SMALL PROGRAMMING EXERCISES 10

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Our two new exercises are graph problems. In the first one the Strahler number of a given binary tree has to be computed. It is a nice little exercise allowing a solution that is linear in the size of the tree.

The other exercise involves an acyclic directed graph. Such a graph has sources, i.e. vertices without incoming arcs. We have to determine all vertices that are at least a given ‘distance’ removed from the sources. An unexpected property of this exercise is that although the arcs have weights attached to them, we can still find a solution that is linear in the number of arcs and vertices of the graph.

Exercise 25: Strahler number of a binary tree

A binary tree $T$ is either empty or it consists of a vertex, the root of $T$, and two subtrees $T_0$ and $T_1$, each of which is again a binary tree. The Strahler number $\sigma(T)$ of a binary tree $T$ is defined as follows. $\sigma(T) = 0$ if $T$ is empty. If $T$ has subtrees $T_0$ and $T_1$ the Strahler number is given by

$$\sigma(T) = \begin{cases} \sigma(T_0) \max \sigma(T_1) & \text{if } \sigma(T_0) \neq \sigma(T_1), \\ \sigma(T_0) + 1 & \text{if } \sigma(T_0) = \sigma(T_1). \end{cases}$$

We have to compute $\sigma(T)$ of a given binary tree $T$ of $N$ vertices, $N \geq 1$. The vertices are numbered 0 through $N - 1$. Vertex 0 is the root of $T$. As in Exercise 17, the tree is recorded in an integer array $v(i): 1 \leq i < N$:

(Ai: $1 \leq i < N$: the tree with vertex $v(i)$ as its root has a subtree with vertex $i$ as its root)

The functional specification is

$$[[N: \text{int}; \{N \geq 1\}]$$
$$v(i: 1 \leq i < N): \text{array of int;}$$
$$\{v \text{ represents binary tree } T\}$$
$$[[p: \text{int};$$
$$S$$
$$\{p = \sigma(T)\}$$
$$]]$$
Exercise 26: K-sequel of an acyclic digraph

Given is an acyclic directed graph \( G \) each arc of which has a positive weight. The weight of a path is the sum of the weights of its arcs. A vertex without predecessors is called a source. For each \( K \geq 1 \) the \( K \)-sequel of \( G \) is defined as the set of all vertices \( j \) for which the weight of each path from a source to \( j \) is at least \( K \).

Graph \( G \) is represented in arrays \( b \) and \( e \) in the usual way. The weights are given by an array \( w(i: 0 \leq i < M) \): the arc from vertex \( j \) to vertex \( e(i), b(j) \leq i < b(j+1) \), has weight \( w(i) \).

We are requested to find a statement list \( S \) such that

\[
\begin{align*}
[N, M, K: \text{int}; & \{N \geq 1 \land M \geq 0 \land K \geq 1\} \\
b(j: 0 \leq j \leq N): \text{array of int}; & \\
e, w(i: 0 \leq i < M): \text{array of int}; & \\
\{\text{suc}(G, b, e) \land G \text{acyclic} \land (Ai: 0 \leq i < M: w(i) \geq 1)\} & \\
[a(j: 0 \leq j < N): \text{array of bool}; & \\
S & \\
\{(Aj: 0 \leq j < N: a(j) \equiv (j \text{ in the } K\text{-sequel of } G))\} & \\
\}\end{align*}
\]

Solution of Exercise 23 (problem of the masks)

A \( P(i: 0 \leq i < M) \)-mask in \( X(j: 0 \leq j < N) \) is an increasing integer sequence \( r(i: 0 \leq i < M) \) that satisfies

\[(Ai: 0 \leq i < M: 0 \leq r(i) < N \land P(i) = X(r(i))).\]

We have to determine \( S \) such that

\[
\begin{align*}
[M, N: \text{int}; & \{M \geq 1 \land N \geq 0\} \\
P(i: 0 \leq i < M), X(j: 0 \leq j < N): \text{array of int}; & \\
\{(Ai: 0 \leq i < M: (Nh: 0 \leq h < M: P(h) = i) = 1)\} & \\
[a: \text{int}; & \\
S & \\
\{a = (\text{number of } P(i: 0 \leq i < M)\text{-masks in } X(j: 0 \leq j < N))\} & \\
\}\end{align*}
\]
If we replace in the postcondition the constant $N$ by a variable $n$ we get the following invariant:

$$a = \text{(number of } P(i: 0 \leqslant i < M)\text{-masks in } X(j: 0 \leqslant j < n))}$$

$$\forall 0 \leqslant n \leqslant N.$$

It can be initialized with $n, a = 0, 0$. In order to determine the number of $P(i: 0 \leqslant i < M)$-masks in $X(j: 0 \leqslant j < n + 1)$ we need to know the number of $P(i: 0 \leqslant i < M - 1)$-masks in $X(j: 0 \leqslant j < n)$, since variable $a$ has to be increased by that number if $P(M - 1) = X(n)$. Consequently, rather than a single count of masks we need a whole array $b(h: 0 \leqslant h \leqslant M)$ of them. The invariant then becomes

$$P: \quad (Ah: 0 \leqslant h \leqslant M: b(h) = B(h, n))$$

$$\forall 0 \leqslant n \leqslant N$$

in which $B(h, n)$ denotes the number of $P(i: 0 \leqslant i < h)$-masks in $X(j: 0 \leqslant j < n)$.

The proper initialization for $n = 0$ and the way in which array $b$ should be changed when increasing the value of $n$ follow directly from the recurrence relation for $B(h, n)$:

$$B(0, n) = 1,$$
$$B(h, 0) = 0 \quad \text{for } h \geqslant 1$$

and for $h \geqslant 0$ and $n \geqslant 0$,

$$B(h + 1, n + 1) = \begin{cases} B(h + 1, n) + B(h, n) & \text{if } X(n) = P(h), \\ B(h + 1, n) & \text{if } X(n) \neq P(h). \end{cases}$$

If $0 \leqslant X(n) < M$ there exists one $h$ such that $X(n) = P(h)$. In order to determine that $h$ we need the inverse of $P$. To that end, we introduce an integer array $q(h: 0 \leqslant h < M)$ and establish

$$(Ah: 0 \leqslant h < M: q(P(h)) = h)$$

The solution is now straightforward.

$$S: \begin{array}{l}
\mid n: \text{int;} \\
q(h: 0 \leqslant i < M), b(h: 0 \leqslant h \leqslant M): \text{array of int;} \\
b: (0) = 1 \\
; \mid [h: \text{int}; h \leftarrow 0 \\
\quad ; \text{do } h \neq M \rightarrow q: (P(h)) = h \; ; h \leftarrow h + 1 \; ; b: (h) = 0 \text{ od} \\
\mid \\
; n \leftarrow 0
\end{array}$$
Our solution has a computation time that is linear in $M$ and $N$.

Solution of Exercise 24 (recognizing $h$-sequences)

An $h$-sequence is a sequence of zeros and ones generated by the grammar

$$(h\text{-seq}) := 0 \mid 1(h\text{-seq})(h\text{-seq}).$$

We have to solve $S$ in

$$\begin{align*}
&\begin{array}{l}
\{N: \text{int}; \{N \geq 0\} \\
H(i: 0 \leq i < N): \text{array of int}; \\
\{(A_i: 0 \leq i < N: H(i) = 0 \lor H(i) = 1)\} \\
[b: \text{bool}; \\
S \\
\{b = (H(i: 0 \leq i < N) \text{ is an } h\text{-sequence})\}
\end{array}
\end{align*}$$

In the second part of the grammar a concatenation of two $h$-sequences occurs, so we should be looking at the problem of recognizing concatenations of $h$-sequences. Since a proper prefix of an $h$-sequence is not an $h$-sequence, a sequence of zeros and ones can in at most one way be partitioned into a concatenation of $h$-sequences.

A 0 by itself is an $h$-sequence. Consequently, a sequence of zeros and ones that starts with a 0 is a concatenation of $m, m \geq 1$, $h$-sequences if and only if the rest of the sequence is a concatenation of $m - 1$ $h$-sequences. A 1 followed by two
\(h\)-sequences is an \(h\)-sequence. Consequently, a sequence of zeros and ones that starts with a 1 is a concatenation of \(m, m \geq 1\), \(h\)-sequences if and only if the rest of the sequence is a concatenation of \(m + 1\) \(h\)-sequences.

The program follows immediately from these two observations. Its invariant is

\[
H(i: 0 \leq i < n) \text{ followed by } m \text{ } h\text{-sequences is an } h\text{-sequence}
\]

\[\land 0 \leq n \leq N \land m \geq 0.\]

\(S: \quad \left[ m, n: \text{int}; m, n := 1, 0 \right.
\]
\[; \text{do } m \neq 0 \land n \neq N \]
\[\rightarrow \text{if } H(n) = 0 \rightarrow m := m - 1 \]
\[\quad \Box H(n) = 1 \rightarrow m := m + 1 \]
\[\text{fi} \]
\[; \quad n := n + 1 \]
\[\text{od} \]
\[; \quad b := (m = 0 \land n = N) \]
\]}

The program has an execution time that is proportional to \(N\).