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$D^*(\mathbb{R};\leqslant)$ does not imply D_1^*

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Abstract

We prove that the cyclic monotonically normal space T of Rudin is not a J_2-space. Consequently, T has the monotone extension property but does not have D_1^* or $D^*(\mathbb{R};+;\text{cch})$. This answers some questions of van Douwen. © 1998 Elsevier Science B.V.

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In memory of Kiiti Morita

Throughout, we use the terminology of [2]. The definition of a J_2-space and J_2-function appears in Definition 1 of [2]. The definitions of $D^*(\mathbb{R};\leqslant)$, D_1^* and $D^*(\mathbb{R};+;\text{cch})$ appear in the introduction of [2, pp. 82, 83].

Theorem 1. *There exists a monotonically normal space T which is not a J_2-space.*

Proof. The space T is the space described by Rudin in [6]. Since our argument is clearly gleaned from the proof in [6], we will use the same notation and simply point out the necessary changes. For convenience, let $(x_i)_j = x_{ij}$, and let τ be the topology on T described in [6].

Let $k:\tau|Y \to \tau$ be a K_1-function such that $U, V \in \tau|Y$ and $U \subset V$ imply that $k(U) \subset k(V)$. Then, from [6, p. 305], we get the following fact:

(a) There is $p \in X$ such that, for all $x > p$ in X and $i < 3$, $U_{pxi} \subset k(U_{xi} \cap Y)$.

Henceforth, we fix p to satisfy (a).

(b′) If $p < t \in X$ and $i < 3$, then $U_{pti} \subset \bigcap_{j<3} k(U_{t_ij} \cap Y)$. (The proof of this follows immediately from the proof of (b) in [6].)

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If $n < \omega$ and $p < x \in X$, let V_{nx1} be the set of $v \in X$ such that $x_1 \leqslant v$, n is the number of terms of X between x_1 and v and all these terms are of the form u_1, for some $x_1 \leqslant u < v$. Let S'_n be the statement that

$$V_{nx1} \subset k(U_{x_1 0} \cap Y) \cap k(U_{x_1 1} \cap Y), \quad \text{for all } x \in X.$$

(c′) S'_n holds for all $n < w$.

Proof of (c′). S'_0 holds since $V_{0x1} = \{x_1\}$ and

$$x_1 \in U_{px1} \subset \bigcap_{j<3} k(U_{x_1 j} \cap Y) \subset k(U_{x_1 0} \cap Y) \cap k(U_{x_1 1} \cap Y),$$

by (b′). Suppose S'_n holds and let $v \in V_{(n+1)x1}$, for some $x \in X$. Note that $x_{11} \leqslant v$; hence, $v \in V_{nx_1 1}$. Then by inductive hypothesis,

$$v \in k(U_{x_{11} 0} \cap Y) \cap k(U_{x_{11} 1} \cap Y) \subset k(U_{x_1 1} \cap Y),$$

because k is monotone. Also, $v \in U_{px_1 0} \subset k(U_{x_1 0} \cap Y)$, by (a). Hence,

$$v \in k(U_{x_1 0} \cap Y) \cap k(U_{x_1 1} \cap Y),$$

and this completes the induction. □

(d′) k is not a J_2-function.

Proof of (d′). Fix x and choose $y \in Y$ which extends x_1 such that if $w \in X$ and $x_1 < w < y$ then $w = u_1$ for some $x_1 \leqslant u < y$. Since k is a K_1-function, there exists $z \in X$ such that $x_1 < z < y$ and $B_z(y) \subset k(B_{x_1}(y) \cap Y)$. Pick $z < v < y$, and note that, since $v \in$ some V_{nx1},

$$v \in k(U_{x_1 0} \cap Y) \cap k(U_{x_1 1} \cap Y),$$

by (c′). Hence,

$$v \in k\big(B_{x_1}(y) \cap Y\big) \cap k(U_{x_1 0} \cap Y) \cap k(U_{x_1 1} \cap Y) \neq \emptyset,$$

but

$$B_{x_1}(y) \cap (U_{x_1 0} \cap U_{x_1 1}) \cap Y = B_{x_1}(y) \cap U_{rx1} \cap Y = \emptyset,$$

and this proves that k is not a J_2-function. □

Theorem 2. *The space T has $D^*(\mathbb{R}; \leqslant)$ but it does not have D^*_1 or $D^*(\mathbb{R}; +; \mathrm{cch})$.*

Proof. Since T is monotonically normal, T has $D^*(\mathbb{R}; \leqslant)$ by Theorem 3.3 of [5]. By Theorem 9 of [1], T does not have the property D^*_1. From the diagram of Theorem 10 of [1] one immediately gets that T does not have the property $D^*(\mathbb{R}; +; \mathrm{cch})$. □

Theorem 2 answers the question on lines 9 and 10 [3, p. 31]; it also answers the first question on [3, p. 37]; see also line 10 on [4, p. 300].

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