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The Total Chromatic Number of Nearly Complete Bipartite Graphs

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The *total chromatic number* $\chi_T(G)$ of a graph G is the least number of colours needed to colour the edges and vertices of G so that no two adjacent vertices receive the same colour, no two edges incident with the same vertex receive the same colour, and no edge receives the same colour as either of the vertices it is incident with. Let $n \geqslant 1$, let J be a subgraph of $K_{n,n}$, let $e = |E(J)|$, and let $j(J)$ be the maximum size (i.e., number of edges) of a matching in J. Then

$$\chi_T(K_{n,n} \backslash E(J)) = n + 2$$

if and only if $e + j \leqslant n - 1$.

1. INTRODUCTION

A *vertex-colouring* of a graph G is a map ψ: $V(G) \to \mathscr{C}$, where $\mathscr{C}$ is a set of colours, such that no two adjacent vertices receive the same colour. *The chromatic number* $\chi(G)$ of G is the least value of $|\mathscr{C}|$ for which G has a vertex colouring. It is well-known, and easy to show, that $\chi(G) \leqslant \Delta(G) + 1$, where $\Delta(G)$ is the maximum degree of G.

An *edge-colouring* of a graph G is a map ϕ: $E(G) \to \mathscr{C}$, where $\mathscr{C}$ is a set of colours, such that no two adjacent edges receive the same colour. The *chromatic index* (or edge-chromatic number) $\chi'(G)$ of G is the least value of $|\mathscr{C}|$ for which G has an edge-colouring. A famous theorem of Vizing [9] states that $\Delta(G) \leqslant \chi'(G) \leqslant \Delta(G) + 1$, where $\Delta(G)$ is the maximum degree of G, G being a simple graph.

A *total-colouring* of G is a map θ: $E(G) \cup V(G) \to \mathscr{C}$ such that no incident or adjacent pair of elements of $E(G) \cup V(G)$ receive the same colour. Thus a total-colouring of G incorporates both a vertex-colouring and an edge-colouring of G, and satisfies the additional condition that no vertex receives the same colour as an edge incident with the vertex. *The total-chromatic*

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number $\chi_T(G)$ is the least value of $|\mathscr{C}|$ for which G has a total-colouring. A long-standing conjecture of Behzad [1] is that $\Delta(G)+1 \leqslant \chi_T(G) \leqslant \Delta(G)+2$ if G is a simple graph. The lower bound here is trivial, but whether the upper bound is true is today still unknown. The only upper bounds of this type to date, $\chi_T(G) \leqslant (11/6)|\Delta(G)| + O(\Delta(G))$ and $\chi_T(G) \leqslant (9/5)|\Delta(G)|$, both under some additional restriction on G, are due to Bollobás and Harris [2], and Chetwynd and Häggkvist [3], respectively. Hind [7] showed that $\chi_T(G) \leqslant \chi'(G) + 2\sqrt{\chi(G)}$. Chetwynd and Hilton [4] showed that if G is regular and satisfies $d(G) \geqslant (6/7)|V(G)|$, then $\chi_T(G) \leqslant \Delta(G)+2$, and obtained a number of other similar results for regular graphs of high degree.

It is well-known that $\chi_T(K_{n,n}) = n+2$. Our object is to prove the following theorem.

THEOREM 1. *Let $n \geqslant 1$, let J be a subgraph of $K_{n,n}$, let $e = |E(J)|$, and let $j(J)$ be the maximum size (i.e., number of edges) of a matching in J. Then*

$$\chi_T(K_{n,n} \backslash E(J)) = n+2$$

if and only if $e + j \leqslant n-1$.

A similar result, with K_{2n} instead of $K_{n,n}$, was proved in [6]. The result here seems to be somewhat harder to prove.

2. PROOF OF THEOREM 1

We first prove the necessity.

Proof of Necessity. It is easy to see that $\chi_T(K_{n,n}) = n+2$, and so it follows that $\chi_T(K_{n,n} \backslash E(J)) \leqslant n+2$. We show that, if $e + j \leqslant n-1$ then $\chi_T(K_{n,n} \backslash E(J)) = n+2$. We do this by assuming instead that $\chi_T(K_{n,n} \backslash E(J)) = n+1$ and showing that then $e + j \geqslant n$. So suppose that $K_{n,n} \backslash E(J)$ is totally coloured with $n+1$ colours $c_1, ..., c_{n+1}$.

Let the two sets of independent vertices of $K_{n,n}$ be L and R. For $1 \leqslant i \leqslant n+1$, let c_i be used to colour p_i vertices of L and q_i vertices of R. Let

$$j_i = \min\{p_i, q_i\}$$

and

$$f_i = \max\{p_i, q_i\} - \min\{p_i, q_i\} = \max\{p_i, q_i\} - j_i.$$

Then

$$p_i + q_i = f_i + 2j_i,$$

so

$$(f_1+\cdots+f_{n+1})+2(j_1+\cdots+j_{n+1})=2n.$$

There are clearly j_i independent edges of J joining vertices coloured c_i, and so, as each vertex receives exactly one colour, J has at least $(j_1+\cdots+j_{n+1})$ independent edges. Thus $j \geqslant j_1+\cdots+j_{n+1}$, and so

$$f_1+\cdots+f_{n+1} \geqslant 2n-2j.$$

Call a pair (c, v), where c is a colour, v is a vertex, and c is used on v itself or on an edge incident with v, a *colour-vertex* pair. There are at most $2n-f_i$ colour-vertex pairs where the colour is c_i, and so there are at most

$$2n(n+1)-(f_1+\cdots+f_{n+1})$$

colour-vertex pairs altogether. But the number of colour-vertex pairs equals the number of vertices plus twice the number of edges, and so is $2n^2+2n-2e$. Thus

$$\begin{aligned} 2n^2+2n-2e &\leqslant 2n(n+1)-(f_1+\cdots+f_{n+1}) \\ &\leqslant 2n(n+1)-2n+2j \\ &= 2n^2+2j. \end{aligned}$$

Therefore

$$n \leqslant e+j$$

as required.

We defer the proof of the sufficiency of Theorem 1 until we have proved a number of lemmas. But we do observe that, in order to prove the sufficiency, by adding in edges if necessary, we may without loss of generality suppose that either $e+j=n$ or n is odd and J consists of $(1/2)(n+1)$ independent edges (so that $e+j=n+1$), and then show that $K_{n,n}\backslash E(J)$ can be totally coloured with $n+1$ colours.

We first consider the case when $e+j=n$ and prove the following seven lemmas.

LEMMA 1. *Let $e+j=n$ and let $H=K_{n,n}\backslash E(J)$. Let H be totally coloured with $n+1$ colours, $c_1, \ldots, c_{n+1}$. Then there are non-negative integers $j_1, \ldots, j_{n+1}$ such that H has exactly $n+j_i$ elements coloured c_i, in one of L, R exactly j_i vertices are coloured c_i, in the other of L, R at least j_i vertices are coloured c_i, $j_1+\cdots+j_{n+1}=j$, and, for some matching M of J with $|M|=j$, there are j_i edges of M such that c_i is used to colour the vertices at each end of these edges.*

Proof. Suppose that, for each $i \in \{1, ..., n+1\}$, colour c_i occurs on j_i vertices of one of L and R, and on at least j_i vertices of the other of L and R. Then c_i occurs on at most $n+j_i$ elements (edges and vertices) of H. Thus H contains at most $(n+1)\,n+j_1+\cdots+j_{n+1}$ elements altogether. Since we know that the number of elements that H contains is $n^2-e+2n=(n+1)\,n+(n-e)=(n+1)\,n+j$, it follows that $j \leqslant j_1+\cdots+j_{n+1}$.

For each $i \in \{1, ..., n+1\}$, a set of j_i vertices of L and a set of j_i vertices of R are at each end of a set of j_i independent edges of J. Moreover, if $i_1 \neq i_2$, then the set of j_{i_1} edges of J corresponding to the colour c_{i_1} is independent from the set of j_{i_2} edges of J corresponding to the colour c_{i_2}. Thus J contains a matching M consisting of $j_1+\cdots+j_{n+1}$ edges. Since the maximum size of a matching of J is j, it follows that $j \geqslant j_1+\cdots+j_{n+1}$. Therefore $j=j_1+\cdots+j_{n+1}$. It follows now that, for each $i \in \{1, ..., n+1\}$, c_i occurs on exactly $n+j_i$ elements of H.

This proves Lemma 1.

Whenever we have a vertex-colouring with colours $c_1, ..., c_{n+1}$ of a bipartite graph $H=K_{n,n}\backslash E(J)$, with vertex sets denoted by L and R, for $1 \leqslant i \leqslant n+1$ we let p_i denote the number of vertices coloured c_i in L, we let q_i denote the number of vertices coloured c_i in R, and we let $j_i=\min\{p_i, q_i\}$.

We first consider a bipartite multigraph H^* constructed from H by introducing two further vertices l^* and r^*, joining l^* to each vertex v of R by $d_J(v)$ edges, and joining r^* to each vertex v of L by $d_J(v)$ edges. Then H^* is regular of degree n except that l^* and r^* have degree e. Let S^* be a submultigraph of H^*. We say that S^* is *almost totally coloured* if all edges of S^* and all vertices of $S^*\backslash\{l^*, r^*\}$ are coloured, and the colouring is such that no two adjacent vertices are assigned the same colour, and, for $v \in V(S^*)\backslash\{l^*, r^*\}$, no two edges incident with v have the same colour, and no edge incident with v has the same colour as v. Thus it is possible for a colour to be used on more than one edge incident with l^* or r^*, but the edges of a multiple edge all have different colours.

A reason why it might be helpful to consider H^* instead of H is given succinctly in the following lemma.

LEMMA 2. *Let $e+j=n$ and let $H=K_{n,n}\backslash E(J)$. H can be totally coloured with $n+1$ colours if and only if H^* can be almost totally coloured with $n+1$ colours.*

Proof of Lemma 2. If H^* is almost totally coloured with $n+1$ colours, then this incorporates a total-colouring of H.

To prove the converse, suppose that H is totally coloured with $c_1, ..., c_{n+1}$. Then, by Lemma 1, J has a matching M with $|M|=j$, there are integers $j_1, ..., j_{n+1}$ such that $j=j_1+\cdots+j_{n+1}$, and, for $1 \leqslant i \leqslant n+1$, there are j_i edges of M whose end vertices are both coloured c_i. Therefore, for $1 \leqslant i \leqslant n+1$, R has $q_i - j_i$ further vertices coloured c_i, and thus L has $q_i - j_i$ vertices at which colour c_i is not used. Colour an edge joining r^* to each such vertex with c_i. Then altogether $\sum_{i=1}^{n+1} (q_i - j_i) = n - j = e$ such edges are coloured. There are therefore no further edges from r^* to vertices of L. Similarly, for $1 \leqslant i \leqslant n+1$, colour an edge joining l^* to vertices of R at which c_i is absent with c_i. This is the required almost total-colouring of H^*.

This proves Lemma 2.

LEMMA 3. *Let $e+j=n$ and let $H = K_{n,n} \backslash E(J)$. Let H have a vertex-colouring with colours $c_1, ..., c_{n+1}$. Let S^* be an induced subgraph of H^* such that l^*, r^*, and each end-vertex of each edge of J is in S^*. Let the vertex sets of S^* be $\{l^*, l_1, ..., l_\lambda\}$ and $\{r^*, r_1, ..., r_\rho\}$. Let the vertex-colouring of S^* be part of an almost total-colouring of S^* with $c_1, ..., c_{n+1}$, and, for $1 \leqslant i \leqslant n+1$, let N_i be the number of elements (vertices and edges) of S^* which are coloured c_i. Then the almost total-colouring of $S^* \cup L \cup R$ can be extended to an almost total-colouring of H^* with $c_1, ..., c_{n+1}$ if and only if*

(i) $N_i \geqslant \lambda + \rho - n + \max\{p_i, q_i\}$ $(\forall i \in \{1, ..., n+1\})$,

and

(ii) *for $1 \leqslant i \leqslant n+1$, in S^* there are $q_i - j_i$ edges coloured c_i incident with r^*, and there are $p_i - j_i$ edges coloured c_i incident with l^*.*

Proof. Necessity (of Lemma 3). Suppose that an almost total-colouring of $S^* \cup L \cup R$ with $c_1, ..., c_{n+1}$ is extended to an almost total-colouring of H^* with $c_1, ..., c_{n+1}$. Then, by Lemma 1, for $1 \leqslant i \leqslant n+1$, c_i occurs on $n + j_i$ elements of H. There are q_i vertices of R coloured c_i, and so it follows from Lemma 1 that there are $q_i - j_i$ vertices of L which, in H, have no edge coloured c_i incident with them, and are not themselves coloured c_i. Therefore in H^* there are $q_i - j_i$ edges coloured c_i incident with r^*, and there are similarly $p_i - j_i$ edges coloured c_i incident with l^*. This proves (ii).

Altogether there are $n + j_i + \max\{p_i - j_i, q_i - j_i\} = n + \max\{p_i, q_i\}$ elements of H^* coloured c_i. There are $n - \lambda$ elements of H^* coloured c_i incident with, or equal to, elements of $L \backslash V(S^*)$ and there are at most a further $n - \rho$ elements coloured c_i incident with, or equal to, elements of $R \backslash V(S^*)$. Therefore $N_i \geqslant n + \max\{p_i, q_i\} - (n - \lambda) - (n - \rho) = \rho + \lambda - n + \max\{p_i, q_i\}$. This proves (i).

Sufficiency (of Lemma 3). Suppose that we have an almost total-colouring with $c_1, ..., c_{n+1}$ of $S^* \cup L \cup R$, and that this incorporates a vertex-colouring of H. Suppose moreover that (i) and (ii) are satisfied.

If $\lambda < n$ then we shall extend the almost total-colouring of $S^* \cup L \cup R$ to an almost total-colouring of $T^* \cup L \cup R$, where T^* is the subgraph of H^* induced by $\{l^*, l_1, ..., l_n, r^*, r_1, ..., r_\rho\}$, in such a way that

$$N_i' \geqslant \rho + \max\{p_i, q_i\},$$

where N_i' denotes the number of elements of T^* coloured c_i [this is what (i) becomes when $n = \lambda$].

As an aid to doing this, we construct an auxilliary bipartite graph B. The vertex sets of B are $\{c_1', ..., c_{n+1}'\}$ and $\{r^{*\prime}, r_1', ..., r_\rho'\}$. In B, c_i' is joined to r_k' by an edge if in S^* colour c_i is neither on the vertex r_k nor on any edge incident with r_k; c_i' is joined to $r^{*\prime}$ by h_i edges $(1 \leqslant i \leqslant n+1,\ 1 \leqslant k \leqslant \rho)$, where h_i is the number of vertices coloured c_i in $\{l_{\lambda+1}, ..., l_n\}$.

For $1 \leqslant k \leqslant \rho$, in S^* the vertex r_k has $\lambda + 1$ colours used either on it, or on an edge incident with it. There are therefore $(n+1) - (\lambda+1) = n - \lambda$ colours not used on it or at it, so the degree in B of r_k' is $n - \lambda$. The degree in B of $r^{*\prime}$ is $\sum_{i=1}^{n+1} h_i = n - \lambda$.

For $1 \leqslant i \leqslant n+1$, the colour c_i is used on N_i elements of S^*, including, by (ii), $q_i - j_i$ edges incident with r^*, and $p_i - h_i$ vertices in $\{l_1, ..., l_\lambda\}$, and so it is not used on or at

$$\rho - N_i + (q_i - j_i) + (p_i - h_i)$$

of the vertices $r_1, ..., r_\rho$. Therefore in B the degree of c_i' is

$$\begin{aligned}
&\rho - N_i + (q_i - j_i) + (p_i - h_i) + h_i \\
&= \rho - N_i + p_i + q_i - j_i \\
&\leqslant \rho - (\lambda + \rho - n + \max\{p_i, q_i\}) + p_i + q_i - j_i \\
&= n - \lambda - \max\{p_i, q_i\} + (p_i + q_i - j_i) \\
&= n - \lambda.
\end{aligned}$$

By König's theorem [8] we may now edge-colour B with $n - \lambda$ colours, $l_{\lambda+1}', ..., l_n'$. The edges incident with $r^{*\prime}$ will all get different colours, so we may suppose without loss of generality that the edge $r^{*\prime}c_i'$ of B is coloured with a colour l_m', where in $S^* \cup L \cup R$ c_i is the colour used on the vertex $l_m (1 \leqslant i \leqslant n+1)$.

We use the edge-colouring of B to extend the almost total-colouring of $S^* \cup L \cup R$ to $T^* \cup L \cup R$ as follows. For $1 \leqslant i \leqslant n+1$, $1 \leqslant k \leqslant \rho$, $\lambda + 1 \leqslant m \leqslant n$, if edge $c_i'r_k'$ of B is coloured l_m', then we colour edge $l_m r_k$ of H^* with colour c_i.

It is easy to check that this is an almost total-colouring of $T^* \cup L \cup R$. The colouring of $L \cup R$ is a vertex-colouring of H. For $\lambda + 1 \leqslant m \leqslant n$, $1 \leqslant k_1 < k_2 \leqslant \rho$, two edges $l_m r_{k_1}$ and $l_m r_{k_2}$ of H^* receive different colours because in B the edges coloured l_m' incident with the vertices r_{k_1}' and r_{k_2}' do not have a common vertex. Similarly for $\lambda+1 \leqslant m_1 < m_2 \leqslant n$, $1 \leqslant k \leqslant \rho$, the edges $l_{m_1} r_k$ and $l_{m_2} r_k$ of H^* have different colours because in B different edges incident with r'_k were coloured l'_{m_1} and l'_{m_2}. For $1 \leqslant k \leqslant \rho$, $\lambda+1 \leqslant m \leqslant n$, the edge $l_m r_k$ and the vertex l_m in H^* have different colours because the edges on $r^{*\prime}$ and r'_k in B coloured l'_m do not have a common vertex. Finally for $1 \leqslant k \leqslant \rho$, $1 \leqslant m_1 \leqslant \lambda < m_2 \leqslant n$, the edges $r_k l_{m_1}$ and $r_k l_{m_2}$ of H^* have different colours because of the definition of the graph B.

If $\rho = n$ the argument is complete as we now have an almost total-colouring of H^*. If $\rho < n$ then $T^* \cup L \cup R$ has an almost total-colouring, and condition (ii) is satisfied. Moreover, for $1 \leqslant i \leqslant n+1$, since the degree in B of c'_i was $\rho - N_i + p_i + q_i - j_i$, it follows that c_i occurs on

$$(\rho - N_i + p_i + q_i - j_i) + N_i = \rho + p_i + q_i - j_i = \rho + \max\{p_i, q_i\}$$

edges and vertices of T^*. In other words, $N'_i \geqslant \rho + \max\{p_i, q_i\}$, as required. We then repeat the earlier argument with λ, $l_1, ..., l_\lambda$, ρ, $r_1, ..., r_\rho$ replaced by ρ, $r_1, ..., r_\rho$, n, $l_1, ..., l_n$, respectively.

This concludes the proof of Lemma 3.

For any bipartite simple graph J with vertex set $L \cup R$, where each edge joins a vertex of L to a vertex of R, let $\Delta_L(J)$ and $\Delta_R(J)$ denote the maximum degree in J of the vertices of L and of R, respectively. The next two lemmas are needed for the proof of Lemma 6.

Lemma 4. *Let J be a bipartite simple graph with vertex set $L \cup R$, where each edge joins a vertex of L to a vertex of R. Let the maximum size of a matching in J be j, and let the number of edges of J be e. Then*

$$\Delta_L(J) + \Delta_R(J) \leqslant e + 3 - j.$$

Proof. The j edges of a maximum sized matching in J contribute at most two to $\Delta_L(J) + \Delta_R(J)$. The remaining $e-j$ edges of J contribute at most $e-j+1$ to $\Delta_L(J) + \Delta_R(J)$, since at most one can contribute twice. Thus

$$\Delta_L(J) + \Delta_R(J) \leqslant (e-j+1) + 2 = e - j + 3.$$

Lemma 5. *Let $e+j=n$ and let $H = K_{n,n} \backslash E(J)$. Then*

$$\Delta_L + \Delta_R \leqslant n + 1 - j.$$

Proof. If $j \geqslant 2$ then, by Lemma 4.

$$\Delta_L + \Delta_R \leqslant e + 3 - j = n + 3 - 2j \leqslant n + 1 - j.$$

If $j=1$, then clearly

$$\Delta_L+\Delta_R\leqslant e+1=n=n+1-j.$$

This proves Lemma 5.

For the next lemma we need the following notation. For $1\leqslant i\leqslant n+1$, let s_i be the number of vertices of J which are in L and have degree in J i or more; similarly let t_i be the number of vertices of J which are in R and have degree in J i or more. Clearly $s_1\geqslant s_2\geqslant \cdots \geqslant s_{n+1}$ and $t_1\geqslant t_2\geqslant \cdots \geqslant t_{n+1}$.

LEMMA 6. *Let $e+j=n$ and $H=K_{n,n}\backslash E(J)$. Then H^* has an almost total-colouring with $n+1$ colours.*

Proof. First we describe a particular vertex-colouring of H which, as we shall show, can be extended to an almost total-colouring of H^*. Let M be a matching of J with $|M|=j$. For $1\leqslant i\leqslant j$, let c_i be placed on the vertices at each end of one of the edges of J. Then, for $1\leqslant i\leqslant n+1-j$, let t_i further vertices of L have colour c_{i+j}. By Lemma 5, $\Delta_R\leqslant n+1-j$, so

$$\left(\sum_{i=1}^{n+1-j} t_i\right)+j=e+j=n,$$

and so all vertices of L receive a colour. We may suppose that y is such that $t_y>0$, $t_{y+1}=\cdots=t_{n+1}=0$ $(y=\Delta_R(J))$.

Now for $1\leqslant i\leqslant n+1-y-j$, let s_i further vertices of R (apart from the vertices incident with edges of M) be colored c_{j+y+i}. By Lemma 5, $s_{n+1-y-j}=\cdots=s_{n+1}=0$ and

$$\left(\sum_{i=1}^{n+1-y-j} s_i\right)+j=e+j=n.$$

Thus all vertices of R receive a colour.

Now let S^* be the subgraph of H^* induced by l^*, r^*, and the vertices at the ends of J. This graph has at most $2+j+e=n+2$ vertices. Let n_L and n_R be the number of vertices of L and R, respectively, which are incident with edges of J. Then $n_L+n_R\leqslant e+j=n$.

We now have a vertex-colouring of H, and we let p_i, q_i, and j_i have their standard meanings. Note that $p_1=\cdots=p_j=q_1=\cdots=q_j=j_1=\cdots=j_i=1$ and that $p_{i+j}=t_i$ and $q_{i+j}=s_j$ for $1\leqslant i\leqslant n+1-j$. We colour the edges incident with l^* and r^* as follows. For $1\leqslant i\leqslant n+1-y-j$, we colour an edge joining r^* to each of s_i vertices of $V(J)\cap L$ with colour c_{j+y+i}. Similarly, for $1\leqslant i\leqslant n+1-j$ we colour an edge joining l^* to each of t_i vertices of $V(J)\cap R$ with colour c_{i+j}. These colourings are done in accordance with the almost total-colouring rules, and, for $1\leqslant i\leqslant n+1$, q_i-j_i edges coloured c_i are incident with r^*, and $p_i - j_i$ edges coloured c_i are incident with l^*, so that (ii) of Lemma 3 is satisfied. Moreover notice that, for $1 \leqslant i \leqslant n+1$, colour c_i is used on $V(J)$ or on an edge incident with l^* or r^* at least $\max\{p_i, q_i\}$ times altogether.

We now colour all the remaining edges of S^* in accordance with the almost total-colouring rules. We colour these edges one by one. If a particular edge e^* is not coloured so far, then the number of colours used on or at the vertex of e^* in L is at most $(n_R - 1) + 1 = n_R$, and the number of colours used on or at the vertex of e^* in R is at most n_L. Thus the number of colours available to colour e^* with is at least

$$n + 1 - n_L - n_R \geqslant 1,$$

so e^* can in fact be coloured. When this colouring of S^* is completed, then, of course, for $1 \leqslant i \leqslant n+1$, colour c_i is still used on at least $\max\{p_i, q_i\}$ elements.

In the notation of Lemma 3 we have $\lambda = n_L$, $\rho = n_R$, so that, for $1 \leqslant i \leqslant n+1$,

$$\lambda + \rho - n + \max\{p_i, q_i\} \leqslant \max\{p_i, q_i\} \leqslant N_i.$$

Thus, by Lemma 3, the almost total-colouring of $S^* \cup L \cup R$ with $c_1, \ldots, c_{n+1}$ can be extended to an almost total-colouring of H^* with $c_1, \ldots, c_{n+1}$.

This proves Lemma 6.

LEMMA 7. *Let $e + j = n$ and $H = K_{n,n} \backslash E(J)$. Then H can be totally coloured with $n+1$ colours.*

Proof. This now follows from Lemmas 2 and 6.

The other main case we have to consider in order to prove the sufficiency in Theorem 1 is the case $e + j = (n+1)$ when n is odd and J consists of $(1/2)(n+1)$ independent edges.

LEMMA 8. *Let $n \geqslant 3$ be odd and let J consist of $(1/2)(n+1)$ independent edges. Let $H = K_{n,n} \backslash E(J)$. Then H can be totally coloured with $n+1$ colours.*

Proof. If $n = 3$ the lemma may easily be verified, so suppose that $n \geqslant 5$. Let J consist of edges $l_1 r_1, \ldots, l_{(1/2)(n+1)} r_{(1/2)(n+1)}$, where $l_1, \ldots, l_{(1/2)(n+1)} \in L$ and $r_1, \ldots, r_{(1/2)(n+1)} \in R$. Let J^+ be obtained from J by removing the edge $l_{(1/2)(n+1)} r_{(1/2)(n+1)}$ and replacing it by $l_{(1/2)(n-1)} r_{(1/2)(n+1)}$.

By Lemma 7, the graph $H^+ = K_{n,n} \backslash J^+$ can be totally coloured with $n+1$ colours, $c_1, \ldots, c_{n+1}$. Moreover, if we follow the procedure described in the proof of Lemma 6, then, for $1 \leqslant i \leqslant (1/2)(n-1)$, we will have l_i and r_i

coloured c_i. For $(1/2)(n+1) \leqslant i \leqslant n$, l_i will be coloured $c_{(1/2)(n+1)}$, one of $r_{(1/2)(n+1)}, ..., r_n$ will be coloured $c_{(1/2)(n+5)}$, the rest will be coloured $c_{(1/2)(n+3)}$. We may assume that $r_{(1/2)(n+1)}$ is coloured $c_{(1/2)(n+5)}$, and that, for $(1/2)(n+3) \leqslant i \leqslant n$, r_i is coloured $c_{(1/2)(n+3)}$. Notice that one of the two edges joining r^* to $l_{(1/2)(n-1)}$ will be coloured $c_{(1/2)(n+5)}$.

When, in the proof of Lemma 6, the edges of S^* are coloured, after the edges incident with l^* and r^* are coloured we can colour the edges incident with $r_{(1/2)(n+1)}$ without using c_{n+1}. Then, in the resulting total-colouring of H^+, the colour c_{n+1} will be used on an edge $l_i r_{(1/2)(n+1)}$ for some $i \in \{(1/2)(n+1), ..., n+1\}$. We may without loss of generality, suppose that the edge $l_{(1/2)(n+1)}\, r_{(1/2)(n+1)}$ is coloured c_{n+1}.

Finally from H^+ we remove the edge $l_{(1/2)(n+1)}\, r_{(1/2)(n+1)}$, we restore the edge $l_{(1/2)(n-1)}\, r_{(1/2)(n+1)}$ coloured $c_{(1/2)(n+5)}$, and we recolour the vertices $l_{(1/2)(n+1)}$, $r_{(1/2)(n+1)}$ with c_{n+1}. It is easy to check that this is a total-colouring of H (the "before" and "after" colourings are shown in Fig. 1).

This proves Lemma 8.

Proof of the Sufficiency in Theorem 1. As remarked earlier, by adding in edges if necessary, we can without loss of generality assume that either $e+j=n$, or n is odd, $e+j=(n+1)$, and J consists of $(1/2)(n+1)$ independent edges. In the first case the sufficiency follows from Lemma 7, and in the second it follows from Lemma 8.

3. Concluding Remarks

We have the following corollary to Theorem 1.

THEOREM 2. *Let $n \geqslant 1$, let J be a subgraph of $K_{n,n}$, let $e = |E(J)|$, and let $j(J)$ be the maximum size of a matching in J. Let $H = K_{n,n} \backslash J$. Then*

$$\chi_T(H) = \begin{cases} \Delta(H)+2 & \text{if } e+j \leqslant n-1 \\ \Delta(H)+1 & \text{if } 2n-1 \geqslant e+j \geqslant n. \end{cases}$$

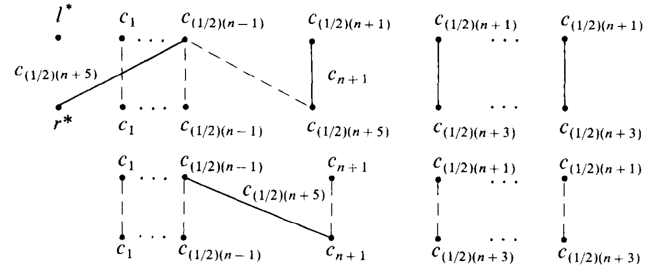

FIG. 1. $(H^+)^*$, then H.

Proof. If $e+j \leqslant 2n-1$, then J has at most $2n-1$ vertices, and so $\Delta(H)=n$.

We also remark that in [5] the author and A. G. Chetwynd have formulated a conjecture which could place Theorem 1 in a much wider context.

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