SMALL PROGRAMMING EXERCISES 10

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Our two new exercises are graph problems. In the first one the Strahler number of a given binary tree has to be computed. It is a nice little exercise allowing a solution that is linear in the size of the tree.

The other exercise involves an acyclic directed graph. Such a graph has sources, i.e. vertices without incoming arcs. We have to determine all vertices that are at least a given ‘distance’ removed from the sources. An unexpected property of this exercise is that although the arcs have weights attached to them, we can still find a solution that is linear in the number of arcs and vertices of the graph.

Exercise 25: Strahler number of a binary tree

A binary tree $T$ is either empty or it consists of a vertex, the root of $T$, and two subtrees $T_0$ and $T_1$, each of which is again a binary tree. The Strahler number $\sigma(T)$ of a binary tree $T$ is defined as follows. $\sigma(T) = 0$ if $T$ is empty. If $T$ has subtrees $T_0$ and $T_1$ the Strahler number is given by

$$\sigma(T) = \begin{cases} \sigma(T_0) \max \sigma(T_1) & \text{if } \sigma(T_0) \neq \sigma(T_1), \\ \sigma(T_0) + 1 & \text{if } \sigma(T_0) = \sigma(T_1). \end{cases}$$

We have to compute $\sigma(T)$ of a given binary tree $T$ of $N$ vertices, $N \geq 1$. The vertices are numbered 0 through $N-1$. Vertex 0 is the root of $T$. As in Exercise 17, the tree is recorded in an integer array $v(i: 1 \leq i < N)$:

$(Ai: 1 \leq i < N$: the tree with vertex $v(i)$ as its root has a subtree with vertex $i$ as its root)$

The functional specification is

$\llbracket N: \text{int}; \{N \geq 1\}$

$v(i: 1 \leq i < N): \text{array of int};$

{v represents binary tree $T}$

$\llbracket p: \text{int};$

$S$

{p = \sigma(T)}$

$\rrbracket$

$\rrbracket$
Exercise 26: K-sequl of an acyclic digraph

Given is an acyclic directed graph \( G \) each arc of which has a positive weight. The weight of a path is the sum of the weights of its arcs. A vertex without predecessors is called a source. For each \( K \geq 1 \) the \( K \)-sequl of \( G \) is defined as the set of all vertices \( j \) for which the weight of each path from a source to \( j \) is at least \( K \).

Graph \( G \) is represented in arrays \( b \) and \( e \) in the usual way. The weights are given by an array \( w(i: 0 \leq i < M) \): the arc from vertex \( j \) to vertex \( e(i), b(j) \leq i < b(j+1) \), has weight \( w(i) \).

We are requested to find a statement list \( S \) such that

\[
\begin{align*}
&[[N, M, K: \text{int}; \{N \geq 1 \land M \geq 0 \land K \geq 1\}] \\
&\quad b(j: 0 \leq j \leq N): \text{array of int} \\
&\quad e, w(i: 0 \leq i < M): \text{array of int} \\
&\quad \{\text{suc}(G, b, e) \land G \text{ acyclic} \land (Ai: 0 \leq i < M: w(i) \geq 1)\} \\
&\quad [[a(j: 0 \leq j < N): \text{array of bool}] \\
&\quad S \\
&\quad \{(Aj: 0 \leq j < N: a(j) = (j \text{ in the } K\text{-sequl of } G))] \\
&\]} 
\]

Solution of Exercise 23 (problem of the masks)

A \( P(i: 0 \leq i < M) \)-mask in \( X(j: 0 \leq j < N) \) is an increasing integer sequence \( r(i: 0 \leq i < M) \) that satisfies

\[(Ai: 0 \leq i < M: 0 \leq r(i) < N \land P(i) = X(r(i))).\]

We have to determine \( S \) such that

\[
\begin{align*}
&[[M, N: \text{int}; \{M \geq 1 \land N \geq 0\}] \\
&\quad P(i: 0 \leq i < M), X(j: 0 \leq j < N): \text{array of int} \\
&\quad \{(Ai: 0 \leq i < M: (Nh: 0 \leq h < M: P(h) = i) = 1)\} \\
&\quad [[a: \text{int}] \\
&\quad S \\
&\quad \{a = (\text{number of } P(i: 0 \leq i < M)\text{-masks in } X(j: 0 \leq j < N))\} \\
&\]} 
\]
If we replace in the postcondition the constant $N$ by a variable $n$ we get the following invariant:

$$a = \text{(number of } P(i: 0 \leq i < M)\text{-masks in } X(j: 0 \leq j < n))}
\land 0 \leq n \leq N.$$

It can be initialized with $n, a = 0, 0$. In order to determine the number of $P(i: 0 \leq i < M)$-masks in $X(j: 0 \leq j < n + 1)$ we need to know the number of $P(i: 0 \leq i < M - 1)$-masks in $X(j: 0 \leq j < n)$, since variable $a$ has to be increased by that number if $P(M - 1) = X(n)$. Consequently, rather than a single count of masks we need a whole array $b(h: 0 \leq h \leq M)$ of them. The invariant then becomes

$$P: \quad (\forall h: 0 \leq h \leq M: b(h) = B(h, n))
\land 0 \leq n \leq N$$

in which $B(h, n)$ denotes the number of $P(i: 0 \leq i < h)$-masks in $X(j: 0 \leq j < n)$.

The proper initialization for $n = 0$ and the way in which array $b$ should be changed when increasing the value of $n$ follow directly from the recurrence relation for $B(h, n)$:

$$B(0, n) = 1,$$

$$B(h, 0) = 0 \quad \text{for } h \geq 1$$

and for $h \geq 0$ and $n \geq 0$,

$$B(h + 1, n + 1) = \begin{cases} B(h + 1, n) + B(h, n) & \text{if } X(n) = P(h), \\ B(h + 1, n) & \text{if } X(n) \neq P(h). \end{cases}$$

If $0 \leq X(n) < M$ there exists one $h$ such that $X(n) = P(h)$. In order to determine that $h$ we need the inverse of $P$. To that end, we introduce an integer array $q(h: 0 \leq h < M)$ and establish

$$\forall h: 0 \leq h < M: q(P(h)) = h$$

The solution is now straightforward.

$$S: | \begin{array}{l}
[n: \text{int}; \\
q(h: 0 \leq i < M), b(h: 0 \leq h \leq M): \text{array of int}; \\
b: (0) = 1 \\
; |[h: \text{int}; h := 0 \\
; \quad \text{do } h \neq M \rightarrow q: (P(h)) = h ; h := h + 1 ; b: (h) = 0 \text{ od} \\
]; \\
; n := 0
\end{array}$$
\begin{verbatim}
do n \neq N
   \rightarrow \textbf{if } 0 \leq X(n) \land X(n) < M
   \rightarrow [[h: \texttt{int}; h := q(X(n)); b: (h + 1) = b(h + 1) + b(h)]]
   \square 0 \geq X(n) \lor X(n) \geq M
   \rightarrow \textbf{skip}
   \textbf{fi}
   ; n := n + 1
\end{verbatim}

Our solution has a computation time that is linear in \(M\) and \(N\).

Solution of Exercise 24 (recognizing \(h\)-sequences)

An \(h\)-sequence is a sequence of zeros and ones generated by the grammar

\[
(h\text{-seq}) := 01 (h\text{-seq})(h\text{-seq}).
\]

We have to solve \(S\) in

\[
[[N: \texttt{int}; \{N \geq 0\}]
\hspace{1cm} H(i: 0 \leq i < N): \texttt{array of int} ;
\hspace{1cm} \{(A_i: 0 \leq i < N: H(i) = 0 \lor H(i) = 1)\}
\hspace{1cm} [[b: \texttt{bool};
\hspace{1cm} S
\hspace{1cm} \{b = (H(i: 0 \leq i < N) \text{ is an } h\text{-sequence})\}]
\hspace{1cm}]]
\]

In the second part of the grammar a concatenation of two \(h\)-sequences occurs, so we should be looking at the problem of recognizing concatenations of \(h\)-sequences. Since a proper prefix of an \(h\)-sequence is not an \(h\)-sequence, a sequence of zeros and ones can in at most one way be partitioned into a concatenation of \(h\)-sequences.

A 0 by itself is an \(h\)-sequence. Consequently, a sequence of zeros and ones that starts with a 0 is a concatenation of \(m, m \geq 1\), \(h\)-sequences if and only if the rest of the sequence is a concatenation of \(m - 1\) \(h\)-sequences. A 1 followed by two
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A sequence of zeros and ones that starts with a 1 is a concatenation of $m$, $m \geq 1$, h-sequences if and only if the rest of the sequence is a concatenation of $m + 1$ h-sequences.

The program follows immediately from these two observations. Its invariant is

$$H(i: 0 \leq i < n) \text{ followed by } m \text{ h-sequences is an h-sequence}$$

$$\land 0 \leq n \leq N \land m \geq 0.$$ 

**S:**

```plaintext
[m, n: int; m, n := 1, 0
; do m ≠ 0 ∧ n ≠ N
    → if H(n) = 0 → m := m - 1
    □ H(n) = 1 → m := m + 1
    fi
    ; n := n + 1
    od
; b := (m = 0 ∧ n = N)
]
```

The program has an execution time that is proportional to $N$. 
