unstable and 3-period solution occurs. The evidence for cascade of period doubling bifurcations leading to chaos can be seen for \( 4.44 < T < 4.66 \) and \( 5.11 < T < 5.33 \), which are both followed by a cascade of period-halving bifurcations from chaos to cycles. A typical chaotic oscillation is captured when \( T = 6.7 \) (see Fig. 5).

From the bifurcation diagram of Fig. 4, we note that several types of attractor may coexist, that is different attractors can coexist with the same \( T \). Which one of attractors is reached depends on the initial values. For example, in Fig. 6, a \( T \)-periodic solution suddenly changes to a \( 3T \)-periodic solution when \( T = 4.15 \), and in Fig. 7a \( 4T \)-periodic solution coexists with another \( 4T \)-periodic solution when \( T = 5.11 \).

We also investigate the effect of the release amount \( \mu \) of predator on the unforced system (2.1). Fig. 8 shows bifurcation diagrams obtained by stroboscopically sampling the pest population \( x_1 \) and the predator population \( x_2 \) for \( 0.01 < \mu < 9.5 \) keeping the rest parameters as \( p_1 = 0.1, p_2 = 0.01, T = 6, k = 0.5 \). The figures also exhibit complex dynamic behavior including period-doubling, period-halving, chaos and non-unique dynamics.

If we select the parameters as \( d = 0.2, r = 4, b = 1, \beta = 1, c = 0.9, K = 6 \), the unforced system (2.1) still has a stable limit cycle. Fig. 9(a) shows the bifurcation diagram for the pest population as a function of \( \mu \) in the range \( 0.01 < \mu < 4.75 \) keeping other parameters as \( p_1 = 0.2, p_2 = 0.001, T = 6, k = 0.5 \). However, the complexities in Fig. 9(a) are not completely similar to that in Fig. 8, and Fig. 9(a) only contains period windows and quasiperiodic solutions with frequency-lockings. A quasiperiodic solution is given in Fig. 9(b).
Fig. 3. Dynamical behavior of the pest–predator system (2.1) (without impulse) with \( d = 0.2, r = 8, b = 1, \beta = 1, c = 0.95, K = 1.6, x_1(0) = 1.5, x_2(0) = 1.5 \): (a) time-series of the pest population and (b) phase portrait.

Fig. 4. Bifurcation diagram of system (2.2) showing the effects of \( T \) with \( d = 0.2, r = 8, b = 1, \beta = 1, c = 0.95, K = 1.6, p_1 = 0.1, p_2 = 0.01, \mu = 5.5, T = 3.3, k = 0.5, x_1(0) = 1.5, x_2(0) = 1.5 \) and \( x_1 \) are plotted for 420 values of \( T \) over \([3.4, 7.17]\).

Fig. 5. A strange attractor: (a) time-series of the pest population and (b) phase portrait.
Fig. 6. Coexistence of $T$-period solution with $3T$-period when $T = 4.15$: (a) solution with initial value $x_1(0) = 1, x_2(0) = 0.2$ will finally tend to a $T$-period solution and (b) solution with initial value $x_1(0) = 1.5, x_2(0) = 1.5$ will finally tend to a $3T$-period solution.

Fig. 7. Coexistence of $4T$-period solution with $4T$-period when $T = 5.11$: (a) solution with initial value $x_1(0) = 1, x_2(0) = 0.2$ will finally tend to a $4T$-period solution and (b) solution with initial value $x_1(0) = 1.5, x_2(0) = 1.5$ will finally tend to another $4T$-period solution.

Fig. 8. Bifurcation diagrams of system (2.2) showing the effect of $\mu$ with $d = 0.2, r = 8, b = 1, \beta = 1, c = 0.95, K = 1.6, p_1 = 0.1, p_2 = 0.01, T = 6, k = 0.5, x_1(0) = 1.5, x_2(0) = 1.5$: (a) $x_1$ are plotted for 475 values of $\mu$ over $[0.01, 9.5]$. (b) $x_2$ are plotted for 475 values of $\mu$ over $[0.01, 9.5]$. 
Fig. 9. (a) Bifurcation diagram of system (2.2) showing the effect of $\mu$ with $d=0.2$, $r=4$, $b=1$, $\beta=1$, $c=0.9$, $K=6$, $p_1=0.2$, $p_2=0.001$, $T=6$, $k=0.5$, $x_1(0)=1.5$, $x_2(0)=1.5$ and $x_1$ are plotted for 950 values of $\mu$ over $[0.01, 4.75]$. (b) A quasiperiodic solution at $\mu=0.18$.

All the above results show that the impulsive perturbations destroy the stable limit cycle of the unforced system (2.1) and make the dynamic behavior of system (2.2) more complex.

5. Discussion

In this paper, we have investigated the dynamic behaviors of a Holling II functional response predator–prey system concerning impulsive control strategy for pest control in detail. We have shown that there exists an asymptotically stable pest-eradication periodic solution if the impulsive period is less than some threshold. When the stability of pest-eradication periodic solution is lost, system (2.2) is permanent, which is in line with reality from a biological point of view. Numerical results show that system (2.2) can take on various kinds of periodic fluctuations and several types of attractor coexistence and is dominated by periodic, quasiperiodic and chaotic solutions, which implies that the presence of pulse makes the dynamic behavior more complex.

If we choose our impulsive control strategy, which uses a combination of biological and chemical tactics, for the purpose of suppressing the abundance of the pest, from Theorem 3.1 and Remark 3.1, we know that the pest-eradication periodic solution $(0, \tilde{x}_2(t))$ is asymptotically stable when $T < T_{\text{max}}$. In order to drive the pest to extinction, we can determine the impulsive period $T$ according to the effect of the chemical pesticides on the population and the cost of the releasing natural enemies such that $T < T_{\text{max}}$. In some cases, pesticides can be successfully integrated into pest control strategy with little harm to natural enemies. This may be accomplished by using selective pesticides such as *Bacillus thuringiensis* (B.t.), or placing the pesticides in a location where natural enemies will not contact it, so in these cases $p_2$ is smaller by far than $p_1$. If we choose parameters as $d=0.4$, $r=1$, $b=2$, $\beta=2$, $c=0.9$, $K=8$, $p_1=0.2$, $p_2=0.01$, $\mu=0.3$, $k=0.5$, then we have $T_{\text{max}} \approx 0.96$, so we can make the impulsive period smaller than 0.96 in order to eradicate pests. Now we can compare validity of our impulsive control strategy with the classical methods (only biological control or chemical control). If we only choose the biological control (i.e., $p_1 = p_2 = 0$) and other parameters are the same, then we have $T_{\text{max}} = 0.75$, which implies that we must release more natural enemies to eradicate the pests. If we only choose the chemical control (i.e., $\mu \equiv 0$) and other parameters are the same, similar to the analysis of pest-eradication periodic solution, we can prove both
the pests and natural enemies go to extinct ultimately when \( T < \frac{1}{r} \ln \frac{1}{1 - p_1} \approx 0.223 \), and it is not desirable. When \( 0.223 < T < 0.29 \), by numerical simulations there exists a stable predator-free periodic solution \( (x_1^*(t), 0) \) (see Fig. 10). In the absence of natural enemies, the pest population are able to increase much more rapidly. This can result in greater reliance on pesticide sprays after the natural enemies are eliminated. Since insect pests may quickly become resistant to chemical pesticides and we must use higher rates and more toxic materials to combat pests. In fact, it is expected that the pests are killed while the natural enemies should be preserved. That is to say, we need fewer pesticide applications and help to preserve natural enemy population while being more compatible with biological control. Therefore, our impulsive strategy is more effective than the classical one if we take chemical control efficiently.

There are some interesting problems: If we take into account the life cycle of the pest, i.e., the model we considered has periodically varying parameters (such as fertility efficiency, mortality) or is a stage population model, how do we determine the impulsive period in order to drive the pest to extinction and how does the pulse affect the dynamics of the system? In a real world, the numbers of releasing natural enemies often change, if we release natural enemies stochastically, how does this stochastic noise affect the permanence and extinction of this system? We will continue to study these problems in the future.

**Appendix**

**Proof of Lemma 3.2.** Define \( V(t) = V(t, x(t)) = (c/b) x_1(t) + x_2(t) \). Then when \( t \neq (n + k - 1)T, t \neq nT \), choose \( 0 < l < d \), we have

\[
D^+ V(t) + lV = \frac{c}{b} (r + l)x_1(t) - \frac{rc}{bK} x_1^2(t) + (l - d) x_2(t) \\
\leq \frac{c}{b} (r + l)x_1(t) - \frac{rc}{bK} x_1^2(t) \leq M_0 \quad \text{where} \quad M_0 = \frac{cK(r + l)^2}{4b}.
\]

When \( t = (n + k - 1)T \), \( V((n + k - 1)T^+) \leq V((n + k - 1)T), t = nT \), \( V(nT^+) \leq V(nT) + \mu \).
By Lemma 2.2 of Lakshmikantham et al. [11], for \( t \geq 0 \) we have
\[
V(t) \leq V(0)e^{-lt} + \frac{M_0}{l}(1 - e^{-lt}) + \mu \frac{e^{-l(t-T)}}{1 - e^{lT}} + \mu \frac{e^{lT}}{e^{lT} - 1}
\]
\[
\rightarrow \frac{M_0}{l} + \mu \frac{e^{lT}}{e^{lT} - 1} \quad \text{as} \ t \rightarrow \infty.
\]
So \( V(t) \) is uniformly ultimately bounded. Hence, by the definition of \( V(t) \), we have there exists a constant \( M > 0 \) such that \( x_i(t) \leq M, i = 1, 2 \), for \( t \) large enough. The proof is completed. \( \square \)

**Proof of Theorem 3.1.** The local stability of periodic solution \((0, \tilde{x}_2(t))\) may be determined by considering the behavior of small amplitude perturbations of the solution. Defining \( x_1(t) = u(t), x_2(t) = \tilde{x}_2(t) + v(t) \), it may be written
\[
\begin{pmatrix}
u(t) \\
v(t)
\end{pmatrix}
= \Phi(t)
\begin{pmatrix}
u(0) \\
v(0)
\end{pmatrix},
\]
where \( \Phi \) satisfies
\[
\frac{d\Phi}{dt} = \begin{pmatrix}
r - \frac{b}{\beta} & 0 \\
\frac{c}{\beta} & -d
\end{pmatrix} \Phi(t)
\]
and \( \Phi(0) = I \), the identity matrix. Hence the fundamental solution matrix is
\[
\Phi(t) = \begin{pmatrix}
\exp\left[\int_0^t \left(r - \frac{b}{\beta} \tilde{x}_2(s) \right) ds\right] & 0 \\
\ast & \exp(-dt)
\end{pmatrix}.
\]
There is no need to calculate the exact form of \( \ast \) as it is not required in the analysis that follows.

The resetting impulsive conditions of (2.2) becomes
\[
\begin{pmatrix}
u((n+k-1)T^+) \\
v((n+k-1)T^+)
\end{pmatrix}
= \begin{pmatrix}
1 - p_1 & 0 \\
0 & 1 - p_2
\end{pmatrix}
\begin{pmatrix}
u((n+k-1)T) \\
v((n+k-1)T)
\end{pmatrix}
\]
and
\[
\begin{pmatrix}
u((nT)^+) \\
v((nT)^+)
\end{pmatrix}
= \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
u(nT) \\
v(nT)
\end{pmatrix}.
\]

The stability of the periodic solution \((0, \tilde{x}_2(t))\) is determined by the eigenvalues of
\[
M = \begin{pmatrix}
1 - p_1 & 0 \\
0 & 1 - p_2
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix} \Phi(T)
\]
which are
\[
\mu_1 = (1 - p_2)e^{-dT} < 1, \quad \mu_2 = (1 - p_1) \exp\left[\int_0^T \left(r - \frac{b}{\beta} \tilde{x}_2(t) \right) dt\right],
\]
according to Floquet theory (see [2]), \((0, \tilde{x}_2(t))\) is locally stable if \(|\mu_2| < 1\), i.e., (3.3) holds true. This completes the proof. □

**Proof Theorem 3.2.** Suppose \(x(t)\) is a solution (2.2) with \(x(0) > 0\). By Lemma 3.2, we have proved there exists a constant \(M > 0\) such that \(x_i(t) \leq M, i = 1, 2\) for \(t\) large enough. We may assume \(x_i(t) \leq M\) and \(M > r\beta/b, t \geq 0, i = 1, 2\).

From system (2.2) we note that \(d\tilde{x}_2(t)/dt \geq -dx_2(t)\), consider the following impulsive differential equation:

\[
\begin{align*}
\frac{dy(t)}{dt} &= -dy(t), \quad t \neq (n + k - 1)T, \quad t \neq nT, \\
y(t^+) &= (1 - p_2)y(t), \quad t = (n + k - 1)T, \\
y(t^+) &= y(t) + \mu, \quad t = nT, \\
y(0^+) &= x_2(0^+). \quad (A.1)
\end{align*}
\]

From Lemma 3.3 and comparison theorem of impulsive equation (see [11, Theorem 3.1.1]), we have \(x_2(t) \geq y(t)\) and \(y(t) \to \tilde{x}_2(t)\) as \(t \to \infty\). Hence for some \(\epsilon > 0\), we have

\[
x_2(t) \geq y(t) > \tilde{x}_2(t) - \epsilon \quad (A.2)
\]

for all \(t\) large enough. So \(x_2(t) \geq (\mu(1 - p_2) \exp(-dT))/((1 - (1 - p_2) \exp(-dT)) - \epsilon \leq m_2\) for \(t\) large enough. Thus, we only need to find an \(m_1 > 0\) such that \(x_1(t) \geq m_1\) for \(t\) large enough. We will do it in the following two steps:

1. From condition (3.4), let \(0 < m_3 < d\beta/c, \epsilon_1 > 0\) be small enough such that

\[
\eta \triangleq (1 - p_1) \exp \left\{ rT - rKm_3T - \frac{b}{\beta} \epsilon_1 T \right\} - \frac{b\mu}{\beta} \left[ (1 - p_2) \exp\left( \left(-d + \frac{cm_3}{\beta}\right)kT \right) - (1 - p_2) \exp\left( \left(-d + \frac{cm_3}{\beta}\right)T \right) \right] \left[ 1 - (1 - p_2) \exp\left( \left(-d + \frac{cm_3}{\beta}\right)T \right) \right] > 1,
\]

we will prove \(x_1(t) < m_3\) cannot hold for all \(t \geq 0\). Otherwise,

\[
\frac{dx_2(t)}{dt} \leq x_2(t) \left(-d + \frac{cm_3}{\beta}\right), \quad t \neq (n + k - 1)T, \quad t \neq nT, \\
x_2(t^+) = (1 - p_2)x_2(t), \quad t = (n + k - 1)T, \\
x_2(t^+) = x_2(t) + \mu, \quad t = nT.
\]
So we have $x_2(t) \leq z(t)$ and $z(t) \to \tilde{z}(t), t \to \infty$, where $z(t)$ is the solution of

\[
\frac{dz(t)}{dt} = z(t) \left(-d + \frac{cm_3}{\beta}\right), \quad t \neq (n + k - 1)T, \quad t \neq nT,
\]

\[
z(t^+) = (1 - p_2)z(t), \quad t = (n + k - 1)T,
\]

\[
z(t^+) = z(t) + \mu, \quad t = nT,
\]

\[
z(0^+) = x_2(0^+)
\]

and

\[
\tilde{z}(t) = \begin{cases} 
\frac{\mu \exp((-d + cm_3/\beta)(t - (n - 1)T))}{1 - (1 - p_2) \exp((-d + cm_3/\beta)T)}, & (n - 1)T < t \leq (n + k - 1)T, \\
\frac{\mu(1 - p_2) \exp((-d + cm_3/\beta)(t - (n - 1)T))}{1 - (1 - p_2) \exp((-d + cm_3/\beta)T)}, & (n + k - 1)T < t \leq nT.
\end{cases}
\]

Therefore, there exists a $T_1 > 0$ such that

\[
x_2(t) \leq z(t) < \tilde{z}(t) + \varepsilon_1
\]

and

\[
\frac{dx_1(t)}{dt} \geq x_1(t) \left( r - \frac{r}{K} m_3 - \frac{b}{\beta} (\tilde{z}(t) + \varepsilon_1) \right), \quad t \neq (n + k - 1)T,
\]

\[
x_1(t^+) = (1 - p_1)x_1(t), \quad t = (n + k - 1)T
\]

for $t \geq T_1$. Let $N \in \mathbb{Z}_+$ and $(N + k - 1)T \geq T_1$. Integrating (A4) on $((n + k - 1)T, (n + k)T], n \geq N$, we have

\[
x_1((n + k)T) \geq x_1((n + k - 1)T)(1 - p_1) \exp \left[ \int_{(n + k - 1)T}^{(n + k)T} \left( r - \frac{r}{K} m_3 - \frac{b}{\beta} (\tilde{z}(t) + \varepsilon_1) \right) dt \right]
\]

\[
= x_1((n + k - 1)T) \eta.
\]

Then $x_1((N + n + k)T) \geq x_1((N + k)T) \eta^n \to \infty$ as $n \to \infty$, which is a contradiction to the boundedness of $x_1(t)$. Hence, there exists a $t_1 > 0$ such that $x_1(t_1) \geq m_3$.

2. If $x_1(t) \geq m_3$ for all $t \geq t_1$, then our aim is achieved. Otherwise, let $t^* = \inf_{t > t_1} \{x_1(t) < m_3\}$, there are two possible case for $t^*$.

Case (i): $t^* = (n_1 + k - 1)T, n_1 \in \mathbb{Z}_+$. Then $x_1(t) \geq m_3$ for $t \in [t_1, t^*]$ and $(1 - p_1)m_3 \leq x_1(t^*) = (1 - p_1)x_1(t^*) < m_3$. Choose $n_2, n_3 \in \mathbb{Z}_+$ such that

\[
(n_2 - 1)T > \ln \left( \frac{\varepsilon_1}{M + \mu} \right) / \left(-d + \frac{cm_3}{\beta}\right),
\]

\[
(1 - p_1)^{n_2} \eta^{n_3} \exp(n_2 \eta_1 T) > (1 - p_1)^{n_2} \eta^{n_3} \exp[(n_2 + 1)\eta_1 T] > 1,
\]

\[
\Rightarrow (1 - p_1)^{n_2} \eta^{n_3} \exp(n_2 \eta_1 T) > (1 - p_1)^{n_2} \eta^{n_3} \exp[(n_2 + 1)\eta_1 T] > 1.
\]
where $\eta_1 = r - (r/K)m_3 - (b/\beta)M < 0$. Let $T' = n_2 T + n_3 T$, we claim that there must be a $t_2 \in (t^*, t^* + T')$ such that $x_1(t_2) > m_3$. Otherwise, consider $(A_3)$ with $z(n_1 T^+) = x_2(n_1 T^+)$, we have

$$z(t) = \begin{cases} 
0 
(1 - p_2)^n - \left( \frac{\mu}{1 - (1 - p_2) \exp((-d + cm_3)T)} \right) \exp((-d + cm_3)T) 
\times (t - n_1 T), + \tilde{z}(t), \quad (n - 1)T < t \leq (n + k - 1)T, 
\end{cases}$$

and $n_1 + 1 \leq n \leq n_1 + n_2 + n_3$. Then

$$|z(t) - \tilde{z}(t)| < (M + \mu) \exp \left[ \left( -d + \frac{cm_3}{\beta} \right) (t - n_1 T) \right]$$

$$< \epsilon_1 \text{ and } x_2(t) \leq z(t) \leq \tilde{z}(t) + \epsilon_1,$n_1 T + (n_2 - 1)T \leq t \leq t^* + T'$$

which implies that $(A4)$ holds for $t^* + n_2 T \leq t \leq t^* + T'$. So as in step 1, we have

$$x_1(t^* + T') \geq x_1(t^* + n_2 T) \eta_1^{n_3}.$$

Again from system (2.2), we get

$$\frac{dx_1(t)}{dt} \geq m_3 \left( r - \frac{r}{K} m_3 - \frac{b}{\beta} M \right), \quad t \neq nT,$$

$$m_1'(t) = (1 - p_1)x_1(t), \quad t = nT,$$

for $t \in [t^*, t^* + n_2 T]$. Integrating $(A6)$ on $[t^*, t^* + n_2 T]$, we have

$$x_1(t^* + n_2 T) \geq m_3(1 - p_1)^{n_2} \exp(n_2 \eta_1 T).$$

Thus, we have

$$x_1(t^* + T') \geq m_3(1 - p_1)^{n_2} \exp(n_2 \eta_1 T) \eta_1^{n_3} > m_3,$$

which is a contradiction.

Let $\tilde{t} = \inf_{t > t^*} \{x_1(t) > m_3\}$. Then for $t \in (t^*, \tilde{t})$, $x_1(t) \leq m_3$ and $x_1(\tilde{t}) = m_3$. For $t \in (t^*, \tilde{t})$, we have $x_1(t) \geq m_3(1 - p_1)^{n_2 + n_3} \exp(n_2 + n_3)\eta_1 T$. Let $m_1'(t) = m_3(1 - p_1)^{n_2 + n_3} \exp[(n_2 + n_3)\eta_1 T]$, so we have $x_1(t) \geq m_1'$ for $t \in (t^*, \tilde{t})$. For $t > \tilde{t}$, the same arguments can be continued since $x_1(\tilde{t}) \geq m_3$.

Case (ii): $t^* \neq (n + k - 1)T, n \in Z_+$. Then $x_1(t) \geq m_3$ for $t \in [t_1, t^*)$ and $x_1(t^*) = m_3$, suppose $t^* \in ((n_1' + k - 1)T, (n_1' + k)T)$, $n_1' \in Z_+$. There are two possible case for $t \in (t^*, (n_1' + k)T)$.

If $x_1(t) \leq m_3$ for all $t \in (t^*, (n_1' + k)T)$, similar to Case (i), we can prove there must be a $t_2' \in ((n_1' + k)T, (n_1' + k)T + T')$ such that $x(t_2') > m_3$. Here we omit it.

Let $\hat{t} = \inf_{t > t^*} \{x_1(t) > m_3\}$, then $x_1(t) \leq m_3$ for $t \in (t^*, \hat{t})$ and $x_1(\hat{t}) = m_3$. For $t \in (t^*, \hat{t})$, we have $x_1(t) \geq m_3(1 - p_1)^{n_2 + n_3} \exp[(n_2 + n_3 + 1)\eta_1 T]$. Let $m_1(t) = m_3(1 - p_1)^{n_2 + n_3} \exp[(n_2 + n_3 + 1)\eta_1 T] < m_1'$, so $x_1(t) \geq m_1$ for $t \in (t^*, \hat{t})$. For $t > \hat{t}$, the same arguments can be continued since $x_1(\hat{t}) \geq m_3$.

If there exists a $t \in (t^*, (n_1' + k)T)$ such that $x_1(t) > m_3$. Let $\tilde{t} = \inf_{t > t^*} \{x_1(t) > m_3\}$, then $x_1(t) \leq m_3$ for $t \in (t^*, \tilde{t})$ and $x_1(\tilde{t}) = m_3$. For $t \in (t^*, \tilde{t})$, we have $x_1(t) \geq x_1(t^*) \exp[\eta_1(t - t^*)] \geq m_3 \exp(\eta_1 T) > m_1$. 
Since $x_1(t) \geq m_3$ for $t > \hat{t}$, the same arguments can be continued. Hence $x_1(t) \geq m_1$ for all $t \geq t_1$. The proof is completed. □

References